ON ALGEBRAIC POLYNOMIALS WITH RANDOM COEFFICIENTS

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ABSTRACT. The expected number of real zeros and maxima of the curve representing algebraic polynomial of the form $a_0 \binom{n-1}{0}^{1/2} + a_1 \binom{n-1}{1}^{1/2} x + a_2 \binom{n-1}{2}^{1/2} x^2 + \cdots + a_{n-1} \binom{n-1}{n-1}^{1/2} x^{n-1}$ where $a_j, j = 0, 1, 2, \dots, n-1$, are independent standard normal random variables, are known. In this paper we provide the asymptotic value for the expected number of maxima which occur below a given level. We also show that most of the zero crossings of the curve representing the polynomial are perpendicular to the x axis. The results show a significant difference in mathematical behaviour between our polynomial and the random algebraic polynomial of the form $a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1}$ which was previously the most studied.

1. INTRODUCTION

The random algebraic polynomial is commonly defined as

$$Q(x) \equiv Q_n(x, \omega) = \sum_{j=0}^{n-1} a_j(\omega) x^j,$$

where $(\Omega, \mathcal{A}, \text{Pr})$ is a fixed probability space, and $\{a_j(\omega)\}_{j=0}^{n-1}$ is a sequence of independent random variables defined on Ω . The previous works on $Q(x)$ mainly assume identical distribution for the coefficients a_j 's. They include the pioneer works of Littlewood and Offord [7] and [8] and recent works of Wilkins [9] and Farahmand [5]. It is known that for identical standard normally distributed coefficients and n sufficiently large the expected number of real zeros of $Q(x)$ is asymptotic to $(2/\pi) \log n$. However, there is little known about random polynomials with non-identical coefficients. Motivated by their close relation with physics, reported by Edelman and Kostland [1], as well as their mathematical interest, we assume that the coefficients $a_j, j = 0, 1, 2, \dots, n-1$, have means zero and non-identical variances $\binom{n-1}{j}$. This is the same as considering polynomials of the form

$$P(x) \equiv P_n(x, \omega) = \sum_{j=0}^{n-1} a_j(\omega) \binom{n-1}{j}^{1/2} x^j$$

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with the same assumption of identically normal standard distribution for a_j 's as above. Let $N(a, b)$ be the number of real zeros and $M_u(a, b)$ the number of maxima of $P(x)$ which occurs below a level u , in the interval (a, b) . In [6] it is shown that for n sufficiently large $EN(-\infty, \infty) \sim EM_\infty(-\infty, \infty) \sim \sqrt{n}$. This is interesting as it shows that the curve representing $P(x)$ has a significantly larger expected number of real zero crossings and therefore oscillates more frequently than $Q(x)$. Also, unlike $Q(x)$, since $EN(-\infty, \infty) \sim EM_u(-\infty, \infty)$ asymptotically all the oscillations of $P(x)$ are between two zero crossings. Therefore, in order to obtain a better understanding of the mathematical behaviour of $P(x)$, it is of special interest to study the number of local maxima which occurs below a given level as well as the slope or the nature of crossings. To this end, as in [4] and [5], we define $S_u(a, b)$ as the number of up-crossings of $P(x)$ which possess a slope greater than u or down-crossings with a slope less than $-u$. We define these crossings as u -sharp. Theorem 1 shows that there is no significant number of maxima which occurs below the x -axis while Theorem 2 gives a high level below which asymptotically all the maxima occur. By letting $u \rightarrow \infty$ as $n \rightarrow \infty$, Theorem 3 shows that asymptotically all the crossings are sharp. We prove the following theorems:

