AN INEQUALITY FOR THE NORM OF A POLYNOMIAL FACTOR

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ABSTRACT. Let $p(z)$ be a monic polynomial of degree n , with complex coefficients, and let $q(z)$ be its monic factor. We prove an asymptotically sharp inequality of the form $\|q\|_E \leq C^n \|p\|_E$, where $\|\cdot\|_E$ denotes the sup norm on a compact set E in the plane. The best constant C_E in this inequality is found by potential theoretic methods. We also consider applications of the general result to the cases of a disk and a segment.

1. INTRODUCTION

Let $p(z)$ be a monic polynomial of degree n , with complex coefficients. Suppose that $p(z)$ has a monic factor $q(z)$, so that

$$p(z) = q(z)r(z),$$

where $r(z)$ is also a monic polynomial. Define the uniform (sup) norm on a compact set E in the complex plane $\mathbb{C}$ by

$$(1.1) \quad \|f\|_E := \sup_{z \in E} |f(z)|.$$

We study the inequalities of the following form:

$$(1.2) \quad \|q\|_E \leq C^n \|p\|_E, \quad \deg p = n,$$

where the main problem is to find the best (the smallest) constant C_E , such that (1.2) is valid for *any* monic polynomial $p(z)$ and *any* monic factor $q(z)$.

In the case $E = \overline{D}$, where $D := \{z : |z| < 1\}$, the inequality (1.2) was considered in a series of papers by Mignotte [9], Granville [7] and Glesser [6], who obtained a number of improvements on the upper bound for $C_{\overline{D}}$. D. W. Boyd [3] made the final step here, by proving that

$$(1.3) \quad \|q\|_{\overline{D}} \leq \beta^n \|p\|_{\overline{D}},$$

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with

$$(1.4) \quad \beta := \exp \left(\frac{1}{\pi} \int_0^{2\pi/3} \log \left(2 \cos \frac{t}{2} \right) dt \right).$$

The constant $\beta = C_{\overline{D}}$ is asymptotically sharp, as $n \rightarrow \infty$, and it can also be expressed in a different way, using Mahler's measure. This problem is of importance in designing algorithms for factoring polynomials with integer coefficients over integers. We refer to [5] and [8] for more information on the connection with symbolic computations.

A further development related to (1.2) for $E = [-a, a]$, $a > 0$, was suggested by P. B. Borwein in [1] (see Theorems 2 and 5 there or see Section 5.3 in [2]). In particular, Borwein proved that if $\deg q = m$, then

$$(1.5) \quad |q(-a)| \leq \|p\|_{[-a,a]} a^{m-n} 2^{n-1} \prod_{k=1}^m \left(1 + \cos \frac{2k-1}{2n} \pi \right),$$

where the bound is attained for a monic Chebyshev polynomial of degree n on $[-a, a]$ and a factor q . He also showed that, for $E = [-2, 2]$, the constant in the above inequality satisfies

$$\begin{aligned} & \limsup_{n \rightarrow \infty} \left(2^{m-1} \prod_{k=1}^m \left(1 + \cos \frac{2k-1}{2n} \pi \right) \right)^{1/n} \\ & \leq \lim_{n \rightarrow \infty} \left(2^{[2n/3]-1} \prod_{k=1}^{[2n/3]} \left(1 + \cos \frac{2k-1}{2n} \pi \right) \right)^{1/n} \\ & = \exp \left(\int_0^{2/3} \log (2 + 2 \cos \pi x) dx \right) = 1.9081 \dots, \end{aligned}$$

which hints that

$$(1.6) \quad C_{[-2,2]} = \exp \left(\int_0^{2/3} \log (2 + 2 \cos \pi x) dx \right) = 1.9081 \dots$$

We find the asymptotically best constant C_E in (1.2) for a rather arbitrary compact set E . The general result is then applied to the cases of a disk and a line segment, so that we recover (1.3)–(1.4) and confirm (1.6).

2. RESULTS

Our solution of the above problem is based on certain ideas from the logarithmic potential theory (cf. [12] or [13]). Let $\text{cap}(E)$ be the *logarithmic capacity* of a compact set $E \subset \mathbb{C}$. For E with $\text{cap}(E) > 0$, denote the *equilibrium measure* of E (in the sense of the logarithmic potential theory) by μ_E . We remark that μ_E is a positive unit Borel measure supported on E , $\text{supp } \mu_E \subset E$ (see [13, p. 55]).

