ON AN ASYMPTOTIC BEHAVIOR OF ELEMENTS OF ORDER p IN IRREDUCIBLE REPRESENTATIONS OF THE CLASSICAL ALGEBRAIC GROUPS WITH LARGE ENOUGH HIGHEST WEIGHTS

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ABSTRACT. The behavior of the images of a fixed element of order p in irreducible representations of a classical algebraic group in characteristic p with highest weights large enough with respect to p and this element is investigated. More precisely, let G be a classical algebraic group of rank r over an algebraically closed field K of characteristic $p > 2$. Assume that an element $x \in G$ of order p is conjugate to that of an algebraic group of the same type and rank $m < r$ naturally embedded into G . Next, an integer function σ_x on the set of dominant weights of G and a constant c_x that depend only upon x , and a polynomial d of degree one are defined. It is proved that the image of x in the irreducible representation of G with highest weight ω contains more than $d(r - m)$ Jordan blocks of size p if m and $r - m$ are not too small and $\sigma_x(\omega) \geq p - 1 + c_x$.

Asymptotic lower estimates for the number of Jordan blocks of size p in the images of a fixed element of order p in irreducible representations of a classical algebraic group in characteristic p with highest weights large enough with respect to p and this element are obtained. More precisely, let G be a classical algebraic group of rank r over an algebraically closed field K of characteristic $p > 2$. Assume that an element $x \in G$ of order p is conjugate to that of an algebraic group G_m of the same type and rank $m < r$ naturally embedded into G . Set

$$d(r - m) = \begin{cases} r - m & \text{for } G = A_r(K), \\ 2r - 2m & \text{for other types.} \end{cases}$$

Let Δ_x be the labelled Dynkin diagram of the conjugacy class containing x in the sense of Bala and Carter [1] and let c_x be the sum of the labels at Δ_x for $G \neq A_r(K)$ and the half of this sum for $G = A_r(K)$. For brevity, throughout the article we refer to Δ_x as the labelled Dynkin diagram of x . Next, an integer function σ_x on the set of dominant weights of G that depends only upon Δ_x is defined. For p -restricted weights σ_x coincides with the canonical homomorphism determined by

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Δ_x . It is proved that the image of x in the irreducible representation of G with highest weight ω contains more than $d(r-m)$ Jordan blocks of size p if m and $r-m$ are not too small and $\sigma_x(\omega) \geq p-1+c_x$.

We need some more notation to formulate the main results. Let ω_i and α_i be the fundamental weights and the simple roots of G (with respect to a fixed maximal torus T) labelled as in [2]. Denote by δ_i , $1 \leq i \leq r$, the label on Δ_x corresponding to its i th node. We have $0 \leq \delta_i \leq 2$. In what follows $\mathbb{Z}$ is the set of integers, $\mathbf{X} = \mathbf{X}(G)$ is the set of weights of G , $\mathbf{X}^+ \subset \mathbf{X}$ is the set of dominant weights, $\text{Irr} = \text{Irr}(G)$ is the set of irreducible rational representations of G (considered up to the equivalence) and $\omega(\varphi)$ is the highest weight of a representation φ . There exists a uniquely determined homomorphism $\tau_x : \mathbf{X} \rightarrow \mathbb{Z}$ such that $\tau_x(\alpha_i) = \delta_i$. The weight $\mu \in \mathbf{X}^+$ is called p -restricted if $\mu = \sum_{i=1}^r a_i \omega_i$ with all $a_i < p$. Each weight $\omega \in \mathbf{X}^+$ can be represented in the form $\sum_{j=0}^s p^j \omega^j$ where ω^j are p -restricted. Set $\sigma_x(\omega) = \tau_x(\sum_{j=0}^s \omega^j)$. Now we can state our main result.

Theorem 1. *Let $\varphi \in \text{Irr}$ and $\sigma_x(\omega(\varphi)) \geq p-1+c_x$. Assume that $m > 1$ for $G = B_r(K)$ or $D_r(K)$, $r-m > 1$ for $G = A_r(K)$, and > 3 for $G = B_r(K)$ or $D_r(K)$. Then the element $\varphi(x)$ has more than $d(r-m)$ Jordan blocks of size p .*

Proposition 2 below shows that one cannot weaken the inequality for $\sigma_x(\omega(\varphi))$ in Theorem 1 and that the estimates obtained are asymptotically exact.

