CONTINUOUS SELECTIONS AND C -SPACES

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ABSTRACT. A characterization of paracompact C -spaces via continuous selections avoiding Z_∞ -sets is given. The result is applied to prove a countable sum theorem for paracompact C -spaces, and to obtain a new partial solution of a question raised by E. Michael.

1. INTRODUCTION

We apply a recent characterization of paracompact C -spaces given by Uspenskij [16] to obtain a selection theorem of Michael's type characterizing this class of spaces. More precisely, the following theorem will be proved.

Theorem 1.1. *For a paracompact space X the following conditions are equivalent:*

- (a) X is a C -space.
- (b) Let Y be a Banach space and let $\varphi : X \rightarrow \mathcal{F}_c(Y)$ be an l.s.c. mapping. Then, for every sequence of set-valued mappings $\psi_n : X \rightarrow \mathcal{F}(Y)$ such that each ψ_n has a closed graph and $\psi_n(x) \cap \varphi(x)$ is a Z_∞ -set in $\varphi(x)$ for every $x \in X$ and $n \in \mathbb{N}$, there exists a single-valued continuous map $f : X \rightarrow Y$ with $f(x) \in \varphi(x) \setminus \bigcup \{\psi_n(x) : n \in \mathbb{N}\}$ for each $x \in X$.
- (c) Let Y be a Banach space and let $\varphi : X \rightarrow \mathcal{F}_c(Y)$ be an l.s.c. mapping. Then, for every closed $F \subset Y$ there exists a single-valued continuous selection for φ avoiding F , provided that $F \cap \varphi(x)$ is a Z_∞ -set in $\varphi(x)$ for each $x \in X$.

Here, 2^Y is the family of all nonempty subsets of Y , $\mathcal{F}(Y)$ that of all closed $F \in 2^Y$, and $\mathcal{F}_c(Y)$ denotes the convex elements of $\mathcal{F}(Y)$ (the same notation will be used when Y is a subset of a vector space). A set-valued mapping $\theta : X \rightarrow 2^Y$ is *lower semi-continuous*, or l.s.c., if $\theta^{-1}(U) = \{x \in X : \theta(x) \cap U \neq \emptyset\}$ is open in X for every open $U \subset Y$, and it has a *closed graph* if $\text{Graph}(\theta) = \{(x, y) \in X \times Y : y \in \theta(x)\}$ is closed in $X \times Y$. A map $f : X \rightarrow Y$ is a *selection* for $\theta : X \rightarrow 2^Y$ if $f(x) \in \theta(x)$ for every $x \in X$, and it *avoids* a set $F \subset Y$ provided $f(x) \notin F$ for every $x \in X$.

We adopt the following definition for a Z_∞ -set: a closed subset F of Y (possibly empty) is said to be a Z_n -set in Y , $n \in \mathbb{N}$, if the set $C(\mathbb{I}^n, Y \setminus F)$ is dense in $C(\mathbb{I}^n, Y)$, and we say that F is a Z_∞ -set in Y provided it is a Z_n -set in Y for all $n \in \mathbb{N}$ (see [2] and [15]). The collection of all Z_∞ -sets (resp., Z_n -sets) in Y will be denoted by $\mathcal{Z}_\infty(Y)$ (resp., $\mathcal{Z}_n(Y)$).

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The C -space property was originally defined by W. Haver [6] for compact metric spaces. Addis and Gresham [1] reformulated Haver's definition for arbitrary spaces: A space X has property C (or X is a C -space) if for any sequence $\{\omega_n : n \in \mathbb{N}\}$ of open covers of X there exists a sequence $\{\gamma_n : n \in \mathbb{N}\}$ of open disjoint families in X such that each γ_n refines ω_n and $\bigcup\{\gamma_n : n \in \mathbb{N}\}$ is a cover of X . Note that, in fact, a C -space X is paracompact if and only if it is countably paracompact and normal. Every finite-dimensional paracompact space, as well as every countable-dimensional metric space, is a C -space [1], but there exists a compact metric C -space which is not countable-dimensional [14].

Our first application of Theorem 1.1 is a countable sum theorem for paracompact C -spaces, i.e. if a paracompact space X is a countable union of its closed C -subspaces, then X also has property C (see Theorem 4.1). Therefore, every strongly countable-dimensional paracompact space (a countable union of closed finite-dimensional subsets) has property C . Theorem 4.1 allows one to prove also that a paracompact space is a C -space provided it has locally property C (see Theorem 4.2). The latter result improves [1, Theorem 2.16] by dropping the assumption on the space under consideration to be hereditarily collectionwise normal. Let us point out here that Theorem 4.1 should be compared with another countable sum theorem [1, Theorem 2.7] which works without the assumption that the sets are closed but for hereditarily collectionwise normal spaces.

