A RANDOM VARIATIONAL PRINCIPLE WITH APPLICATION TO WEAK HADAMARD DIFFERENTIABILITY OF CONVEX INTEGRAL FUNCTIONALS

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ABSTRACT. We present a random version of the Borwein-Preiss smooth variational principle, stating that under suitable conditions, the set of minimizers of a perturbed function depending on a random variable, admits a measurable selection. Two applications are given. The first one shows that if E is a superreflexive Banach space, then any convex continuous integral functional on $L^1(T, \mu; E)$ from a certain class (in particular the usual L^1 norm), is weak Hadamard differentiable on a subset whose complement is σ -very porous. The second application is a random version of the Caristi fixed point theorem for multifunctions.

Ekeland's variational principle and its smooth analogues are useful tools in the study of non-linear problems in various areas of mathematics (see for instance [E1], [E2], [B-P], [D-G-Z1], [D-G-Z2]).

In this paper we present a random version of the smooth variational principle of Borwein-Preiss [B-P]. Namely, we prove that a suitable perturbed function of a given one admits, as in [B-P], a minimum point, which in our setting is a measurable function of a random variable.

We give two applications of our random smooth variational principle. The first one is about weak Hadamard differentiability of some convex integral functionals in the Lebesgue-Bochner space $L^1(T, \mu; E)$.

Borwein and Fitzpatrick [B-F] have shown that in $L^1(T, \mu)$, where μ is sigma finite, there exists an equivalent weak Hadamard differentiable norm; hence using Preiss-Phelps-Namioka's theorem [P-P-N], they establish that $L^1(T, \mu)$ is a weak Hadamard Asplund space. Examination of their proofs will convince the reader that the results of Borwein-Fitzpatrick remain valid in the space $L^1(T, \mu; E)$, provided E is a reflexive Banach space and μ is sigma finite, since the Dunford-Pettis criterium for weak compactness in L^1 is crucial in their proofs.

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Here we prove that if E is a separable Banach space with a uniformly Fréchet differentiable norm, then any convex continuous integral functional on $L^1(T, \mu; E)$ from a certain class (in particular the usual L^1 norm) is weak Hadamard differentiable on a subset whose complement is σ -very porous. The proof of this result is direct and, unlike in [B-F], does not rely on the deep theorem in [P-P-N].

As a second application, a random version of the Caristi fixed point theorem for multifunctions is obtained.

Let $(E, \|\cdot\|)$ be a Banach space with dual E^* and let $S = \{x \in E : \|x\| = 1\}$.

The norm $\|\cdot\|$ of E is said to be *uniformly Fréchet differentiable* if for every $x \in S$ there exists an element $\nabla\|x\| \in E^*$ such that the following condition holds: for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\frac{\|x + th\| - \|x\|}{t} - \langle \nabla\|x\|, h \rangle < \varepsilon \quad \text{for every } x \in S, h \in S, t \in (0, \delta).$$

The norm $\|\cdot\|$ of E is said to be *weak Hadamard differentiable* at $x \in E$ if there exists an element $\nabla\|x\| \in E^*$ such that for every weakly compact subset W of E the following holds: for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\frac{\|x + th\| - \|x\|}{t} - \langle \nabla\|x\|, h \rangle < \varepsilon \quad \text{for every } h \in W \text{ and } t \in (0, \delta).$$

About measurability we retain notation and terminology of Himmelberg [H].

The following theorem is a random version of the Borwein-Preiss smooth variational principle.

Theorem 1. *Suppose that $(E, \|\cdot\|)$ is a separable Banach space and $(T, \mathcal{A}, \mu)$ is a measurable space with a complete σ -finite measure μ . Let $F : T \rightarrow 2^E$ be a measurable multifunction with non-empty closed values and $f : T \times E \rightarrow \mathbf{R}$ be a function with the following properties:*

- (1) $\inf_{x \in F(t)} f(t, x) > -\infty$, for every $t \in T$;
- (2) $f(\cdot, x)$ is measurable for every $x \in E$;
- (3) $f(t, \cdot)$ is continuous for every $t \in T$.