Proposition 2. *Let $\varphi \in \text{Irr}$ and $\omega = \omega(\varphi) = a\omega_1$ with $a < p$. Assume that m and r are such as in Theorem 1 and x is a regular unipotent element in G_m . Set $c = m$ for $G = A_r(K)$, $2m$ for $G = B_r(K)$, $2m-1$ for $G = C_r(K)$, and $2m-2$ for $G = D_r(K)$. Suppose that $p > c$. Then $|x| = p$, $\sigma_x(\omega) = ac$ and $c_x = c$. There exist constants $N_G(a, m, p)$ and $Q_G(a, m, p)$ that depend upon the type of G , a , m , and p and do not depend upon r such that $\varphi(x)$ contains at most $N_G(a, m, p)$ Jordan blocks of size p if $p < ac < p+c-1$ and at most $d(r-m) + Q_G(a, m, p)$ such blocks if $ac = p+c-1$.*

Put $l = [(m+2)/2]$ for $G = A_r(K)$ and $l = m$ otherwise. By Lemma 3 below, $c_x = \sum_{i=1}^l \delta_i$ and $c_x \leq p-1$. Hence $c_x \leq 2l$.

For $\varphi \in \text{Irr}$ define the weight $\bar{\omega}(\varphi)$ as follows: write down the p -adic expansion for the weight $\omega = \omega(\varphi)$ considered before the statement of Theorem 1 and set $\bar{\omega}(\varphi) = \sum_{j=0}^s \omega^j$. So $\sigma_x(\omega(\varphi)) = \tau_x(\bar{\omega}(\varphi))$.

The study of an asymptotic behavior of elements of order p in representations of the classical groups in characteristic p was begun by the author in [12] where a notion of a p -large representation was introduced. In our present notation a representation $\varphi \in \text{Irr}$ is p -large if and only if $\sigma_x(\omega(\varphi)) \geq p$ for a long root element $x \in G$ (an equivalent definition from [12]: the value of $\bar{\omega}(\varphi)$ on the maximal root is $\geq p$). The common goal of [12] and the present article is to investigate the behavior of elements of order p in irreducible representations in characteristic p for a fixed p and $r \rightarrow \infty$ and to discover asymptotic regularities which are specific for prime characteristics but do not (or almost do not) depend upon p . Such properties can find applications in recognizing representations and linear groups. According to [12, Theorem 1.1], the image of any element of order p in a p -large representation has

at least $f(r)$ Jordan blocks of size p where

$$f(r) = \begin{cases} 2r - 2 & \text{for } G = A_r(K), \\ 6r - 7 & \text{for } G = B_r(K), p = 3, \\ 8r - 10 & \text{for } G = B_r(K), p > 3, \\ 4r - 4 & \text{for } G = C_r(K), \\ 4r - 8 & \text{for } G = D_r(K), p = 2, \\ 6r - 10 & \text{for } G = D_r(K), p = 3, \\ 8r - 16 & \text{for } G = D_r(K), p > 3. \end{cases}$$

In [12, Theorem 1.3] for $p > 2$ and all types of the classical groups examples of representations $\varphi \in \text{Irr}$ such that $\varphi(x)$ has only one Jordan block of size p for a long root element x and $\sigma_x(\omega(\varphi)) = p - 1$ are given. It is also shown [12, Theorem 1.4] that the estimates in [12, Theorem 1.1] are asymptotically exact for the groups of type A , B , and D provided $p > 2$ in the last two cases.