Theorem 2.1. *Let $E \subset \mathbb{C}$ be a compact set, $\text{cap}(E) > 0$. Then the best constant C_E in (1.2) is given by*

$$(2.1) \quad C_E = \frac{\max_{u \in \partial E} \exp \left(\int_{|z-u| \geq 1} \log |z-u| d\mu_E(z) \right)}{\text{cap}(E)}.$$

Furthermore, if E is regular, then

$$(2.2) \quad C_E = \max_{u \in \partial E} \exp \left(- \int_{|z-u| \leq 1} \log |z-u| d\mu_E(z) \right).$$

The above notion of regularity is to be understood in the sense of the exterior Dirichlet problem (cf. [13, p. 7]). Note that the condition $\text{cap}(E) > 0$ is usually satisfied for all applications, as it only fails for very *thin* sets (see [13, pp. 63–66]), e.g., finite sets in the plane. But if E consists of finitely many points, then the inequality (1.2) cannot be true for a polynomial $p(z)$ with zeros at every point of E and for its linear factors $q(z)$. On the other hand, Theorem 2.1 is applicable to any compact set with a connected component consisting of more than one point (cf. [13, p. 56]).

One can readily see from (1.2) or (2.1) that the best constant C_E is invariant under the rigid motions of the set E in the plane. Therefore we consider applications of Theorem 2.1 to the family of disks $D_r := \{z : |z| < r\}$, which are centered at the origin, and to the family of segments $[-a, a]$, $a > 0$.

Corollary 2.2. *Let D_r be a disk of radius r . Then the best constant $C_{\overline{D_r}}$, for $E = \overline{D_r}$, is given by*

$$(2.3) \quad C_{\overline{D_r}} = \begin{cases} \frac{1}{r}, & 0 < r \leq 1/2, \\ \frac{1}{r} \exp \left(\frac{1}{\pi} \int_0^{\pi - 2 \arcsin \frac{1}{2r}} \log \left(2r \cos \frac{x}{2} \right) dx \right), & r > 1/2. \end{cases}$$

Note that (1.3)–(1.4) immediately follow from (2.3) for $r = 1$. The graph of $C_{\overline{D_r}}$, as a function of r , is in Figure 1.

FIGURE 1. $C_{\overline{D_r}}$ as a function of r .

Corollary 2.3. *If $E = [-a, a]$, $a > 0$, then*

$$(2.4) \quad C_{[-a,a]} = \begin{cases} \frac{2}{a}, & 0 < a \leq 1/2, \\ \frac{2}{a} \exp \left(\int_{1-a}^a \frac{\log(t+a)}{\pi \sqrt{a^2 - t^2}} dt \right), & a > 1/2. \end{cases}$$

Observe that (2.4), with $a = 2$, implies (1.6) by the change of variable $t = 2 \cos \pi x$. We include the graph of $C_{[-a,a]}$, as a function of a , in Figure 2.

FIGURE 2. $C_{[-a,a]}$ as a function of a .

We now state two general consequences of Theorem 2.1. They explain some interesting features of C_E , which the reader may have noticed in Corollaries 2.2 and 2.3. Let

$$\text{diam}(E) := \max_{z, \zeta \in E} |z - \zeta|$$

be the Euclidean diameter of E .

Corollary 2.4. *Suppose that $\text{cap}(E) > 0$. If $\text{diam}(E) \leq 1$, then*

$$(2.5) \quad C_E = \frac{1}{\text{cap}(E)}.$$

It is well known that $\text{cap}(D_r) = r$ and $\text{cap}([-a, a]) = a/2$ (see [12, p. 135]), which clarifies the first lines of (2.3) and (2.4) by (2.5).

The next corollary shows how the constant C_E behaves under dilations of the set E . Let αE be the dilation of E with a factor $\alpha > 0$.

Corollary 2.5. *If E is regular, then*

$$(2.6) \quad \lim_{\alpha \rightarrow +\infty} C_{\alpha E} = 1.$$

Thus Figures 1 and 2 clearly illustrate (2.6).

We remark that one can deduce inequalities of the type (1.2), for various L_p norms, from Theorem 2.1, by using relations between L_p and L_∞ norms of polynomials on E (see, e.g., [11]).

