ON THE NUMBER OF MAXIMAL SUM-FREE SETS

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ABSTRACT. It is shown that the set $\{1, 2, \dots, n\}$ contains at most $2^{n/2-2^{-28}n}$ maximal sum-free subsets, provided n is large enough.

A set $A \subseteq [n] = \{1, 2, \dots, n\}$ is sum-free if for any two elements $a, b \in A$ we have $a + b \notin A$. A sum-free set $A \subseteq [n]$ is maximal if it is not contained in any other sum-free subset of $[n]$. Let $s(n)$ and $s_{\max}(n)$ denote the number of sum-free and maximal sum-free subsets of $[n]$, respectively. Since the set of odd numbers is sum-free, and so is each of its subsets, $s(n) \geq 2^{\lceil n/2 \rceil}$. It is conjectured that, in fact, we have $s(n) \leq c2^{n/2}$ for some constant $c > 0$ but at this moment we know only (see Calkin [2] and Alon [1]) that the value of the exponent is close to $n/2$, i.e. the following holds.

Theorem 1. $s(n) = 2^{n/2+o(n)}$. □

In this note we study the behaviour of $s_{\max}(n)$. Cameron and Erdős [4] observed that $s_{\max}(n) \geq 2^{\lfloor n/4 \rfloor}$ and asked if $s_{\max}(n) = o(s(n))$, or, maybe, even $s_{\max}(n) \leq 2^{n/2-\epsilon n}$, holds for some constant $\epsilon > 0$. Our main result states that this is indeed the case.

Theorem 2. *There exists n_0 such that for $n \geq n_0$ we have $s_{\max}(n) \leq 2^{n/2-2^{-28}n}$.*

For a maximal sum-free set $A \subseteq [n]$ and $B \subseteq A$ let

$$h_A(B) = |[n] \setminus [(A \setminus B) \cup ((A \setminus B) + (A \setminus B)) \cup ((A \setminus B) - (A \setminus B)) \cup (A \setminus B)/2]|,$$

i.e. $h_A(B)$ denotes the number of elements $i \in [n]$ one can add to $A \setminus B$ so that $\{i\} \cup (A \setminus B)$ remains sum-free. Our argument relies on the following result of probabilistic flavor.

Lemma 3. *Let β be a constant such that $0 < \beta < 1/2$. Then there exists n_0 such that for $n \geq n_0$, every maximal sum-free set $A \subseteq [n]$ with $|A| = m \geq n/11$ contains at least $n^{-5\sqrt{n}} \binom{m}{k}$ subsets B of $k = \lfloor \beta m \rfloor$ elements for which $h_A(B) \leq 2\beta n$.*

Proof. For each $i \in [n] \setminus A$ let us choose a pair $R_i = \{a'_i, a''_i\}$ of two, not necessarily distinct, elements of A such that either $i = a'_i + a''_i$, or $i = a'_i - a''_i$, or, maybe, $2i = a'_i = a''_i$ (since A is a maximal sum-free set such a pair always exists). Let

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A' be a set of all $a \in A$ which belong to at most $\sqrt{n}$ from the sets R_i , $i \in [n] \setminus A$. Clearly, $m' = |A'| \geq |A| - 2\sqrt{n}$. Let

$$W = \{i : R_i \cap A' \neq \emptyset\},$$

and denote by $\mathbf{B}$ a set chosen uniformly at random among all subsets of A' with k elements. We shall study the behaviour of the random variable $\mathbf{X} = \sum_{i \in W} \mathbf{X}_i$, where for $i \in W$

$$\mathbf{X}_i = \begin{cases} 0 & \text{if } R_i \cap \mathbf{B} = \emptyset, \\ 1 & \text{if } R_i \cap \mathbf{B} \neq \emptyset. \end{cases}$$

For the expectation of $\mathbf{X}$, we get

$$\begin{aligned} \mathbb{E} \mathbf{X} &= \sum_{i \in W} \mathbb{E} \mathbf{X}_i \leq |W| \left(1 - \min \left\{ \frac{\binom{m'-2}{k}}{\binom{m'}{k}}, \frac{\binom{m'-1}{k}}{\binom{m'}{k}} \right\} \right) \\ &\leq (1 + o(1))(n - m)(1 - (1 - (k/m')^2)) \leq 2\beta n - 2\beta m. \end{aligned}$$

In order to estimate the variance $\text{Var } \mathbf{X}$ note that

$$\text{Var } \mathbf{X} \leq \sum_{i,j: R_i \cap R_j \neq \emptyset} |\text{Cov}(\mathbf{X}_i, \mathbf{X}_j)| + \sum_{i,j: R_i \cap R_j = \emptyset} |\text{Cov}(\mathbf{X}_i, \mathbf{X}_j)|.$$

As a crude upper bound for the first sum one can take

$$\sum_{i,j: R_i \cap R_j \neq \emptyset} |\text{Cov}(\mathbf{X}_i, \mathbf{X}_j)| \leq \sum_{i,j: R_i \cap R_j \neq \emptyset} 1 \leq 2n\sqrt{n},$$

while the second sum is not larger than

$$\left(\sum_{i \in W} \mathbb{E} \mathbf{X}_i \right) m' \left[\left| \frac{\binom{m'-2}{k}}{\binom{m'}{k}} - \frac{\binom{m'-4}{k-2}}{\binom{m'-2}{k-2}} \right| + \left| \frac{\binom{m'-1}{k}}{\binom{m'}{k}} - \frac{\binom{m'-3}{k-1}}{\binom{m'-1}{k-1}} \right| \right] \leq \mathbb{E} \mathbf{X} m' \frac{11}{m'} \leq 11 \mathbb{E} \mathbf{X}.$$