In what follows $\mathbb{C}$ is the complex field, $G_{\mathbb{C}}$ is the simple algebraic group over $\mathbb{C}$ of the same type and rank as G , and $\text{Irr}_{\mathbb{C}}$ is the set of irreducible rational representations of $G_{\mathbb{C}}$ (considered up to the equivalence). For $\rho \in \text{Irr}$ or $\text{Irr}_{\mathbb{C}}$ and a unipotent element $z \in G$ or $G_{\mathbb{C}}$ denote by $k_{\rho}(z)$ the degree of the minimal polynomial of $\rho(z)$. It is well known that $k_{\rho}(z)$ is equal to the maximal size of a Jordan block of $\rho(z)$. If $\varphi \in \text{Irr}$, then $\varphi_{\mathbb{C}}$ is the irreducible representation of $G_{\mathbb{C}}$ with highest weight $\bar{\omega}(\varphi)$. For unipotent $x \in G$ put $k_{\varphi_{\mathbb{C}}}(x) = k_{\varphi_{\mathbb{C}}}(y)$ where $y \in G_{\mathbb{C}}$ is a unipotent element with the labelled Dynkin diagram Δ_x (this is correctly determined). Now let $|x| = p$. By the results [11, Theorem 1.1, Lemma 2.5, and Proposition 2.12], $k_{\varphi_{\mathbb{C}}}(x) = \sigma_x(\omega(\varphi)) + 1$ and $k_{\varphi}(x) = \min\{p, k_{\varphi_{\mathbb{C}}}(x)\}$. Hence if z is a long root element, then $k_{\varphi}(z) = k_{\varphi_{\mathbb{C}}}(z)$ if and only if φ is not p -large. The results of [11] imply that for not very small p and r there exists a wide class of representations $\varphi \in \text{Irr}$ such that $k_{\varphi}(z) = k_{\varphi_{\mathbb{C}}}(z) < p$ for a long root element $z \in G$, but $k_{\varphi}(x) = p < k_{\varphi_{\mathbb{C}}}(x)$ for many other elements $x \in G$ of order p . In this connection in [13, Section 2] a notion of a p -large representation for a given element x of order p was introduced. A representation $\varphi \in \text{Irr}$ was called p -large for x if $\sigma_x(\omega(\varphi)) \geq p$. It has been conjectured ([13, Conjecture 1] that if $x \in G_m$, r is large enough with respect to m and φ is p -large for x , then $\varphi(x)$ has at least $F(r)$ blocks of size p where F is an increasing function. Our Proposition 2 formally disproves this conjecture, but Theorem 1 actually proves a refined version of it with a stronger assumption on $\sigma_x(\omega(\varphi))$. Thus for arbitrary elements x there is a gap between the class of representations $\varphi \in \text{Irr}$ with $k_{\varphi}(x) = k_{\varphi_{\mathbb{C}}}(x)$ and that of representations where asymptotic estimates for the number of Jordan blocks of size p in $\varphi(x)$ hold. Perhaps for some classes of elements of order p stronger estimates than those of Theorem 1 are possible, but now it is not clear how to determine such classes.

The case $p = 2$ is not considered here, but in this situation $\varphi \in \text{Irr}$ is 2-large if $\omega(\varphi) \neq 2^j \omega_i$. For 2-large representations the estimates from [12, Theorem 1.1] are available. For remaining representations certain estimates could be obtained as well, but this article does not seem a proper place for this. We plan to handle this question in a subsequent paper which will be devoted to refining some estimates in [12].

The results of this article as well as those of [12] can be easily transferred to irreducible K -representations of finite classical groups in characteristic p .

1. NOTATION AND PRELIMINARY COMMENTS

Throughout the article for a semisimple algebraic group S the symbols $\text{Irr}(S)$, $\mathbf{X}(S)$, and $\mathbf{X}^+(S)$ mean the same as the similar ones for G introduced earlier; $R(S)$ is the set of roots of S , $\langle S_1, \dots, S_j \rangle$ is the subgroup in S generated by subgroups $S_1, \dots, S_j$; $\text{Irr}_p(S) \subset \text{Irr}(S)$ is the set of p -restricted representations, i.e. irreducible representations with p -restricted highest weights; $\mathbf{X}(\varphi)$ ($\mathbf{X}(M)$) is the set of weights of a representation φ (a module M); $\dim M$ is the dimension of M ; $M(\omega)$ is the irreducible S -module with highest weight ω ; L is the Lie algebra of G ; $R = R(G)$, $R^+ \subset R$ is the set of positive roots; $\text{Irr}_p = \text{Irr}_p(G)$; $\mathcal{X}_\beta \subset G$ and $X_\beta \in L$ are the root subgroup and the root element associated with $\beta \in R$, $\mathcal{X}_{\pm i} = \mathcal{X}_{\pm \alpha_i}$, and $X_{\pm i} = X_{\pm \alpha_i}$. Set $H(\beta_1, \dots, \beta_j) = \langle \mathcal{X}_{\beta_1}, \mathcal{X}_{-\beta_1} \dots, \mathcal{X}_{\beta_j}, \mathcal{X}_{-\beta_j} \rangle$. For $\omega \in \mathbf{X}(S)$ and $\alpha \in R(S)$ denote by $\langle \omega, \alpha \rangle$ the value of the weight ω on the root α . For an S -module M and a unipotent element $x \in S$ define $k_M(x)$ similarly to $k_\varphi(x)$. If $|x| = p$, then $n_\varphi(x)$ is the number of Jordan blocks of size p of the matrix $\varphi(x)$ for a representation φ of S and $n_M(x)$ denotes the same number for a module M affording φ .