Another application of Theorem 1.1 concerns the following Michael's problem [11, Problem 396]: *Let X be a paracompact space, E a Banach space, $Y \subset E$ a convex G_δ -subset, and let $\varphi : X \rightarrow \mathcal{F}_c(Y)$ be l.s.c. Does there exist a single-valued continuous selection for φ ?*

The first author has provided a positive answer to this question if X is either a countable-dimensional metric space [5, Corollary 1.2] or a strongly countable-dimensional paracompact space [5, Corollary 1.3] (actually, these results were proved under the less restrictive assumption that Y is G_δ and φ is l.s.c. closed- and convex-valued in Y). Using Theorem 1.1, we demonstrate that the restrictions on X can be weakened to " X is a paracompact C -space" (see Theorem 4.4).

A few words about the paper. Section 2 is devoted to some preliminary results. The proof of Theorem 1.1 is given in Section 3. The final Section 4 contains applications of Theorem 1.1.

2. PRELIMINARIES

We agree on the following notation. For a metric space (Y, d) , a point $y \in Y$ and a positive number δ , the *open* ball in Y with center y and radius δ is denoted by $B_\delta(y)$. If Y is a normed space, then d stands for the metric generated by the norm of Y .

In what follows, to a pair of set-valued mappings $\varphi : X \rightarrow 2^Y$ and $\psi : X \rightarrow 2^Y$ we associate a set-valued mapping $\Delta_{(\varphi, \psi)} : X \times Y \rightarrow 2^{\mathbb{R}} \cup \{\emptyset\}$ such that, for every $(x, y) \in X \times Y$, the value $\Delta_{(\varphi, \psi)}(x, y)$ consists of all $\delta > 0$ for which there exists a neighbourhood U_δ of x with the property that

$$(1) \quad B_\delta(y) \cap \varphi(z) \neq \emptyset \text{ and } B_\delta(y) \cap \psi(z) = \emptyset, \text{ whenever } z \in U_\delta.$$

Also, we associate another set-valued mapping $\Phi_{(\varphi, \psi)} : X \rightarrow 2^Y \cup \{\emptyset\}$ defined by

$$(2) \quad \Phi_{(\varphi, \psi)}(x) = \{y \in Y : \Delta_{(\varphi, \psi)}(x, y) \neq \emptyset\}, \quad x \in X,$$

and functions $u_{(\varphi,\psi)}, \ell_{(\varphi,\psi)} : \text{Graph}(\Phi_{(\varphi,\psi)}) \rightarrow \mathbb{R}$ defined for $(x, y) \in \text{Graph}(\Phi_{(\varphi,\psi)})$ by

$$(3) \quad u_{(\varphi,\psi)}(x, y) = \inf \Delta_{(\varphi,\psi)}(x, y) \quad \text{and} \quad \ell_{(\varphi,\psi)}(x, y) = \sup \Delta_{(\varphi,\psi)}(x, y).$$

Finally, let us recall that a function $r : X \rightarrow \mathbb{R}$ is *lower semi-continuous* (resp., *upper semi-continuous*), or merely lsc (usc), if $r^{-1}(a, +\infty)$ (resp., $r^{-1}(-\infty, a)$) is open in X for every $a \in \mathbb{R}$.

Lemma 2.1. *Let (Y, d) be a metric space, and let $\varphi : X \rightarrow 2^Y$ and $\psi : X \rightarrow 2^Y$ be set-valued mappings such that φ is l.s.c., ψ is with a closed graph and $\varphi(x) \setminus \psi(x) \neq \emptyset$ for every $x \in X$. Then,*

- (i) $\varphi(x) \setminus \psi(x) \subset \Phi_{(\varphi,\psi)}(x)$ for every $x \in X$;
- (ii) $u_{(\varphi,\psi)}(x, y) < \ell_{(\varphi,\psi)}(x, y)$ for every $(x, y) \in \text{Graph}(\Phi_{(\varphi,\psi)})$;
- (iii) $\text{Graph}(\Phi_{(\varphi,\psi)})$ is open in $X \times Y$;
- (iv) $u_{(\varphi,\psi)}$ is usc, while $\ell_{(\varphi,\psi)}$ is lsc.