Theorem 1. *For all sufficiently large n , the expected number of maxima which occurs below the x -axis is*

$$EM_0(-\infty, \infty) \sim O(1).$$

Theorem 2. *For $f_n \equiv f$ as any function of n such that $f \rightarrow \infty$ as $n \rightarrow \infty$ and u such that $u/f^{2n}\sqrt{n} \rightarrow \infty$ as $n \rightarrow \infty$*

$$EM_u(-\infty, \infty) \sim \frac{\sqrt{n}}{2}.$$

Theorem 3. *For any u such that $u/\sqrt{n} \rightarrow 0$ as $n \rightarrow \infty$*

$$ES_u(-\infty, \infty) \sim \frac{\sqrt{n}}{2}.$$

2. MOMENTS

In order to be able to obtain estimates for the expected number of maxima below a level and the expected number of sharp crossings we need the following identities ((2.1) is well known and (2.2)-(2.6) are easily derived by consecutive differentiation of (2.1)):

$$(2.1) \quad A^2 = \text{var}\{P(x)\} = \sum_{j=0}^{n-1} \binom{n-1}{j} x^{2j} = (x^2 + 1)^{n-1},$$

$$(2.2) \quad \begin{aligned} B^2 &= \text{var}\{P'(x)\} = \sum_{j=0}^{n-1} j^2 \binom{n-1}{j} x^{2j-2} \\ &= (n-1)(x^2 + 1)^{n-3}(nx^2 - x^2 + 1), \end{aligned}$$

$$\begin{aligned}
 C^2 &= \text{var}\{P''(x)\} = \sum_{j=0}^{n-1} j^2(j-1)^2 \binom{n-1}{j} x^{2j-4} \\
 (2.3) \quad &= (n-1)(n-2)(x^2+1)^{n-5} \{(n-1)(n-2)x^4 + 4(n-2)x^2 + 2\},
 \end{aligned}$$

$$(2.4) \quad D = \text{cov}\{P(x), P'(x)\} = \sum_{j=0}^{n-1} j \binom{n-1}{j} x^{2j-1} = (n-1)x(x^2+1)^{n-2},$$

$$\begin{aligned}
 E &= \text{cov}\{P(x), P''(x)\} = \sum_{j=0}^{n-1} j(j-1) \binom{n-1}{j} x^{2j-2} \\
 (2.5) \quad &= (n-1)(n-2)x^2(x^2+1)^{n-3},
 \end{aligned}$$

and

$$\begin{aligned}
 F &= \text{cov}\{P'(x), P''(x)\} = \sum_{j=0}^{n-1} j^2(j-1) \binom{n-1}{j} x^{2j-3} \\
 (2.6) \quad &= (n-1)(n-2)x(x^2+1)^{n-4} \{(n-1)x^2 + 2\}.
 \end{aligned}$$

Then as is seen in [3] and [2]

$$EM_u(a, b) = \int_a^b \int_{-\infty}^u \int_{-\infty}^0 |z| p_x(t, 0, z) \, dz \, dt \, dx,$$

where $p_x(t, y, z)$ denotes the three-dimensional density function for $P(x)$, $P'(x)$ and $P''(x)$. Since from (2.1)-(2.6) the determinant of the covariance matrix of the above density function is

$$\begin{aligned}
 |\Sigma| &= A^2 B^2 C^2 - A^2 F^2 - B^2 E^2 - C^2 D^2 + 2DEF \\
 (2.7) \quad &= 2(n-1)^2(n-2)(x^2+1)^{3n-9}
 \end{aligned}$$

from [3] and [2] we have

$$(2.8) \quad EM_u(a, b) = \frac{1}{\pi} \int_a^b \frac{d}{2e\sqrt{2a}} \left\{ \Phi(u\sqrt{2a}) - \frac{b}{2\sqrt{ac}} \Phi\left(\frac{ub}{\sqrt{2c}}\right) \exp\left(-\frac{aeu^2}{c}\right) \right\} du$$

where from (2.1)-(2.7)

$$(2.9) \quad a = \frac{B^2 C^2 - F^2}{2|\Sigma|} = \frac{2nx^2 - 2x^2 + n^2x^4 - 3nx^4 + 2x^4 + 2}{4(x^2+1)^{n-1}},$$