An element $x \in G$ of order p can be embedded into a closed connected subgroup Γ of type A_1 whose labelled diagram coincides with Δ_x (see [6, Theorem 4.2]). Set $\mathbf{X}_1 = \mathbf{X}(A_1(K))$ (the simply connected group of this type) and identify $\mathbf{X}_1$ with $\mathbb{Z}$ mapping $a\omega_1 \in \mathbf{X}_1$ into $a \in \mathbb{Z}$. Then $\mathbf{X}(\Gamma)$ can be identified with a subset of $\mathbb{Z}$. The canonical homomorphism τ_x can be obtained as the restriction of weights from a maximal torus $T \subset G$ to a maximal torus $T_1 \subset \Gamma$ such that $T_1 \subset T$. From now on we fix the tori T and T_1 , and all weights and roots of G and Γ are considered with respect to T and T_1 . Throughout the text ε_i with $1 \leq i \leq r+1$ for $G = A_r(K)$ and $1 \leq i \leq r$ otherwise are weights of the standard realization of G labelled as in [3, ch. VIII, §13]. Set $e_i = \tau_x(\varepsilon_i)$. One can choose Γ , T and T_1 such that the restriction to Γ of the natural representation of G is a direct sum of irreducible components with p -restricted highest weights (see comments in [14, Section 3]); $e_i \geq e_j$ for $i < j$; $e_i \geq 0$ if $G = A_r(K)$ and $i \leq (r+1)/2$; and $e_i \geq 0$ for all $i \leq r$ if $G \neq A_r(K)$. If $H \subset G$ is a semisimple subgroup generated by some root subgroups, then $T_H = T \cap H$ is a maximal torus in H . If $T_1 \subset T_H$, we denote by the same symbol τ_x the homomorphism $\mathbf{X}(H) \rightarrow \mathbb{Z}$ determined by restricting weights from T_H to T_1 . This causes no confusion. If an element v of some G -module is an eigenvector for T , we denote its weights with respect to T , T_H , and T_1 by $\omega(v)$, $\omega_H(v)$, and $\omega_\Gamma(v)$. In what follows x is conjugate to an element of G_m , $|x| = p$, m and $r-m$ are such as in the assertion of Theorem 1, and $\delta_i = \tau_x(\alpha_i)$, $1 \leq i \leq r$.

Lemma 3. Set $l = [(m+2)/2]$ for $G = A_r(K)$ and $l = m$ otherwise. Then $c_x = \sum_{i=1}^l \delta_i$ and $c_x \leq p-1$.

Proof. Put $k = l-1$ for $G = A_r(K)$, $m = 2t$, and $k = l$ for $G = A_r(K)$, $m = 2t+1$. Our assumptions on e_i , $m-r$, and x imply that $e_i = 0$ for $k < i < r+2-k$ if $G = A_r(K)$ and $e_i = 0$ for $i > m$ otherwise; notice that $e_{k+1} = e_{k+2} = 0$ for $G = A_r(K)$. Now it follows from the definition of c_x and the formulae in [3, ch. VIII, §13] that $c_x = \sum_{i=1}^l \delta_i = e_1 - e_{l+1} = e_1$. As e_1 is a weight of a p -restricted Γ -module, we have $e_1 < p$. This yields the lemma. $\square$

Proof of Theorem 1. Set $\omega = \omega(\varphi)$ and let $\omega = \sum_{i=1}^r a_i \omega_i$. It is clear that $\omega \neq 0$ as $\tau_x(\omega) \neq 0$. Define subgroups H_1 and $H_2 \subset G$ as follows. For $G = A_r(K)$ set

$u = r - t + 2$ if $m = 2t$ and $r - t + 1$ if $m = 2t + 1$, $\beta = \varepsilon_{t+1} - \varepsilon_u$,

$$H_1 = H(\alpha_1, \dots, \alpha_t, \beta, \alpha_u, \dots, \alpha_r), \quad H_2 = H(\alpha_{t+2}, \dots, \alpha_{u-2})$$

(we have $H_1 = H(\alpha_1, \varepsilon_2 - \varepsilon_{r+1})$ for $m = 2$ and $H_1 = H(\varepsilon_1 - \varepsilon_{r+1})$ for $m = 1$). For $G = B_r(K)$, $C_r(K)$, or $D_r(K)$ put $\beta = \varepsilon_m$, $2\varepsilon_m$, or $\varepsilon_{m-1} + \varepsilon_m$, respectively, and

$$H_1 = H(\alpha_1, \dots, \alpha_{m-1}, \beta).$$

Next, set

$$H_2 = H(\alpha_{m+1}, \dots, \alpha_{r-1}, \varepsilon_{r-1} + \varepsilon_r)$$