Hence

$$\text{Var } \mathbf{X} \leq 11 \mathbb{E} \mathbf{X} + 2n\sqrt{n} \leq 3n\sqrt{n} = o(n^2),$$

and so Chebyshev's inequality implies that, for n large enough, with probability at least $1/2$ we have $\mathbf{X} \leq 2\beta n - \beta m$. Thus, since $h_A(\mathbf{B}) \leq \mathbf{X} + |\mathbf{B}|$, there exist at least

$$\frac{1}{2} \binom{m'}{k} \geq n^{-5\sqrt{n}} \binom{m}{k}$$

subsets B of A with k elements for which $h_A(B) \leq 2\beta n$. □

Proof of Theorem 2. Let us first estimate the number $s'_{\max}(n)$ of all maximal sum-free subsets of $[n]$ with at least $n/11$ elements. Set $\beta = 2^{-23}$ and consider the number t of all pairs $(A, A \setminus B)$, where A is a maximal sum-free set of at least $n/11$ elements and $B \subseteq A$ is such that $|B| = \lfloor \beta |A| \rfloor$ and $h_A(B) \leq 2\beta n$. Then, due to Lemma 3, for n large enough we have

$$t \geq s'_{\max}(n) n^{-5\sqrt{n}} \binom{|A|}{|B|} \geq s'_{\max}(n) n^{-5\sqrt{n}} \binom{\lfloor n/11 \rfloor}{\lfloor \beta n/11 \rfloor} \geq s'_{\max}(n) \beta^{-\beta n/11}.$$

On the other hand, there are at most $2^{2\beta n}$ ways to enlarge a set $A \setminus B$ to a maximal sum-free set. Thus, $t \leq s(n)2^{2\beta n}$ and using Theorem 1 we infer that

$$s'_{\max}(n) \leq s(n)2^{2\beta n}\beta^{\beta n/11} \leq 2^{n/2+o(n)}(4^{11}\beta)^{\beta n/11} \leq 2^{n/2-2^{-27}n}.$$

Hence

$$s_{\max}(n) \leq s'_{\max}(n) + \sum_{i=1}^{n/11} \binom{n}{i} \leq 2^{n/2-2^{-27}n} + 2^{49n/100} \leq 2^{n/2-2^{-28}n}. \quad \square$$

Note that in the proof of Theorem 2 we used no arithmetic properties of the natural numbers. Thus, our result can be generalized as follows. Let $\oplus : \Omega \times \Omega \rightarrow \Omega$ be an operation in A . We say that $A \subseteq \Omega$ is $\oplus$ -free if $a \oplus b \notin A$ for $a, b \in A$ and define a maximal $\oplus$ -free set accordingly. Furthermore, for $A \subseteq \Omega$, let $\sigma(A)$ and $\sigma_{\max}(A)$ denote the number of $\oplus$ -free and maximal $\oplus$ -free subsets of A , respectively. Then one can mimic our argument to show the following result.

Theorem 4. *For every $\alpha > 0$ there exist $\epsilon > 0$ and n_0 such that the following holds. For every set Ω with operation $\oplus : \Omega \times \Omega \rightarrow \Omega$ and every $A \subseteq \Omega$ for which $|A| = n > n_0$ and $\sigma(A) \geq 2^{\alpha n}$, we have $\sigma_{\max}(A) \leq \sigma(A)2^{-\epsilon n}$.* $\square$

Finally, we remark that Cameron [3] studied the set of all subsets of the natural numbers as a metric space with the distance defined as

$$d(A, B) = 2^{-\min\{A \triangle B\}},$$

where $A \triangle B$ denote the symmetric difference of sets A and B . Then the set of all sum-free subsets of natural numbers has Hausdorff dimension $1/2$ (it follows immediately from Theorem 1 and the fact that all sets which contain only odd numbers are sum-free). Theorem 2 implies that the dimension of the set of all maximal sum-free subsets of the natural numbers is at most $1/2 - 2^{-28} < 1/2$.

REFERENCES

1. N. Alon, *Independent sets in regular graphs and sum-free subsets of finite groups*, Israel J. Math **73** (1991), 247–256. MR **92k**:11024
2. N. Calkin, *On the number of sum-free sets*, Bull. London Math. Soc. **22** (1990) 141–144. MR **91b**:11015
3. P. Cameron, *Portrait of a typical sum-free set*. In “Survey in Combinatorics 1987” (C. Whitehead, ed.), London Mathematical Lecture Note Ser. **123**, Cambridge University Press, 1987, 13–42. MR **88k**:05138
4. P. Cameron and P. Erdős, *On the number of sets of integers with various properties*. In “Number Theory: Proc. First Conf. Can. Number Th. Ass.” (R.A. Mollin, ed.), Banff, 1988, de Gruyter, 1990, 61–79. MR **92g**:11010

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