$$(2.10) \quad b = \frac{DF - B^2 E}{|\Sigma|} = \frac{x^2}{2(x^2+1)^{n-3}},$$

$$(2.11) \quad c = \frac{A^2 B^2 - D^2}{2|\Sigma|} = \frac{1}{4(n-1)(n-2)(x^2+1)^{n-5}},$$

$$\begin{aligned}
 e &= c - \frac{b^2}{4a} \\
 (2.12) \quad &= \frac{nx^2 - x^2 + 1}{2(n-1)(n-2)(x^2+1)^{n-5}(2nx^2 - 2x^2 + n^2x^4 - 3nx^4 + 2x^4 + 2)},
 \end{aligned}$$

and

$$(2.13) \quad d = \frac{1}{\sqrt{|\Sigma|}}.$$

In the next section we will use (2.8) together with (2.9)-(2.13) to find the asymptotic estimate for $EM_u(-\infty, \infty)$. As far as the expected number of sharp crossings is concerned, from [4] we have

$$(2.14) \quad ES_u(a, b) = \frac{1}{\pi} \int_a^b \frac{\Delta}{A^2} \exp\left(-\frac{u^2 A^2}{2\Delta^2}\right) dx$$

where from (2.1)-(2.3)

$$(2.15) \quad \Delta^2 = A^2 B^2 - D^2 = (n-1)(x^2+1)^{2n-4}.$$

We give the asymptotic value for $ES_u(-\infty, \infty)$ in the final part of the paper. Without loss of generality we only consider the interval $(0, \infty)$. However, since in all above identities the power of x 's are even we obtain our results for the entire real line.

3. MAXIMA BELOW A LEVEL

(i). Level zero

Here we obtain the expected number of maxima below the x -axis. From (2.8) we can easily show

$$(3.1) \quad EM_0(0, \infty) = \frac{\sqrt{n-2}}{4\pi} \times \int_0^\infty \frac{\sqrt{2nx^2 - 2x^2 + n^2x^4 - 3nx^4 + 2x^4 + 2} - \sqrt{(n-1)(n-2)x^2}}{(x^2+1)(nx^2-x^2+1)} dx.$$

First we assume $x > \epsilon$ where $\epsilon = g_n/\sqrt{n}$ and g_n is any function of n such that $g_n \rightarrow \infty$ as $n \rightarrow \infty$. Then since with the above assumption $nx^2 \rightarrow \infty$ as $n \rightarrow \infty$ the first term that appears in the integrand of (3.1) is

$$\begin{aligned} \sqrt{2nx^2 - 2x^2 + n^2x^4 - 3nx^4 + 2x^4 + 2} &= \sqrt{(n-1)(n-2)x^4 + 2(nx^2 - x^2 + 1)} \\ &= \sqrt{(n-1)(n-2)x^2} + Y \end{aligned}$$

where Y is a function of x and n satisfies the relation $2(nx^2 - x^2 + 1) = Y^2 + 2Y\sqrt{(n-1)(n-2)x^2}$ and therefore tends to unity as $n \rightarrow \infty$. Hence from (3.1) letting $x = \tan \theta$ and since $\cot \arctan \epsilon = 1/\epsilon$ we obtain

$$\begin{aligned} EM_0(\epsilon, \infty) &\sim \frac{\sqrt{n-2}}{4\pi} \int_\epsilon^\infty \frac{dx}{(x^2+1)(nx^2-x^2+1)} \\ &\sim \frac{1}{4\pi\sqrt{n}} \int_\epsilon^\infty \frac{dx}{x^2(x^2+1)} \\ &= \frac{1}{4\pi\sqrt{n}} \int_{\arctan \epsilon}^{\pi/2} \cot^2 \theta \, d\theta \\ (3.2) \quad &\sim \frac{1}{4\pi\sqrt{n}\epsilon} \sim \frac{1}{4\pi g_n}. \end{aligned}$$