for $G = B_r(K)$ and

$$H_2 = H(\alpha_{m+1}, \dots, \alpha_r)$$

for $G = C_r(K)$ or $D_r(K)$ (here $H_1 = H(\beta)$ for $G = C_r(K)$ and $m = 1$). One easily observes that the sets of roots in brackets used to define H_1 and H_2 yield bases of the systems $R(H_1)$ and $R(H_2)$, respectively. Denote these bases by $\mathcal{B}_i$. In all cases H_1 is conjugate to G_m in G . We have $H_2 \cong A_{r-m-1}(K)$, $D_{r-m}(K)$, $C_{r-m}(K)$, or $D_{r-m}(K)$ for $G = A_r(K)$, $B_r(K)$, $C_r(K)$, or $D_r(K)$, respectively. It is clear that the subgroups H_1 and H_2 commute. Set $H = H_1 H_2$. Let $U_i = \langle \mathcal{X}_\gamma \mid \gamma \in R^+, \mathcal{X}_\gamma \subset H_i \rangle$, $i = 1, 2$, and $U = U_1 U_2$. It is not difficult to conclude that U_i is a maximal unipotent subgroup in H_i and U is such a subgroup in H . We can assume that $x \in U_1$, $\Gamma \subset H_1$ and $T_1 \subset T_{H_1}$. We shall write a weight $\mu \in \mathbf{X}(H)$ in the form (μ_1, μ_2) where $\mu_i \in \mathbf{X}(H_i)$ is the restriction of μ to T_{H_i} . Set $M = M(\omega)$.

It is clear that $n_V(x) = \dim(x - 1)^{p-1}V$ for each H -module V . Taking this into account, it is not difficult to conclude the following. If $0 \subset W_1 \subset \dots \subset W_t = V$ is a filtration of V , $F_i = W_i/W_{i-1}$, $1 \leq i \leq t$, and $n_{F_i}(x) = n_i$, then

$$(1) \quad n_V(x) \geq \sum_{i=1}^t n_i.$$

First suppose that $\varphi \in \text{Irr}_p$. Since passing to the dual representation does not influence the Jordan form of $\varphi(x)$, one can assume that $a_i \neq 0$ for some $i \leq (r+1)/2$ if $G = A_r(K)$. As for p -large representations the estimates of [12, Theorem 1.1] hold; we also assume that φ is not p -large. Hence $\langle \mu, \alpha \rangle < p$ for all $\mu \in \mathbf{X}(\varphi)$ and long roots α (for all α if $G = A_r(K)$ or $D_r(K)$). By the formulae for the maximal roots of the classical groups in [2, Tables 1-4], this forces that

$$(2) \quad \begin{aligned} a_1 + \dots + a_r &< p && \text{for } G = A_r(K) \text{ or } C_r(K), \\ a_1 + 2a_2 + \dots + 2a_{r-1} + a_r &< p && \text{for } G = B_r(K), \\ a_1 + 2a_2 + \dots + 2a_{r-2} + a_{r-1} + a_r &< p && \text{for } G = D_r(K). \end{aligned}$$

Now we proceed to construct two composition factors M_1 and M_2 of the restriction $M|H$ such that $n_{M_1}(x) \geq d(r-m)$ and $n_{M_2}(x) > 0$. This will be done for almost all ω . In exceptional cases we shall find one factor M_1 such that $n_{M_1}(x) > d(r-m)$. By (1), this would yield the assertion of the theorem.

Let $v \in M$ be a nonzero highest weight vector. Put $\mu_i = \omega_{H_i}(v)$. The vector v generates an indecomposable H -module V_1 with highest weight $\mu = (\mu_1, \mu_2)$. Using (2), one can deduce that $\langle \mu_1, \beta \rangle < p$ for all $\beta \in \mathcal{B}_1$. Here for $G = B_r(K)$ we take into account that $m > 1$. Hence μ_1 is p -restricted. Now assume that either $G \neq B_r(K)$, or $a_i \neq 0$ for some $i < r$. For such representations we construct another weight vector $w \in M$ that is fixed by U . Set $l = t + 1$ for $G = A_r(K)$, $m = 2t$;