3. PROOFS

Proof of Theorem 2.1. The proof of this result is based on the ideas of [3] and [10]. For $u \in \mathbb{C}$, consider a function

$$\rho_u(z) := \max(|z - u|, 1), \quad z \in \mathbb{C}.$$

One can immediately see that $\log \rho_u(z)$ is a subharmonic function in $z \in \mathbb{C}$, which has the following integral representation (see [12, p. 29]):

$$(3.1) \quad \log \rho_u(z) = \int \log |z - t| d\lambda_u(t), \quad z \in \mathbb{C},$$

where $d\lambda_u(u + e^{i\theta}) = d\theta/(2\pi)$ is the normalized angular measure on $|t - u| = 1$. Let $u \in \partial E$ be such that

$$\|q\|_E = |q(u)|.$$

If z_k , $k = 1, \dots, n$, are the zeros of $p(z)$, arranged so that the first m zeros belong to $q(z)$, then

$$\begin{aligned} \log \|q\|_E &= \sum_{k=1}^m \log |u - z_k| \leq \sum_{k=1}^m \log \rho_u(z_k) \leq \sum_{k=1}^n \log \rho_u(z_k) \\ (3.2) \quad &= \sum_{k=1}^n \int \log |z_k - t| d\lambda_u(t) = \int \log |p(t)| d\lambda_u(t), \end{aligned}$$

by (3.1).

We use the well known Bernstein-Walsh lemma about the growth of a polynomial outside of the set E (see [12, p. 156], for example): Let $E \subset \mathbb{C}$ be a compact set, $\text{cap}(E) > 0$, with the unbounded component of $\mathbb{C} \setminus E$ denoted by Ω . Then, for any polynomial $p(z)$ of degree n , we have

$$(3.3) \quad |p(z)| \leq \|p\|_E e^{ng_\Omega(z, \infty)}, \quad z \in \mathbb{C},$$

where $g_\Omega(z, \infty)$ is the Green function of Ω , with pole at ∞ . The following representation for $g_\Omega(z, \infty)$ is found in Theorem III.37 of [13, p. 82]).

$$(3.4) \quad g_\Omega(z, \infty) = \log \frac{1}{\text{cap}(E)} + \int \log |z - t| d\mu_E(t), \quad z \in \mathbb{C}.$$

It follows from (3.1)–(3.4) and Fubini's theorem that

$$\begin{aligned} \frac{1}{n} \log \frac{\|q\|_E}{\|p\|_E} &\leq \int \log \frac{|p(t)|^{1/n}}{\|p\|_E^{1/n}} d\lambda_u(t) \leq \int g_\Omega(t, \infty) d\lambda_u(t) \\ &= \log \frac{1}{\text{cap}(E)} + \int \int \log |z - t| d\lambda_u(t) d\mu_E(z) \\ &= \log \frac{1}{\text{cap}(E)} + \int \log \rho_u(z) d\mu_E(z). \end{aligned}$$

Using the definition of $\rho_u(z)$, we obtain from the above estimate that

$$\begin{aligned} \|q\|_E &\leq \left(\frac{\max_{u \in \partial E} \exp \left(\int \log \rho_u(z) d\mu_E(z) \right)}{\text{cap}(E)} \right)^n \|p\|_E \\ &= \left(\frac{\max_{u \in \partial E} \exp \left(\int_{|z-u| \geq 1} \log |z - u| d\mu_E(z) \right)}{\text{cap}(E)} \right)^n \|p\|_E. \end{aligned}$$

Hence

$$(3.5) \quad C_E \leq \frac{\max_{u \in \partial E} \exp \left(\int_{|z-u| \geq 1} \log |z - u| d\mu_E(z) \right)}{\text{cap}(E)}.$$

In order to prove the inequality opposite to (3.5), we consider the n -th Fekete points $\{a_{k,n}\}_{k=1}^n$ for the set E (cf. [12, p. 152]). Let

$$p_n(z) := \prod_{k=1}^n (z - a_{k,n})$$

be the Fekete polynomial of degree n . Define the normalized counting measures on the Fekete points by

$$\tau_n := \frac{1}{n} \sum_{k=1}^n \delta_{a_{k,n}}, \quad n \in \mathbb{N}.$$

It is known that (see Theorems 5.5.4 and 5.5.2 in [12, pp. 153–155])

$$(3.6) \quad \lim_{n \rightarrow \infty} \|p_n\|_E^{1/n} = \text{cap}(E).$$

Furthermore, we have the following weak* convergence of counting measures (cf. [12, p. 159]):

$$(3.7) \quad \tau_n \xrightarrow{*} \mu_E, \quad \text{as } n \rightarrow \infty.$$