Proof. Let $x \in X$ and $y \in \varphi(x) \setminus \psi(x)$. Since ψ has a closed graph, there exists a neighbourhood U of x and a neighbourhood V of y such that $U \times V \cap \text{Graph}(\psi) = \emptyset$. Then, let $\delta > 0$ be such that $B_\delta(y) \subset V$. Finally, set $U_\delta = \varphi^{-1}(B_\delta(y)) \cap U$ which is a neighbourhood of x because φ is l.s.c. It is now clear that, for every $z \in U_\delta$,

$$B_\delta(y) \cap \varphi(z) \neq \emptyset \quad \text{while} \quad B_\delta(y) \cap \psi(z) = \emptyset.$$

Therefore $\delta \in \Delta_{(\varphi,\psi)}(x, y)$, i.e. $y \in \Phi_{(\varphi,\psi)}(x)$ which is (i).

Turning to the proof of (iii) and (iv), let us first observe that, by (1), the statement of (ii) is a consequence of the lower semi-continuity of φ . Indeed, take $(x, y) \in \text{Graph}(\Phi_{(\varphi,\psi)})$. Then, $\Delta_{(\varphi,\psi)}(x, y) \neq \emptyset$, so, by (1), there exists $\delta > 0$ and a neighbourhood U_δ of x with

$$B_\delta(y) \cap \varphi(z) \neq \emptyset \quad \text{and} \quad B_\delta(y) \cap \psi(z) = \emptyset, \quad \text{whenever } z \in U_\delta.$$

Since $B_\delta(y) \cap \varphi(x) \neq \emptyset$, there exists $\nu > 0$ such that $\nu < \delta$ and $B_\nu(y) \cap \varphi(x) \neq \emptyset$. Then, $U_\nu = U_\delta \cap \varphi^{-1}(B_\nu(y))$ is a neighbourhood of x because φ is l.s.c., and clearly

$$B_\nu(y) \cap \varphi(z) \neq \emptyset \quad \text{and} \quad B_\nu(y) \cap \psi(z) = \emptyset, \quad \text{whenever } z \in U_\nu.$$

Therefore, $\nu \in \Delta_{(\varphi,\psi)}(x, y)$ which finally implies that

$$u_{(\varphi,\psi)}(x, y) = \inf \Delta_{(\varphi,\psi)}(x, y) \leq \nu < \delta \leq \sup \Delta_{(\varphi,\psi)}(x, y) = \ell_{(\varphi,\psi)}(x, y).$$

To show now (iii) and (iv), take a point $(x_0, y_0) \in \text{Graph}(\Phi_{(\varphi,\psi)})$ and an $a \in \mathbb{R}$ with $u_{(\varphi,\psi)}(x_0, y_0) < a < \ell_{(\varphi,\psi)}(x_0, y_0)$. Also, take $\delta_u, \delta_\ell \in \Delta_{(\varphi,\psi)}(x_0, y_0)$, with $\delta_u < a < \delta_\ell$, and a neighbourhood U_0 of x_0 such that

$$(4) \quad B_{\delta_u}(y_0) \cap \varphi(z) \neq \emptyset \quad \text{and} \quad B_{\delta_\ell}(y_0) \cap \psi(z) = \emptyset, \quad z \in U_0.$$

Finally, let $\rho = \min\{(\delta_\ell - a)/2, (a - \delta_u)/2\}$. Take points $x, z \in U_0$ and $y \in B_\rho(y_0)$. Then, by (4),

$$\emptyset \neq B_{\delta_u}(y_0) \cap \varphi(z) \subset B_{\delta_u + \rho}(y) \cap \varphi(z),$$

and

$$B_{\delta_\ell - \rho}(y) \cap \psi(z) \subset B_{\delta_\ell}(y_0) \cap \psi(z) = \emptyset.$$

Therefore $\delta_u + \rho, \delta_\ell - \rho \in \Delta_{(\varphi,\psi)}(x, y)$ which implies $U_0 \times B_\rho(y_0) \subset \text{Graph}(\Phi_{(\varphi,\psi)})$ and that $u_{(\varphi,\psi)}(x, y) < a < \ell_{(\varphi,\psi)}(x, y)$ for every $(x, y) \in U_0 \times B_\rho(y_0)$. $\square$