Let $x_0 : T \rightarrow E$ be a measurable single-valued selection of F such that

$$f(t, x_0(t)) < \inf_{x \in F(t)} f(t, x) + \varepsilon_0(t) \quad \text{for every } t \in T,$$

where $\varepsilon_0 : T \rightarrow (0, +\infty)$ is a given measurable function. Let $p \geq 1$ be given and let $\varepsilon : T \rightarrow (0, +\infty)$ and $\lambda : T \rightarrow (0, +\infty)$ be measurable functions with $\varepsilon(t) > \varepsilon_0(t)$, $t \in T$.

Then there exist measurable selections of F , say $x_n, v : T \rightarrow E$ and measurable functions $\mu_n : T \rightarrow (0, 1)$ such that, for every $t \in T$, we have: $\sum_{n=0}^{\infty} \mu_n(t) = 1$, and

- (4) $x_n(t) \rightarrow v(t)$ as $n \rightarrow \infty$;
- (5) $f(t, v(t)) + \Delta(t, v(t)) \leq f(t, x) + \Delta(t, x)$, for every $x \in F(t)$, where
- (6) $\Delta(t, x) = \frac{\varepsilon(t)}{\lambda(t)^p} \sum_{n=0}^{\infty} \mu_n(t) \|x - x_n(t)\|^p$;
- (7) $\|x_n(t) - v(t)\| < \lambda(t)$ for every $n = 0, 1, 2, \dots$.

Proof. Set $q(t) = \frac{1}{2} \left[\frac{\varepsilon(t) - \varepsilon_0(t)}{\varepsilon(t) + \varepsilon_0(t)} \right]^{1/2}$. For $n = 0, 1, \dots$ and $t \in T, x \in E$, define inductively

$$(8) \quad G_n(t) = \{x \in F(t) : f_n(t, x) \leq \inf_{z \in F(t)} f_n(t, z) + \varepsilon_n(t)\},$$

where

$$(9) \quad f_{n+1}(t, x) = f_n(t, x) + \frac{\varepsilon(t)}{\lambda(t)^p} \mu_n(t) \|x - x_n(t)\|^p, \quad f_0(t, x) = f(t, x),$$

$$\varepsilon_n(t) = \varepsilon_0(t)q(t)^{2n}, \quad \mu_n(t) = (1 - q(t))q(t)^n,$$

and $x_n : T \rightarrow E$ is a measurable function satisfying

$$(10) \quad x_n(t) \in G_n(t)$$

for every $t \in T$.

We shall prove by induction that the definitions of x_n and f_n are correct. For $n = 0$ this is true by assumption. Suppose that x_{n-1} and f_{n-1} are defined. Now, by (9), f_n is well defined. By [H, Theorem 6.5] the multifunction $t \mapsto f_n(t, F(t))$ is weakly measurable (in fact it is measurable, by [H, Theorem 3.5 (iii)]) and by [H, Theorem 6.6] the function $t \mapsto \inf_{z \in F(t)} f_n(t, z)$ is measurable. By [H, Theorem 6.4] and [H, Theorem 3.5 (iii)] the multifunction $t \mapsto G_n(t)$ is measurable and Kuratowski and Ryll-Nardzewski's theorem [K-RN] (see also [H, Theorem 5.1]) produces a measurable function x_n satisfying (10), completing the induction.

We shall prove that $\{x_n(t)\}$ is a fundamental sequence for every $t \in T$. By (9) and (10) we have

$$\begin{aligned} & \frac{\varepsilon(t)}{\lambda(t)^p} \mu_n(t) \|x_{n+1}(t) - x_n(t)\|^p \\ &= f_{n+1}(t, x_{n+1}(t)) - f_n(t, x_{n+1}(t)) \\ &= f_{n+1}(t, x_{n+1}(t)) - f_{n+1}(t, x_n(t)) + f_n(t, x_n(t)) - f_n(t, x_{n+1}(t)) \\ &\leq \varepsilon_{n+1}(t) + \varepsilon_n(t) \\ &= \varepsilon_0(t)q(t)^{2n}(q(t)^2 + 1). \end{aligned}$$