Let $u \in \partial E$ be a point, where the maximum on the right-hand side of (3.5) is attained. Define the factor $q_n(z)$ for $p_n(z)$, with zeros being the n -th Fekete points satisfying $|a_{k,n} - u| \geq 1$. Then we have by (3.7) that

$$\begin{aligned} \lim_{n \rightarrow \infty} \|q_n\|_E^{1/n} &\geq \lim_{n \rightarrow \infty} |q_n(u)|^{1/n} = \lim_{n \rightarrow \infty} \exp \left(\frac{1}{n} \sum_{|a_{k,n} - u| \geq 1} \log |u - a_{k,n}| \right) \\ &= \exp \left(\lim_{n \rightarrow \infty} \int_{|z-u| \geq 1} \log |u - z| d\tau_n(z) \right) \\ &= \exp \left(\int_{|z-u| \geq 1} \log |u - z| d\mu_E(z) \right). \end{aligned}$$

Combining the above inequality with (3.6) and the definition of C_E , we obtain that

$$C_E \geq \lim_{n \rightarrow \infty} \frac{\|q_n\|_E^{1/n}}{\|p_n\|_E^{1/n}} \geq \frac{\exp \left(\int_{|z-u| \geq 1} \log |z - u| d\mu_E(z) \right)}{\text{cap}(E)}.$$

This shows that (2.1) holds true. Moreover, if $u \in \partial E$ is a regular point for Ω , then we obtain by Theorem III.36 of [13, p. 82]) and (3.4) that

$$\log \frac{1}{\text{cap}(E)} + \int \log |u - t| d\mu_E(t) = g_\Omega(u, \infty) = 0.$$

Hence

$$\log \frac{1}{\text{cap}(E)} + \int_{|z-u| \geq 1} \log |u - t| d\mu_E(t) = - \int_{|z-u| \leq 1} \log |u - t| d\mu_E(t),$$

which implies (2.2) by (2.1). $\square$

Proof of Corollary 2.2. It is well known [13, p. 84] that $\text{cap}(\overline{D_r}) = r$ and $d\mu_{\overline{D_r}}(re^{i\theta}) = d\theta/(2\pi)$, where $d\theta$ is the angular measure on ∂D_r . If $r \in (0, 1/2]$, then the numerator of (2.1) is equal to 1, so that

$$C_{\overline{D_r}} = \frac{1}{r}, \quad 0 < r \leq 1/2.$$

Assume that $r > 1/2$. We set $z = re^{i\theta}$ and let $u_0 = re^{i\theta_0}$ be a point where the maximum in (2.1) is attained. On writing

$$|z - u_0| = 2r \left| \sin \frac{\theta - \theta_0}{2} \right|,$$

we obtain that

$$\begin{aligned} C_{\overline{D_r}} &= \frac{1}{r} \exp \left(\frac{1}{2\pi} \int_{\theta_0 + 2\arcsin \frac{1}{2r}}^{2\pi + \theta_0 - 2\arcsin \frac{1}{2r}} \log \left| 2r \sin \frac{\theta - \theta_0}{2} \right| d\theta \right) \\ &= \frac{1}{r} \exp \left(\frac{1}{2\pi} \int_{2\arcsin \frac{1}{2r} - \pi}^{\pi - 2\arcsin \frac{1}{2r}} \log \left(2r \cos \frac{x}{2} \right) dx \right) \\ &= \frac{1}{r} \exp \left(\frac{1}{\pi} \int_0^{\pi - 2\arcsin \frac{1}{2r}} \log \left(2r \cos \frac{x}{2} \right) dx \right), \end{aligned}$$

by the change of variable $\theta - \theta_0 = \pi - x$. □

Proof of Corollary 2.3. Recall that $\text{cap}([-a, a]) = a/2$ (see [13, p. 84]) and

$$d\mu_{[-a, a]}(t) = \frac{dt}{\pi\sqrt{a^2 - t^2}}, \quad t \in [-a, a].$$

It follows from (2.1) that

$$(3.8) \quad C_{[-a, a]} = \frac{2}{a} \exp \left(\max_{u \in [-a, a]} \int_{[-a, a] \setminus (u-1, u+1)} \frac{\log |t - u|}{\pi\sqrt{a^2 - t^2}} dt \right).$$

