ON MINIMAL LENGTHS OF EXPRESSIONS OF COXETER GROUP ELEMENTS AS PRODUCTS OF REFLECTIONS

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ABSTRACT. It is shown that the absolute length $l'(w)$ of a Coxeter group element w (i.e. the minimal length of an expression of w as a product of reflections) is equal to the minimal number of simple reflections that must be deleted from a fixed reduced expression of w so that the resulting product is equal to e , the identity element. Also, $l'(w)$ is the minimal length of a path in the (directed) Bruhat graph from the identity element e to w , and $l'(w)$ is determined by the polynomial $R_{e,w}$ of Kazhdan and Lusztig.

INTRODUCTION

Let (W, S) be a Coxeter system and let $T := \bigcup_{w \in W} wSw^{-1}$ denote the corresponding set of reflections. For $w \in W$, define $l(w)$ (resp., $l'(w)$) to be the minimal number of factors occurring amongst all expressions of w as a product of simple reflections S (resp., reflections). The function l (resp., l') is called the standard (resp., absolute) length function of (W, S) . An expression $w = s_1 \cdots s_n$ with $s_i \in S$ and $n = l(w)$ is called a reduced expression for w .

The standard length function has been extensively studied (see e.g. [2], [11] as general references on Coxeter groups). In case W is a finite reflection group in its natural reflection representation, $l'(w)$ is the codimension of the 1-eigenspace of w (see [4, Lemmas 1–3], which do not use the crystallographic condition), and $\sum_{w \in W} X^{l'(w)} = \prod_{i=1}^n (1 + (d_i - 1)X)$ where $d_1, \dots, d_n$ are the degrees of W ([13]; see also [1] for other natural interpretations of this polynomial). However, this geometric description of $l'(w)$ does not extend to arbitrary infinite W , for the function l' may be unbounded even if S is finite.

In this note, we show that $l'(w)$ is equal to the minimum number of simple reflections that must be deleted from a fixed reduced expression for w so that the resulting product of simple reflections is equal to the identity element e of W . Moreover, $l'(w)$ can also be characterized in terms of combinatorial data related to the well-known Chevalley-Bruhat order (namely the Bruhat graph [8] and the R -polynomial $R_{e,w}$ of [12]). The R -polynomials have been studied quite extensively (cf. [12], [5], [10], [3] for example) because of their close connections to Kazhdan-Lusztig polynomials, which are of interest for their significant applications to Lie representation theory and related geometry and combinatorics.

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1. STATEMENT OF RESULTS

Theorem 1.1. *Let $w = s_1 \dots s_n$ be a reduced expression for $w \in W$. Then $l'(w)$ is the minimum of the natural numbers p for which there exist $1 \leq i_1 < \dots < i_p \leq n$ such that $e = s_1 \dots \widehat{s_{i_1}} \dots \widehat{s_{i_p}} \dots s_n$, where a hat over a factor indicates its omission.*

Write $t_i = s_1 \dots s_{i-1} s_i s_{i-1} \dots s_1$, and fix an expression for e as in the theorem, with $p = l'(w)$. Then $t_{i_1} \dots t_{i_p} w = e$, or equivalently $w = t_{i_p} \dots t_{i_1}$. Thus, $l'(w)$ and an expression of w as a product of $l'(w)$ reflections are effectively computable.

Define the “Bruhat graph” $\Omega = \Omega_{(W,S)}$ to be the directed graph with vertex set W and an edge from $x \in W$ to $y \in W$ iff $l(y) > l(x)$ and there is a reflection t such that $y = tx$ (see [8]¹). It is often convenient to regard Ω as an edge-labelled directed graph, by attaching the label $t \in T$ to an edge (x, tx) . Let $\bar{\Omega}$ be the underlying undirected graph of Ω . A path of length n in Ω (resp., $\bar{\Omega}$) is a sequence $x_0, \dots, x_n$ in W such that for each i there is a directed (resp., undirected) edge of Ω (resp., $\bar{\Omega}$) from x_{i-1} to x_i . Note that $l'(x)$ is the minimal length of a path in $\bar{\Omega}$ from e to x , where e is the identity element of W .

The Chevalley-Bruhat order $\leq$ on W may be defined by $x \leq y$ if there is a path in Ω from x to y ; equivalently, if $y = s_1 \dots s_n$ is a reduced expression for y , then $x \leq y$ iff $x = s_1 \dots \widehat{s_{i_1}} \dots \widehat{s_{i_p}} \dots s_n$ for some $1 \leq i_1 < \dots < i_p \leq n$.