Lemma 2.2. *Let $(Y, \|\cdot\|)$ be a normed space, and let $\varphi : X \rightarrow 2^Y$ and $\psi : X \rightarrow 2^Y$ be set-valued mappings such that φ is l.s.c., ψ is with a closed graph and $\varphi(x) \setminus \psi(x) \neq \emptyset$ for every $x \in X$. Then, for every $x \in X$, we have $[y_0, y_1] \subset \Phi_{(\varphi, \psi)}(x)$ provided $y_1 \in \Phi_{(\varphi, \psi)}(x)$ and $y_0 \in B_{\delta_1}(y_1) \cap \varphi(x)$ for some $\delta_1 \in \Delta_{(\varphi, \psi)}(x, y_1)$.*

Proof. Let $x \in X$, $y_1 \in \Phi_{(\varphi, \psi)}(x)$, $\delta_1 \in \Delta_{(\varphi, \psi)}(x, y_1)$ and $y_0 \in B_{\delta_1}(y_1) \cap \varphi(x)$. Note that, by (i) of Lemma 2.1, $y_0 \in \Phi_{(\varphi, \psi)}(x)$. Then, let $y = (1 - t)y_0 + ty_1$ for some $t \in (0, 1)$. It suffices to show that $\Delta_{(\varphi, \psi)}(x, y) \neq \emptyset$. For the purpose, let us show that $\delta = t\delta_1 \in \Delta_{(\varphi, \psi)}(x, y)$. Towards this end, let U_{δ_1} be as in the definition of “ $\delta_1 \in \Delta_{(\varphi, \psi)}(x, y_1)$ ”, see (1). Next, note that $B_\delta(y) \cap \varphi(x) \neq \emptyset$. Indeed,

$$d(y, y_0) = \|y - y_0\| = t\|y_1 - y_0\| < t\delta_1 = \delta.$$

Then, $U_\delta = \varphi^{-1}(B_\delta(y)) \cap U_{\delta_1}$ is a neighbourhood of x because φ is l.s.c. Thus, to show that $\delta \in \Delta_{(\varphi, \psi)}(x, y)$, it only remains to show that $B_\delta(y) \cap \psi(z) = \emptyset$ for every $z \in U_\delta$. Suppose, if possible, that this fails for some point $z \in U_\delta$. Hence, we can find a point $y_2 \in B_\delta(y) \cap \psi(z)$. Then, $d(y_1, y) = \|y_1 - y\| = (1 - t)\|y_1 - y_0\|$ implies

$$d(y_1, y_2) \leq d(y_1, y) + d(y, y_2) < (1 - t)\delta_1 + \delta = (1 - t)\delta_1 + t\delta_1 = \delta_1.$$

That is, $y_2 \in B_{\delta_1}(y_1) \cap \psi(z)$ which is impossible because $z \in U_{\delta_1}$. $\square$

Lemma 2.3. *Let $F \in \mathcal{Z}_n(A)$ for some $n \in \mathbb{N}$, where A is a closed convex subset of a normed space Y . Also, let $y \in A$ and $H = \overline{A \cap B_\delta(y)}$ for some $\delta > 0$. Then $F \cap H$ is a Z_n -set in H .*

Proof. Let $f \in C(\mathbb{I}^n, H)$. We have to show that, for a given $\varepsilon > 0$, there exists $g \in C(\mathbb{I}^n, H \setminus F)$ with $d(f(x), g(x)) < \varepsilon$ for every $x \in \mathbb{I}^n$. To this end, let $\eta < 1$ be a positive number such that $\eta(\delta + 1) < \varepsilon$. Define $h \in C(\mathbb{I}^n, A \cap B_\delta(y))$ by $h(x) = (1 - \eta)f(x) + \eta y$, $x \in \mathbb{I}^n$. Since $U = A \cap B_\delta(y)$ is open in A , $F \cap U$ is a Z_n -set in U . Therefore, we can find $g \in C(\mathbb{I}^n, U \setminus F)$ such that $d(g(x), h(x)) < \eta$ for every $x \in \mathbb{I}^n$. This g is the required one. $\square$

A space S is called *aspherical* [16] if every continuous map $f : P \rightarrow S$ from a compact polyhedron P can be continuously extended over the cone of P . Note that S is aspherical if and only if any continuous image of an n -sphere, $n \in \mathbb{N}$, in S is contractible in S , i.e. when S is C^n for every $n \in \mathbb{N}$. Finally, we say that a set-valued map $\Phi : X \rightarrow 2^Y$ is *aspherical* if all $\Phi(x)$, $x \in X$, are aspherical subsets of Y .

We conclude this section with the following special case of a result actually proved by Uspenskij [16].