Hence for $m > n$ we obtain

$$(11) \quad \|x_m(t) - x_n(t)\| \leq \lambda(t) \left(\frac{\varepsilon_0(t)}{\varepsilon(t)} \cdot \frac{1 + q(t)^2}{1 - q(t)^2} \right)^{\frac{1}{p}} q(t)^{\frac{n}{p}} < \lambda(t)q(t)^{\frac{n}{p}}.$$

Therefore $\{x_n(t)\}_{n=0}^\infty$ is a fundamental sequence and so converges to $v(t)$ and clearly v is measurable. From (11), letting $m \rightarrow +\infty$, we obtain $\|v(t) - x_n(t)\| < \lambda(t)$, which is (7).

To establish (5), let $\gamma > 0$ be given. As $(f + \Delta)(t, \cdot)$ is continuous, there exists $\delta(t) > 0$ such that

$$(12) \quad f(t, v(t)) + \Delta(t, v(t)) < f(t, x) + \Delta(t, x) + \gamma/3,$$

whenever $\|x - v(t)\| < \delta(t)$. For fixed $t \in T$, choose n sufficiently large so that the following inequalities hold: $\varepsilon_n(t) < \gamma/3$, $\|v(t) - x_k(t)\| < \delta(t)$ for every $k \geq n$, and $\frac{\varepsilon(t)}{\lambda(t)^p} \sum_{k=n}^\infty \mu_k(t) \|x_n(t) - x_k(t)\|^p < \gamma/3$.

For every $x \in F(t)$, using (12), (8), (9) and (10), we can write

$$\begin{aligned} & f(t, v(t)) + \Delta(t, v(t)) \\ & < f(t, x_n(t)) + \Delta(t, x_n(t)) + \gamma/3 \\ & = f_n(t, x_n(t)) + \frac{\varepsilon(t)}{\lambda(t)^p} \sum_{k=n}^{\infty} \mu_k(t) \|x_n(t) - x_k(t)\|^p + \gamma/3 \\ & < f_n(t, x) + \varepsilon_n(t) + \gamma/3 + \gamma/3 \\ & < f(t, x) + \Delta(t, x) + \gamma \end{aligned}$$

and (5) is proved. $\square$

Remark. It is clear that if the conditions of Theorem 1 are satisfied for a.e. $t \in T$, then its conclusions are satisfied for a.e. $t \in T$, too.

Theorem 2 (Random Ekeland's variational principle). *Under the assumptions of Theorem 1, there exists a measurable selection $v : T \rightarrow E$ of F such that:*

- (5') $f(t, v(t)) < f(t, x) + \frac{\varepsilon(t)}{\lambda(t)} \|x - v(t)\|$ for every $t \in T$ and $x \in F(t), x \neq v(t)$;
- (7') $\|x_0(t) - v(t)\| < \lambda(t)$ for every $t \in T$;
- (well posedness) $x_n \rightarrow v(t)$ whenever $f(t, x_n) + \frac{\varepsilon(t)}{\lambda(t)} \|x_n - v(t)\| \rightarrow f(t, v(t))$.

Proof. (5') follows immediately from (5) of Theorem 1 with $p = 1$, and $\varepsilon'(t) = \frac{1}{2}(\varepsilon_0(t) + \varepsilon(t))$ in the place of $\varepsilon(t)$. $\square$

Now we recall the following definition.

A subset P of E is said to be *very porous* if there exists $\alpha > 0$ for which the following holds: for every $x \in E$ and every $r \in (0, \alpha)$ there is $y \in E$ such that

$$B(y, \alpha r) \subset B(x, r) \cap (E \setminus P).$$

P is called *σ -very porous* if it is a countable union of very porous sets.

In the sequel $L^1(T, \mu; E)$ will denote the usual Lebesgue-Bochner space, i.e. the set of all (equivalence classes of) μ -Bochner integrable functions $f : T \rightarrow E$ with the norm $\|f\|_{L^1} = \int_T \|f(t)\| d\mu(t)$.