otherwise take l as in Lemma 3. First suppose that $a_j \neq 0$ for some $j \leq l$ (Case 1). Choose maximal such j and put $w = X_{-l} \dots X_{-(j+1)} X_{-j} v$. Now let $a_j = 0$ for all $j \leq l$ (Case 2). Our assumptions on a_i imply that $a_i \neq 0$ for some $i > l$; furthermore, one can take $i \leq (r+1)/2$ for $G = A_r(K)$ and $i < r$ for $G = B_r(K)$. Choose minimal such i and set $w = X_{-l} \dots X_{-(i-1)} X_{-i} v$ if $G \neq D_r(K)$ or $i < r$ and $w = X_{-l} \dots X_{-(r-3)} X_{-(r-2)} X_{-r} v$ for $G = D_r(K)$ and $i = r$. It follows from [12, Lemma 2.1(iii) and Lemma 2.9] that in all cases $w \neq 0$. Using [10, Lemma 72] and analyzing the roots in $\mathcal{B}_1$ and $\mathcal{B}_2$ and the weight system $\mathbf{X}(\varphi)$, we get that U fixes w in all situations. Here it is essential that the case $G = B_r(K)$ with $\omega = a_r \omega_r$ is excluded. In the latter case we cannot assert that $\mathcal{X}_\beta$ fixes w . Set $\lambda_i = \omega_{H_i}(w)$, $i = 1, 2$. Now it is clear that w generates an indecomposable H -module V_2 with highest weight $\lambda = (\lambda_1, \lambda_2)$. We claim that λ_1 is p -restricted. Write down all the situations where $\langle \lambda_1, \gamma \rangle \neq \langle \mu_1, \gamma \rangle$ for some $\gamma \in \mathcal{B}_1$. We have $\langle \lambda_1, \beta \rangle = \langle \mu_1, \beta \rangle - 1$ in Case 1 if $j = l$ and $G \neq B_r(K)$ or $j = l - 1$ and $G = D_r(K)$ and in Case 2 for $G \neq B_r(K)$ and all i ; and $\langle \lambda_1, \beta \rangle = \langle \mu_1, \beta \rangle - 2$ for $G = B_r(K)$ both in Case 1 with $j = l$ and in Case 2. In Case 1 we also have $\langle \lambda_1, \alpha_{j-1} \rangle = \langle \mu_1, \alpha_{j-1} \rangle + 1$ if $j > 1$ and $\langle \lambda_1, \alpha_j \rangle = \langle \mu_1, \alpha_j \rangle - 1$ if $j < l$. In Case 2 one gets $\langle \lambda_1, \alpha_{l-1} \rangle = \langle \mu_1, \alpha_{l-1} \rangle + 1$ if $l > 1$. In all other situations we have $\langle \lambda_1, \gamma \rangle = \langle \mu_1, \gamma \rangle$. Now apply (2) to conclude that λ_1 is p -restricted.

Set $M_1 = M(\mu)$, $M_2 = M(\lambda)$, $M_1^j = M(\mu_j)$, and $M_2^j = M(\lambda_j)$, $j = 1, 2$. Obviously, M_i is a composition factor of V_i . It is well known that $M_i = M_i^1 \otimes M_i^2$. It is clear that $\tau_x(\mu_1) = \tau_x(\omega) \geq p$. Since $x \in H_1$, we have $\delta_t = 0$ if $\alpha_t \in \mathcal{B}_2$. So by Lemma 3,

$$\tau_x(\lambda_1) = \tau_x(\omega(w)) \geq \tau_x(\omega) - \sum_{i=1}^l \delta_i = \tau_x(\omega) - c_x \geq p - 1.$$

It follows from [11, Theorem 1.1, Lemma 2.5, and Proposition 2.12] that $k_{M_i^1}(x) = p$. Hence $n_{M_i}(x) \geq \dim M_i^2$. One easily observes that M_1^2 and M_2^2 cannot both be trivial H_2 -modules. Our assumptions on $r - m$ and [5, Proposition 5.4.13] imply that the dimension of a nontrivial irreducible H_2 -module is at least $d(r - m)$. In the exceptional case where $G = B_r(K)$ and $\omega = a_r \omega_r$ we need to evaluate $\dim M_1^2$. First let $a_r \neq 1$. As above, $X_r v \neq 0$. This implies that $\mathbf{X}(M_1^2)$ contains a dominant weight $\mu_2 - \alpha_r$ and $\dim M_1^2$ is greater than the size of the orbit of μ_2 under the action of the Weyl group of H_2 . The latter is equal to $2^{r-m-1} \geq d(r - m)$ for our values of $r - m$. By (1), this yields the assertion of the theorem for almost all $\varphi \in \text{Irr}_p$. It remains to consider the case where $G = B_r(K)$ and $\omega = \omega_r$. It is well known that then the restriction $M \downarrow H_1$ is a direct sum of 2^{r-m} H_1 -modules $N = M(\omega_m)$. Since $k_m(x) = p$, we get $k_N(x) = p$ and $n_M(x) \geq 2^{r-m} > d(r - m)$.