If $a \in (0, 1/2]$, then the integral in (3.8) obviously vanishes, so that $C_{[-a, a]} = 2/a$. For $a > 1/2$, let

$$(3.9) \quad f(u) := \int_{[-a, a] \setminus (u-1, u+1)} \frac{\log |t - u|}{\pi\sqrt{a^2 - t^2}} dt.$$

One can easily see from (3.9) that

$$f'(u) = \int_{u+1}^a \frac{dt}{\pi(u-t)\sqrt{a^2 - t^2}} < 0, \quad u \in [-a, 1-a],$$

and

$$f'(u) = \int_{-a}^{u-1} \frac{dt}{\pi(u-t)\sqrt{a^2 - t^2}} > 0, \quad u \in [a-1, a].$$

However, if $u \in (1-a, a-1)$, then

$$f'(u) = \int_{u+1}^a \frac{dt}{\pi(u-t)\sqrt{a^2 - t^2}} + \int_{-a}^{u-1} \frac{dt}{\pi(u-t)\sqrt{a^2 - t^2}}.$$

It is not difficult to verify directly that

$$\int \frac{dt}{\pi(u-t)\sqrt{a^2-t^2}} = \frac{1}{\pi\sqrt{a^2-u^2}} \log \left| \frac{a^2-ut+\sqrt{a^2-t^2}\sqrt{a^2-u^2}}{t-u} \right| + C,$$

which implies that

$$f'(u) = \frac{1}{\pi\sqrt{a^2-u^2}} \log \left(\frac{a^2-u^2+u+\sqrt{a^2-(u-1)^2}\sqrt{a^2-u^2}}{a^2-u^2-u+\sqrt{a^2-(u+1)^2}\sqrt{a^2-u^2}} \right),$$

for $u \in (1-a, a-1)$. Hence

$$f'(u) < 0, \quad u \in (1-a, 0), \quad \text{and} \quad f'(u) > 0, \quad u \in (0, a-1).$$

Collecting all facts, we obtain that the maximum for $f(u)$ on $[-a, a]$ is attained at the endpoints $u = a$ and $u = -a$, and it is equal to

$$\max_{u \in [-a, a]} f(u) = \int_{1-a}^a \frac{\log(t+a)}{\pi\sqrt{a^2-t^2}} dt.$$

Thus (2.3) follows from (3.8) and the above equation. $\square$

Proof of Corollary 2.4. Note that the numerator of (2.1) is equal to 1, because $|z-u| \leq 1$, $\forall z \in E$, $\forall u \in \partial E$. Thus (2.5) follows immediately. $\square$

Proof of Corollary 2.5. Observe that $C_E \geq 1$ for any $E \in \mathbb{C}$, so that $C_{\alpha E} \geq 1$. Since E is regular, we use the representation for C_E in (2.2). Let $T: E \rightarrow \alpha E$ be the dilation mapping. Then $|Tz-Tu| = \alpha|z-u|$, $z, u \in E$, and $d\mu_{\alpha E}(Tz) = d\mu_E(z)$. This gives that

$$\begin{aligned} C_{\alpha E} &= \max_{Tu \in \partial(\alpha E)} \exp \left(- \int_{|Tz-Tu| \leq 1} \log |Tz-Tu| d\mu_{\alpha E}(Tz) \right) \\ &= \max_{u \in \partial E} \exp \left(- \int_{|z-u| \leq 1/\alpha} \log(\alpha|z-u|) d\mu_E(z) \right) \\ &= \max_{u \in \partial E} \exp \left(-\mu_E(\overline{D_{1/\alpha}(u)}) \log \alpha - \int_{|z-u| \leq 1/\alpha} \log |z-u| d\mu_E(z) \right) \\ &< \max_{u \in \partial E} \exp \left(- \int_{|z-u| \leq 1/\alpha} \log |z-u| d\mu_E(z) \right), \end{aligned}$$

where $\alpha \geq 1$. Using the absolute continuity of the integral, we have that

$$\lim_{\alpha \rightarrow +\infty} \int_{|z-u| \leq 1/\alpha} \log |z-u| d\mu_E(z) = 0,$$

which implies (2.6). $\square$

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