Theorem 3. *Let $(E, \|\cdot\|)$ be a separable superreflexive Banach space and let $(T, \mathcal{A}, \mu)$ be a measurable space, with a complete σ -finite measure μ , $\int_T d\mu(t) = 1$. Suppose that $f : T \times E \rightarrow \mathbf{R}$ is a function satisfying conditions (1), (2), (3) of Theorem 1, and*

- a) $f(t, \cdot)$ is convex for every $t \in T$;
- b) there exists a function $L \in L^\infty(T, \mu; \mathbf{R})$ such that for all $t \in T$

$$(13) \quad |f(t, x_1) - f(t, x_2)| \leq L(t) \|x_1 - x_2\| \quad \text{for every } x_1, x_2 \in E;$$

- c) $f(\cdot, 0) \in L^1(T, \mu; \mathbf{R})$.

Define the function $g : L^1(T, \mu; E) \rightarrow \mathbf{R}$ by $g(x) = \int_T f(t, x(t)) d\mu(t)$. Then g is weak Hadamard differentiable on a subset X_0 of $L^1(T, \mu; E)$, whose complement is σ -very porous.

Proof. Since every superreflexive Banach space has an equivalent uniformly Fréchet differentiable norm (see [D-G-Z2, Corollary 4.6, page 152]), we may suppose without loss of generality that the norm $\|\cdot\|$ is uniformly Fréchet differentiable.

In view of b) and c), the definition of g makes sense. The absolute continuity of the Lebesgue integral (see [K-F, Theorem V.5.5]) allows us to define, for every $x, h \in L^1(T, \mu; E)$,

$$(14) \quad \gamma_n(x, h) = \sup \left\{ \gamma > 0 : \int_{\Gamma} \left(f(t, x(t) + h(t)) + f(t, x(t) - h(t)) - 2f(t, x(t)) \right) d\mu(t) \leq \frac{1}{n} \right. \\ \left. \text{for every } \Gamma \subset T \text{ with } \mu(\Gamma) \leq \gamma \|h\|_{L^1} \right\}.$$

Put

$$(15) \quad H_{n,m}(x) = \{h \in L^1(T, \mu; E) : \gamma_n(x, h) \geq \frac{1}{m}\}$$

and

$$(16) \quad \Gamma_{\gamma}(h) = \{t \in T : \|h(t)\| \geq \frac{1}{\gamma}\}.$$

Obviously

$$(17) \quad \mu(\Gamma_{\gamma}(h)) \leq \gamma \int_T \|h(t)\| d\mu(t) = \gamma \|h\|_{L^1} \quad \text{for every } h \in L^1(T, \mu; E).$$

Since the norm $\|\cdot\|$ is uniformly Fréchet differentiable, it is easy to see that there exists $s_{n,m} \in (0, 1/m)$ such that

$$(18) \quad \frac{\|x + s_{n,m}h\|^2 - \|x\|^2}{s_{n,m}} - \langle \nabla \|x\|^2, h \rangle < \frac{1}{n} \quad \text{whenever } \|x\| < 1, \|h\| \leq m.$$

Define the set

$$X_{n,m} = \{x \in L^1(T, \mu; E) : \text{there exists } s \in (0, 1/m) \text{ such that} \\ \frac{g(x + sh) + g(x - sh) - 2g(x)}{s} < 14/n, \text{ for every } h \in H_{n,m}(x)\}.$$

Claim. $L^1(T, \mu; E) \setminus X_{n,m}$ is very porous for every integer n, m .

Assume the contrary. Then for some integer n, m , for $\alpha \in (0, s_{n,m}/2n)$, there exist $x_0 \in L^1(T, \mu, E)$ and $r \in (0, \alpha)$ such that

$$(19) \quad B(v; \alpha r) \cap (L^1(T, \mu; E) \setminus X_{n,m}) \neq \emptyset \quad \text{for every } v \in B(x_0; r).$$

By Himmelberg [H] it follows that the multivalued mapping $F : t \mapsto B[x_0(t); r]$ is measurable.