Theorem 1.2. *For any $w \in W$, $l'(w)$ is the minimal length of a path in Ω from e to w .*

Let $\mathbb{Z}[\alpha]$ be the polynomial ring in an indeterminate α . It is known that there are polynomials $\tilde{R}_{x,y} = \tilde{R}_{x,y}(\alpha) \in \mathbb{Z}[\alpha]$ for $x, y \in W$ uniquely determined by the initial conditions $\tilde{R}_{x,y} = 0$ unless $x \leq y$ and $\tilde{R}_{x,x} = 1$, and the following recurrence relation: for $x, y \in W$ and $s \in S$ with $l(sy) < l(y)$,

$$\tilde{R}_{x,y} = \begin{cases} \tilde{R}_{sx,sy} & \text{if } l(sx) < l(x), \\ \alpha \tilde{R}_{x,sy} + \tilde{R}_{sx,sy} & \text{if } l(sx) > l(x). \end{cases}$$

In fact, $q^{(l(y)-l(x))/2} \tilde{R}_{x,y}(q^{\frac{1}{2}} - q^{-\frac{1}{2}}) = R_{x,y}$ where $R_{x,y}$ is as defined in [12].

Theorem 1.3. *For $w \in W$, $l'(w)$ is the smallest integer m such that the coefficient of α^m in $\tilde{R}_{e,w}$ is non-zero.*

The subgroups of W generated by subsets of the simple reflections are called standard parabolic subgroups, and their conjugates in W are called parabolic subgroups. We record the following simple consequence of the theorems above.

Corollary 1.4. *If W' is a parabolic subgroup of W , then for all $w \in W'$, $l'(w)$ is equal to the minimal number of factors occurring amongst expressions for w as a product of elements of $W' \cap T$.*

Proof. If W' is a standard parabolic subgroup the corollary is clear from Theorem 1.1. The general case follows since the function l' is constant on conjugacy classes. $\square$

¹We take this opportunity to remove an unintended and unnecessary restriction in [8]; the initial sentence of [8, §1] should begin “Let (W, R) be a Coxeter system...”.

2. PROOF OF THE MAIN RESULTS

We will deduce both Theorems 1.1 and 1.3 using 1.2. We begin by recalling from [7], [8] (cf. also [6], in which some of the proofs are simpler) certain properties of reflection subgroups of W which are used in the proofs here.

For a reflection subgroup W' of W (i.e. W' is generated by $W' \cap T$), let

$$\chi(W') := \{t \in W' \cap T \mid l(t't) > l(t) \text{ for all } t' \in W' \cap T \text{ with } t' \neq t\}.$$

Theorem 2.1. *For any reflection subgroup W' of W , $S' = \chi(W')$ is a set of Coxeter generators of W' , and the set of reflections of (W', S') is $W' \cap T$. Moreover, for any $x \in W$, the coset $W'x$ contains a unique element x_0 with $l(x_0)$ minimal, and then the map $v \mapsto vx_0: W' \rightarrow W'x$ induces an isomorphism of directed labelled graphs between $\Omega_{(W', S')}$ and the full subgraph $\Omega_{(W, S)}(W'x)$ of $\Omega_{(W, S)}$ on vertex set $W'x$.*

We say $(W', \chi(W'))$ above is a dihedral reflection subsystem of (W, S) if W' can be generated by two distinct reflections, or, equivalently by [7], if $\chi(W')$ has exactly two elements.

The following lemma immediately establishes Theorem 1.2.

Lemma 2.2. *If there is a path from e to $x \in W$ in $\bar{\Omega}$ of length n , then there is a path from e to x in Ω of length n' for some $n' \leq n$.*

Proof. To prove the lemma, consider a path $e = x_0, x_1, \dots, x_n = x$ in $\bar{\Omega}$ of length n from e to x . We proceed by induction on n , and then for $n > 0$ by induction on $m := l(x_{n-1})$. Replacing n by a smaller integer if necessary, we may assume by the inductive hypothesis that $n \geq 2$ and $x_0, \dots, x_{n-1}$ is a path in Ω . We also assume without loss of generality that $x_n \neq x_{n-2}$ and $l(x_n) < l(x_{n-1})$.

There exist distinct reflections t and t' so $x_n = tx_{n-1}$ and $x_{n-1} = t'x_{n-2}$. Let W' be the dihedral reflection subgroup of W generated by t and t' , $S' := \chi(W')$ and $\Omega' := \Omega_{(W, S)}(W'x_{n-1})$. By Theorem 2.1 and inspection of Bruhat graphs of dihedral Coxeter systems (cf. [8, (1.2)]) it follows that there is a path $x_0, \dots, x_{n-2}, x'_{n-1}, x_n$ in $\bar{\Omega}$ with $x'_{n-1} \in W'x_{n-1}$ which is either a path in Ω (as required) or else such that there is an edge of Ω from x'_{n-1} to x_{n-2} . In the latter case, we have $l(x'_{n-1}) < l(x_{n-2}) < l(x_{n-1}) = m$ so we are done by induction on m . $\square$