Now we find the expected number of maxima in the interval $(0, \epsilon)$. To this end for all sufficiently large n we have

$$\begin{aligned} EM_0(0, \epsilon) &< \frac{\sqrt{n-2}}{4\pi} \int_0^\epsilon \frac{\sqrt{nx^4 + 2nx^2 + 2} - x^2\sqrt{(n-1)(n-2)}}{(x^2+1)(nx^2-x^2+1)} dx \\ (3.3) \quad &= \frac{\sqrt{n-2}}{4\pi} \int_0^\epsilon \frac{\sqrt{(nx^2+1)^2+1} - x^2\sqrt{(n-3/2)^2-1/4}}{(x^2+1)(nx^2-x^2+1)} dx. \end{aligned}$$

Now we can show that both terms appearing in the numerator of the integrand in (3.3) can be written as

$$\sqrt{(nx^2 + 1)^2 + 1} = (nx^2 + 1) + X_1$$

and

$$\sqrt{\left(n - \frac{3}{2}\right)^2 - \frac{1}{4}} = \left(n - \frac{3}{2}\right) + X_2$$

where X_1 and X_2 satisfy relations

$$X_1^2 + 2X_1(nx^2 + 1) - 1 = 0 \quad \text{and} \quad X_2^2 + 2X_2(n - 3/2) + 1/4 = 0.$$

Therefore $X_1 < 1$ and $X_2 > -1/n$ are functions of n and x . Hence from (3.3) letting $(\sqrt{n-1})x = \tan \theta$ we obtain

$$\begin{aligned} EM_0(0, \epsilon) &< \frac{\sqrt{n-2}}{4\pi} \int_0^\epsilon \frac{2 \, dx}{(nx^2 - x^2 + 1)(x^2 + 1)} \\ &< \frac{\sqrt{n-2}}{2\pi} \int_0^\epsilon \frac{dx}{(n-1)x^2 + 1} \\ &\sim \frac{1}{2\pi} \int_0^{\arctan \epsilon \sqrt{n-1}} d\theta \\ &= O(1). \end{aligned}$$

This together with (3.2) completes the proof of Theorem 1.

(ii). High level

Since the second integrand that appears in (2.8) is positive we can obtain an upper limit for $EM_u(0, \infty)$ as

$$\begin{aligned} EM_u(0, \infty) &< \int_0^\infty \frac{\sqrt{(n-2)(2nx^2 - 2x^2 + n^2x^4 - 3nx^4 + 2x^4 + 2)}}{2\pi(1+x^2)(nx^2 - x^2 + 1)} \\ (3.4) \quad &\times \Phi \left\{ \frac{unx^2}{\sqrt{n}(x^2 + 1)^{(n+1)/2}} \right\} dx. \end{aligned}$$

Now in order to evaluate the above integral we divide the positive line into two subintervals $(0, \epsilon)$ and (ϵ, ∞) where $\epsilon = \sqrt{f_n/n}$ and f_n is any function of n smaller than $o(\sqrt{n})$ such that $f_n \rightarrow \infty$ as $n \rightarrow \infty$. Then since in (ϵ, ∞)

$$\frac{nx^2}{(1+x^2)^{n-1}} \sim nf^{-2n},$$

by the assumptions of Theorem 2 the term inside Φ function tends to infinity as $n \rightarrow \infty$. Hence from (3.4) we have

$$\begin{aligned} EM_u(\epsilon, \infty) &< \frac{\sqrt{n-2}}{2\pi} \int_\epsilon^\infty \frac{\sqrt{2nx^2 - 2x^2 + n^2x^4 - 3nx^4 + 2x^4 + 2}}{(1+x^2)(nx^2 - x^2 + 1)} dx \\ &\sim \frac{\sqrt{n-2}}{2\pi} \int_\epsilon^\infty \frac{dx}{1+x^2} \\ (3.5) \quad &\sim \frac{\sqrt{n-2}}{4}. \end{aligned}$$