Without loss of generality we may assume that $\|L\|_{\infty} < 1$. In view of (13) we have

$$\|f(t, x) - f(t, x_0(t))\| \leq r \quad \text{for a.e. } t \in T \text{ and for every } x \in F(t) \text{ } t \in T.$$

By Theorem 1, with the above F and $\varepsilon_0 = r = \varepsilon/2, l = r/2, p = 2$, we obtain measurable selections $v, x_n : T \rightarrow E$ of F , and measurable functions $\mu_n : T \rightarrow (0, 1)$ such that for the function $\Delta(t, x)$ given by (6) (with the above constructed x_n), we have

$$(20) \quad f(t, x) - f(t, v(t)) \leq \Delta(t, x) - \Delta(t, v(t)) \quad \text{for a.e. } t \in T \text{ and for every } x \in F(t),$$

and

$$(21) \quad \|v(t) - x_0(t)\| < \lambda \quad \text{for a.e. } t \in T.$$

So by (19) we can find $z \in B(v; \alpha r) \cap (L^1(T, \mu; E) \setminus X_{n,m})$.

Hence, by definition of $X_{n,m}$, for $s = s_{n,m}r/2$, there exists $h \in H_{n,m}(z)$ such that, setting $\Gamma = \Gamma_{\gamma_n(z,h)}(h)$, we have $\mu(\Gamma) \leq \gamma_n(z,h)\|h\|_{L^1}$ (from (17)) and

$$\begin{aligned} 14/n &\leq \frac{g(z+sh) + g(z-sh) - 2g(z)}{s} \\ &\leq \int_T \frac{f(t, v(t) + sh(t)) + f(t, v(t) - sh(t)) - 2f(t, v(t))}{s} d\mu(t) + 4/n \\ &\quad (\text{by (13) and by the choice of } \alpha \text{ and } s) \\ &< \int_{T \setminus \Gamma} \frac{f(t, v(t) + sh(t)) + f(t, v(t) - sh(t)) - 2f(t, v(t))}{s} d\mu(t) + 5/n \\ &\quad (\text{by (14) and monotonicity of the differential quotient of a convex function}) \\ &\leq \int_{T \setminus \Gamma} \frac{1}{s} \left(\Delta(t, v(t) + sh(t)) + \Delta(t, v(t) - sh(t)) - 2\Delta(t, v(t)) \right) d\mu(t) + 5/n \\ &\quad (\text{by (20), since (16) and (15) imply that } h(t) \leq m \text{ for } t \notin \Gamma) \\ &\quad \text{and (21) implies that } v(t) + sh(t) \in F(t)) \\ &= \frac{\varepsilon}{\lambda} \int_{T \setminus \Gamma} \sum_{n=0}^{\infty} \mu_n \left[\frac{\left\| \frac{v(t)-x_n(t)}{\lambda} + \frac{s}{\lambda} h(t) \right\|^2 - \left\| \frac{v(t)-x_n(t)}{\lambda} \right\|^2}{\frac{s}{\lambda}} \right. \\ &\quad \left. + \frac{\left\| \frac{v(t)-x_n(t)}{\lambda} - \frac{s}{\lambda} h(t) \right\|^2 - \left\| \frac{v(t)-x_n(t)}{\lambda} \right\|^2}{\frac{s}{\lambda}} \right] d\mu(t) + 5/n \\ &< 13/n \quad (\text{by (18), since } \frac{s}{\lambda} \leq s_{n,m}), \end{aligned}$$

which is a contradiction.

Therefore the set $L^1(T, \mu; E) \setminus X_{n,m}$ is very porous.

We need the following.