Proof of Theorem 1.1. Let $w \in W$ have reduced expression $s_1 \cdots s_n$, and define the reflection $t_i = s_1 \cdots s_i \cdots s_1$ for each i . If $e = s_1 \cdots \widehat{s_{i_1}} \cdots \widehat{s_{i_p}} \cdots s_n$, then $t_{i_1} \cdots t_{i_p} w = e$ so $w = t_{i_p} \cdots t_{i_1}$ and $p \geq l'(w)$. On the other hand, by Theorem 1.2 there is a directed path $x_0 = e, x_1, \dots, x_p = w$ of length $p = l'(w)$ in Ω . The strong exchange condition [11] immediately implies that if x has an expression $x = r_1 \cdots r_m$ as a product of simple reflections and there is an edge in Ω from y to x , then $y = r_1 \cdots \widehat{r_i} \cdots r_m$ for some i . It follows immediately by descending induction on i that x_i has an expression obtained by deleting $p - i$ simple reflections from the reduced expression $s_1 \cdots s_n$ for w . For $i = 0$, this is what we had to show. $\square$

Finally, Theorem 1.3 is a consequence of Theorem 1.2 and the following fact from [6]; we provide a more conceptual proof here.

Theorem 2.3. *The coefficient of α^n in the polynomial $\tilde{R}_{x,y}$ is non-zero iff there is a path of length n from x to y in Ω .*

Proof. Recall from [10] that a reflection order of a dihedral Coxeter system (W', S') is a total order $\preceq$ of its reflections such that if $S' = \{r, s\}$ with $r \prec s$, then $r \prec rsr \prec rsrsr \prec \dots \prec srs \prec s$. A reflection order of an arbitrary Coxeter system (W, S) is a total order of the set T of its reflections which restricts to a reflection order on the set of reflections of each dihedral reflection subsystem $(W', \chi(W'))$ of (W, S) ; such reflection orders exist by [10, (2.3)].

We deduce Theorem 2.3 here from the formula [10, (3.4)] for $\tilde{R}_{x,y}$ as the generating function for the set of paths from x to y in Ω with increasing label. Fix a reflection order $\preceq$ of (W, S) . Then

$$(2.1) \quad \tilde{R}_{x,y} = \sum_{n \in \mathbb{N}} \sum_{(t_1, \dots, t_n)} \alpha^n$$

where the inner sum is over those $(t_1, \dots, t_n) \in T^n$ such that $x, t_1x, \dots, t_n \cdots t_1x = y$ is a path in Ω and $t_1 \prec t_2 \prec \dots \prec t_n$.

Hence if the coefficient of α^n in $\tilde{R}_{x,y}$ is non-zero, there is a path of length n from x to y in Ω . Conversely, if there is a path $x = x_0, \dots, x_n = y$ of length n from x to y in Ω , one may write $x_i = t_i \cdots t_1x$ for some reflections $t_1, \dots, t_n$ in W . One may suppose the “label” $(t_1, \dots, t_n) \in T^n$ is chosen to be minimal (in the lexicographic order on T^n induced by $\preceq$) amongst the labels of all paths of length n from x to y in Ω . For each $i = 1, \dots, n-1$, let W_i be the dihedral reflection subgroup of W generated by t_i and t_{i+1} , and let $S_i = \chi(W_i)$. Let y_i be the element of $W_i x_i$ with minimal length $l(y_i)$. The restriction of $\preceq$ to a total order on $W_i \cap T$ is a reflection order of (W_i, S_i) . Now by Theorem 2.1, (t_i, t_{i+1}) is the lexicographically first label (in $(W_i \cap T)^2$) of length two paths in $\Omega_{(W_i, S_i)}$ from $x_{i-1}y_i^{-1}$ to $x_{i+1}y_i^{-1}$. An examination of Bruhat graphs for dihedral groups shows that $t_i \prec t_{i+1}$ for each i . Hence $t_1 \prec \dots \prec t_n$ and the coefficient of α^n in $\tilde{R}_{x,y}$ is non-zero as required. $\square$

Remark 2.4. By Theorem 2.3 and either [3, 6.1] or [5, 1.3 and 5.3], or by the proof of [8, (3.3)], it follows that if $x \leq y$, then there is an integer $m \equiv l(y) - l(x) \pmod{2}$ such that there is a path of length n from x to y in Ω iff $m \leq n \leq l(y) - l(x)$ and $n \equiv l(y) - l(x) \pmod{2}$. If $x = e$, then $m = l'(y)$.

Remark 2.5. The analogue of (2.1) holds for the R -functions of the “twisted Bruhat orders” of [9]; Theorem 2.3 and its proof, and Remark 2.4, extend to these orders mutatis mutandis.

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