Now suppose that $\varphi \in \text{Irr} \setminus \text{Irr}_p$. By the Steinberg tensor product theorem [9, Theorem 1.1], φ can be represented in the form $\bigotimes_{j=1}^s \varphi_j \text{Fr}^j$ where Fr is the Frobenius morphism of G associated with raising elements of K to the p th power and all $\varphi_j \in \text{Irr}_p$. It is clear that the morphism Fr does not influence the Jordan form of $\varphi(x)$. Hence one can assume that $\varphi = \psi \otimes \theta$ where $\theta = \varphi_j \text{Fr}^j$ for some j and both ψ and θ are nontrivial. Set $a = \sigma_x(\omega(\psi))$, $\nu = \omega(\varphi_j)$, $b = \tau_x(\nu)$ and define by μ the restriction of ν to T_H . Now it follows from the definitions of σ_x and τ_x that $\sigma_x(\omega) = a + b$. By [11, Theorem 1.1, Lemma 2.5 and Proposition 2.12], $k_\psi(x) = \min\{a + 1, p\}$ and $k_\theta(x) = \min\{b + 1, p\}$. First suppose that a or $b \geq p - 1$. Set $\rho = \psi$ if $a \geq p - 1$ and $\rho = \theta$ otherwise and denote by π the remaining

representation from the pair (ψ, θ) . Then $k_\rho(x) = p$ and [4, ch. VIII, Lemma 2.2] implies that $n_\varphi(x) \geq \dim \pi$. Let $d(r)$ be the value of $d(r - m)$ if one formally sets $m = 0$. Then by [5, Proposition 5.4.13], $\dim \pi \geq d(r) > d(r - m)$ which settles the case under consideration.

Now assume that both a and $b < p - 1$. Then $k_\psi(x) = a + 1$ and $k_\theta(x) = b + 1$. Since $\sigma_x(\omega) \geq p - 1 + c_x$, we have $b > c_x$. Arguing as for p -restricted φ , we can and shall suppose that $\langle \nu, \alpha_i \rangle \neq 0$ for some $i \leq (r + 1)/2$ if $G = A_r(K)$. Put $M' = M(\nu)$ and construct the composition factors M_i , $i = 1, 2$, of the restriction $M'|H$ as for p -restricted M before. Transfer the notation μ_1 , λ_1 , and M_j^i , $i, j = 1, 2$, to M' . Again we have the exceptional case $G = B_r(K)$ and $\nu = a_r \omega_r$ where we do not construct M_2 and consider M_1 only. Obviously, $\tau_x(\mu_1) = b$. As before, we deduce that $\tau_x(\lambda_1) \geq b - c_x$. By [11, Theorem 1.1, Lemma 2.5, and Proposition 2.12], $k_{M_1}(x) = b + 1$ and $k_{M_2}(x) \geq b + 1 - c_x$. Let n_i be the number of Jordan blocks of the maximal size in the canonical form of x as an element of $\text{End } M_i$, $i = 1, 2$. Looking at the realizations of M_i as tensor products, one easily observes that $n_i \geq \dim M_i^2$. Set $F_1 = M(\omega(\psi)) \otimes M_1$, $F_2 = M(\omega(\psi)) \otimes M_2$ and consider F_i as H -modules in the natural way. In the general case the H -module M has a filtration two of whose quotients are isomorphic to F_1 and F_2 , respectively. In the exceptional case F_1 is a quotient of a submodule in M . Observe that $a + k_{M_2}(x) \geq p$. Using [4, ch. VIII, Theorem 2.7] that describes the canonical Jordan form of a tensor product of unipotent blocks, we obtain that $k_{M_i}(x) = p$ and $n_{F_i}(x) \geq \dim M_i^2$. As for p -restricted M , we show that $n_1 \geq 2^{r-m}$ if $G = B_r(K)$ and $\nu = \omega_r$ and conclude that $\dim M_1^2 + \dim M_2^2 > d(r - m)$ in the general case and $\dim M_1^2 > d(r - m)$ in the exceptional cases with $a_r \neq 1$. Now (1) completes the proof. $\square$