Proposition 4. *If K is a weakly compact subset in $L^1(T, \mu; E)$, then for every integer n and every $v \in E$ there exists an integer m such that $K \subset H_{n,m}(v)$.*

Proof. Assuming the contrary, there exist an integer n and a $v \in E$ such that for every m there exists $h_m \in K \setminus H_{n,m}(v)$, i.e. there exists Γ_m such that $\mu(\Gamma_m) =$

$\frac{1}{m}\|h_m\|$ and

(22)

$$\int_{\Gamma_m} (f(t, v(t) + h_m(t)) + f(t, v(t) - h_m(t)) - 2f(t, v(t))) d\mu(t) \geq \frac{1}{n}.$$

Since K is weakly compact, it is bounded, so $\mu(\Gamma_m) \rightarrow 0$. Now by Dunford-Pettis' theorem of weak compactness in $L^1(T, \mu; E)$ (see [D-U], page 105), K is uniformly integrable, i.e. $\lim_{\mu(E) \rightarrow 0} \int_E \|h(t)\| d\mu(t) = 0$ uniformly for $h \in K$. So, by (22) and (13), when m tends to infinity, we obtain a contradiction. $\square$

The proof of the following proposition is the same as that of [Ph, Proposition 1.23] (which concerns Fréchet differentiability).

Proposition 5. *A convex continuous function $f : E \rightarrow \mathbf{R}$ is weak Hadamard differentiable at $x \in E$ if and only if for every $\varepsilon > 0$ and every weakly compact subset $W \subset E$ there exists $t > 0$ such that*

$$\frac{f(x + th) + f(x - th) - 2f(x)}{t} < \varepsilon \quad \text{for every } h \in W.$$

Now we can complete the proof of Theorem 3. By Proposition 4 and Proposition 5, it follows that g is weak Hadamard differentiable on the set $X_0 = \bigcap_{n,m=1}^{\infty} X_{n,m}$. By the Claim, it follows that the set $L^1(T, \mu; E) \setminus X_0$ is σ -very porous and the theorem is proved.

Comparing the Fréchet and the weak Hadamard differentiability of the usual norm $\|\cdot\|_{L^1}$ of $L^1(T, \mu; E)$, as an interesting corollary of Theorem 3, we obtain that $\|\cdot\|_{L^1}$ is weak Hadamard differentiable on a complement of a σ -very porous subset of $L^1(T, \mu; E)$, while $\|\cdot\|_{L^1}$ is nowhere Fréchet differentiable (see [Ph]).

The following theorem is a random version of Caristi's fixed point theorem [A-E, Theorem 14, Ch.5 Sec.1].

Theorem 6 (Multivalued Caristi's random fixed point theorem). *Suppose that $(E, \|\cdot\|)$ is a separable Banach space and $(T, \mathcal{A}, \mu)$ is a measurable space with a complete σ -finite measure μ . Let X be a closed subset of E , $F : T \times X \rightarrow X$ be a multivalued mapping. Assume that there exists a function $f : T \times X \rightarrow \mathbf{R}$ such that $f(\cdot, x)$ is measurable, $f(t, \cdot)$ is continuous, and for every $x \in X$ and every $t \in T$ there exists $y_{t,x} \in F(t, x)$ such that*

$$(23) \quad f(t, y_{t,x}) + \|x - y_{t,x}\| \leq f(t, x).$$

Then there exists a measurable mapping $v : T \rightarrow X$ such that $v(t) \in F(t, v(t))$, for every $t \in T$.

Proof. Apply Theorem 2 with $\varepsilon_0 < \varepsilon < 1$, $\lambda = 1$ and obtain a measurable mapping $v : T \rightarrow X$ such that

(24)

$$f(t, v(t)) < f(t, x) + \varepsilon \|v(t) - x\| \quad \text{for every } t \in T \text{ and } x \in X, x \neq v(t).$$

By (23), with $x = v(t)$, and by (24), with $x = y_{t,v(t)}$, we obtain

$$\|v(t) - y_{t,v(t)}\| \leq f(t, v(t)) - f(t, y_{t,v(t)}) \leq \varepsilon \|v(t) - y_{t,v(t)}\|$$

for every $t \in T$. Hence $v(t) = y_{t,v(t)}$ and the theorem is proved. $\square$

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