Also from (2.8) we obtain

$$\begin{aligned}
 EM_u(0, \epsilon) &< \frac{\sqrt{n-2}}{2\pi} \int_0^\epsilon \frac{\sqrt{(nx^2 - x^2 + 1)^2 + x^4 + 1}}{(1+x^2)(nx^2 - x^2 + 1)} dx \\
 &= \int_0^\epsilon \frac{\sqrt{n-2}}{2\pi(1+x^2)} \sqrt{1 + \frac{1+x^4}{nx^2 - x^2 + 1}} dx \\
 (3.6) \quad &< \frac{\sqrt{n-2}}{\pi} \int_0^\epsilon \frac{dx}{1+x^2} = \sqrt{n-2} \epsilon = o(f_n).
 \end{aligned}$$

Therefore from (3.5) and (3.6) we have $\sqrt{n}/4$ as an upper limit for $EM_u(0, \infty)$. In order to obtain a lower limit again from (2.8) and for our assumption of u , for all sufficiently large n we can say

$$\begin{aligned}
 EM_u(0, \infty) &> EM_u(\epsilon, \infty) \\
 &\sim \int_\epsilon^\infty \frac{\sqrt{(n-2)(2nx^2 - 2x^2 + n^2x^4 - 3nx^4 + 2x^4 + 2)}}{(1+x^2)(nx^2 - x^2 + 1)} dx \\
 &\sim \frac{\sqrt{n-2}}{2\pi} \int_\epsilon^\infty \frac{dx}{1+x^2} \\
 (3.7) \quad &\sim \frac{\sqrt{n-2}}{4}.
 \end{aligned}$$

Hence from (3.7) and the above upper limit we have the proof of Theorem 2.

4. SHARP CROSSINGS

Since from (2.1), (2.2), (2.4) and (2.5) $\Delta^2/A^2 = (n-1)(x^2+1)^{n-3}$ and $\Delta/A^2 = \sqrt{n-1}/(x^2+1)$, then from (2.14) we have

$$\begin{aligned}
 ES_u(0, \infty) &= \frac{1}{\pi} \int_0^\infty \frac{\Delta}{A^2} \exp\left(-\frac{u^2 A^2}{2\Delta^2}\right) dx \\
 &= \frac{\sqrt{n-1}}{\pi} \int_0^\infty \frac{\exp\{-u^2/(n-1)(x^2+1)^{n-3}\}}{(x^2+1)} dx \\
 (4.1) \quad &= \frac{\sqrt{n-1}}{\pi} \int_0^{\pi/2} \exp\left(-\frac{u^2 \cos^{2n-6} \theta}{n-1}\right) d\theta
 \end{aligned}$$

where $x = \tan \theta$. Now we divide the interval $(0, \pi/2)$ into two subintervals $(0, \epsilon)$ and $(\epsilon, \pi/2)$ where $\epsilon = 1/\sqrt{n}$. Then we can evaluate the integral which appears in (4.1) as

$$\begin{aligned}
 &\int_0^{\pi/2} \exp\left(-\frac{u^2 \cos^{2n-6} \theta}{n-1}\right) d\theta \\
 &> \int_0^\epsilon \exp\left(-\frac{u^2}{n-1}\right) d\theta + \int_\epsilon^{\pi/2} \exp\left(-\frac{u^2 \cos^{2n-6} \epsilon}{n-1}\right) d\theta \\
 &\sim \epsilon \exp\left(-\frac{u^2}{n-1}\right) + \left(\frac{\pi}{2} - \epsilon\right) \exp\left\{-\frac{u^2}{n-1}(1-\epsilon^2)^{n-3}\right\} \\
 &\sim \epsilon \exp\left(-\frac{u^2}{n-1}\right) + \frac{\pi}{2} \exp\left\{-\frac{u^2}{n-1}e^{-(n-3)/n}\right\} \\
 &\sim \frac{\pi}{2}.
 \end{aligned}$$

Hence from (4.1) we obtain $(\sqrt{n-1})/2$ as a lower limit for $ES_u(0, \infty)$. Therefore, since $ES_u(0, \infty) < ES_0(0, \infty) \sim (\sqrt{n-1})/2$, from [6] we have the proof of Theorem 3.

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