Proof of Proposition 2. Let a , x , m , and c be such as in the assertion of the proposition. Assume that $p < ac \leq p + c - 1$. Therefore we have $(a - 1)c \leq p - 1$. Set $M = M(\omega)$ and denote by M_t the weight subspace of weight $t \in \mathbb{Z}$ in the Γ -module M . It is clear that the Weyl group of Γ interchanges M_t and M_{-t} ; hence $\dim M_t = \dim M_{-t}$. Put $e = (a - 1)c$, $V_1 = \bigoplus_{t > e} M_t$, $V_2 = \bigoplus_{t > e} M_{-t}$, and $V = M_e$. Set $f = [(m + 1)/2]$ for $G = A_r(K)$, $f = m$ for $G = B_r(K)$ or $C_r(K)$, and $f = m - 1$ for $G = D_r(K)$. Let $v \in M$ be a nonzero highest weight vector and put $w = X_{-f} \dots X_{-2} X_{-1} v$. By [12, Lemma 2.9], $w \neq 0$. We need a subgroup S which can be defined as follows. Put $I = \{i \mid 1 \leq i \leq r, \delta_i = 0\}$ and $S = \langle \mathcal{X}_i, \mathcal{X}_{-i} \mid i \in I \rangle$. The canonical Jordan forms of x in the standard realizations of G_m and G are well known. We have $|x| = p$ since the dimension of the first realization is at most p due to our assumptions. Taking into account these Jordan forms, one easily obtains the values of δ_i , $1 \leq i \leq r$, and using Lemma 3, deduces the following facts: $I = \{i \mid f + 1 \leq i \leq r - f\}$ for $G = A_r(K)$ and $m = 2f$, $I = \{i \mid f + 1 \leq i \leq r\}$ for $G = B_r(K)$ and $D_r(K)$, and $S = H_2$ in all other cases where H_2 is the subgroup defined in the proof of Theorem 1; $c_x = \sum_{i=1}^f \delta_i = c$, $\tau_x(\omega) = ac$; and $w \in V$. Next, observe that $S \cong A_{r-m}$ for $G = A_r(K)$ and $m = 2f$, $S \cong B_{r-m}(K)$ for $G = B_r(K)$, and $S \cong D_{r-m+1}(K)$ for $G = D_r(K)$. Our construction of the vector w shows that $\mathcal{X}_i$ fixes w if $i \in I$. This forces that w generates an indecomposable S -module M_S with highest weight $\omega_S(w)$. Then one immediately concludes that $M_S \cong M(\omega_1)$. This yields that $\dim M_S = r - m + 1 = d(r - m) + 1$ for $G = A_r(K)$ and $m = 2f$, $\dim M_S = 2(r - m) + 1 = d(r - m) + 1$ for $G = B_r(K)$, $\dim M_S = 2(r - m + 1) = d(r - m) + 2$ for $G = D_r(K)$, and $\dim M_S = d(r - m)$ otherwise. It is clear that $M_S \subset V$. Denote by $\mathbf{X}_f \subset \mathbf{X}(M)$ the subset of weights

of the form $\omega - \sum_{i=1}^f b_i \alpha_i$ and by M_A the irreducible $A_f(K)$ -module with highest weight $a\omega_1$. By Smith's theorem [8], for each $\mu \in \mathbf{X}_f$ the dimension of the weight subspace $M_\mu \subset M$ coincides with that of the weight subspace in M_A whose weight differs from $a\omega_1$ by the same linear combination of the simple roots. Hence $\dim M_\mu$ does not depend upon r . Set $W = \bigoplus_{\mu \in \mathbf{X}_f} M_\mu$. Since M is an irreducible L -module and $p > 2$, observe that M is a linear span of vectors of the form $X_{-i_s} \dots X_{-i_2} X_{-i_1} v$. Now, analyzing the weight structure of M , we conclude that $V_1 \subset W$ and $V = (V \cap W) \oplus M_S$. This implies that $\dim V_1$ ($= \dim V_2$) and $\dim(V \cap W)$ do not depend upon r .

It follows from [10, Lemma 72] that

$$(3) \quad (x-1)^{p-1} M_t \subset \bigoplus_{i \geq t+2p-2} M_i.$$

Let $M_t \not\subset V_2$. Then $t \geq -e$. Obviously, $e < p-1$ if $ac < p-1 + c_x$ and $e = p-1$ for $ac = p-1 + c_x$. Thus (3) implies that

$$(x-1)^{p-1} M_t \subset \bigoplus_{t > p-1} M_t \subset V_1$$

in the first case and

$$(x-1)^{p-1} M_t \subset \bigoplus_{t \geq p-1} M_t \subset V_1 \oplus V$$

in the second case. This forces that $n_M(x) \leq \dim V_2 + \dim V_1 = 2 \dim V_1$ in the first case and $n_M(x) \leq 2 \dim V_1 + \dim(V \cap W) + \dim M_S$ in the second case. We have seen before that $\dim M_S = d(r-m) + u$ with $u = 0, 1$, or 2 . Hence one can take $N_G(a, m, p) = 2 \dim V_1$ and $Q_G(a, m, p) = 2 \dim V_1 + \dim(V \cap W) + u$ to complete the proof. $\square$

Remark 4. For $G = A_r(K)$ or $C_r(K)$ we could give a shorter proof of Proposition 2 using the realization of φ in the a th symmetric power of the standard module (see [7, 1.14 and 8.13]), but we need the proof above for $B_r(K)$ and $D_r(K)$.

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