Proposition 2.4 ([16]). *Let X be a paracompact C -space, let Y be a topological space, and let $\Phi : X \rightarrow 2^Y$ be an aspherical mapping with an open graph. Then, Φ has a single-valued continuous selection.*

3. PROOF OF THEOREM 1.1

We shall prove the implications (a) $\Rightarrow$ (b) and (c) $\Rightarrow$ (a). The implication (b) $\Rightarrow$ (c) is trivial.

(a) $\Rightarrow$ (b). First, we show that this implication holds if the sequence $\{\psi_n : n \in \mathbb{N}\}$ is constant.

Proposition 3.1. *Let X be a paracompact C -space and let Y be a Banach space. If $\varphi : X \rightarrow \mathcal{F}_c(Y)$ is l.s.c. and $\psi : X \rightarrow \mathcal{F}(Y)$ is with a closed graph such that $\psi(x) \cap \varphi(x)$ is a Z_∞ -set in $\varphi(x)$ for every $x \in X$, then there exists a continuous $f : X \rightarrow Y$ with $f(x) \in \varphi(x) \setminus \psi(x)$ for every $x \in X$.*

Proof. Note that each $\varphi(x) \setminus \psi(x) \neq \emptyset$ because $\psi(x) \cap \varphi(x)$ is a Z_∞ -set in $\varphi(x)$. Then, let us consider the mapping $\Phi_{(\varphi, \psi)} : X \rightarrow 2^Y$ defined as in (2).

Claim 1. $\Phi_{(\varphi, \psi)}$ is aspherical.

Let $x_0 \in X$, and let $k : P \rightarrow \Phi_{(\varphi, \psi)}(x_0)$ be a continuous map with P a compact polyhedron. Also, let $u_{(\varphi, \psi)}$ and $\ell_{(\varphi, \psi)}$ be defined as in (3). Then, by Lemma 2.1, $u_{(\varphi, \psi)}(x_0, k(p))$, $p \in P$, is a usc function on P , $\ell_{(\varphi, \psi)}(x_0, k(p))$, $p \in P$, is an lsc function on P , and $u_{(\varphi, \psi)}(x_0, k(p)) < \ell_{(\varphi, \psi)}(x_0, k(p))$ for every $p \in P$. Since P is paracompact, by a result of [3] (see, also, [4, 7]), this implies the existence of a continuous function $r : P \rightarrow \mathbb{R}$ such that

$$(5) \quad u_{(\varphi, \psi)}(x_0, k(p)) < r(p) < \ell_{(\varphi, \psi)}(x_0, k(p)) \quad \text{for every } p \in P.$$

An important property of r is that each $B_{r(p)}(k(p)) \cap \varphi(x_0)$, $p \in P$, is non-empty and its closure doesn't meet $\psi(x_0)$. Further, $\theta(p) = \overline{B_{r(p)}(k(p)) \cap \varphi(x_0)}$, $p \in P$, defines an l.s.c. mapping $\theta : P \rightarrow \mathcal{F}_c(Y)$ because r is continuous. Then, according to the Michael's selection theorem [8, Theorem 3.2''], θ has a continuous selection h . Let

$$\text{Cone}(P) = \{tv + (1-t)p : p \in P \text{ and } t \in [0, 1]\}$$

be the cone of P with a vertex v . Then we have $\text{Cone}(P) = H_1 \cup H_2$, where $H_1 = \{tv + (1-t)p : p \in P, t \in [0, 2^{-1}]\}$ and $H_2 = \{tv + (1-t)p : p \in P, t \in [2^{-1}, 1]\}$. Define $k_1 \in C(H_1, Y)$ by $k_1(tv + (1-t)p) = (1-2t)k(p) + 2th(p)$. Note that, for every $p \in P$, (5) implies $h(p) \in B_{\delta(p)}(k(p)) \cap \varphi(x_0)$ for some $\delta(p) \in \Delta_{(\varphi, \psi)}(x_0, k(p))$. Since $k : P \rightarrow \Phi_{(\varphi, \psi)}(x_0)$, it now follows by Lemma 2.2 that $k_1(z) \in \Phi_{(\varphi, \psi)}(x_0)$ for every $z \in H_1$. Next observe that $k_1|_P = k$ while $\varphi(x_0) \setminus \psi(x_0)$ is an AR because $\psi(x_0) \cap \varphi(x_0) \in \mathcal{Z}_\infty(\varphi(x_0))$ (see [16, Proposition 6.4]; this also follows from [15]). Therefore $k_1|_H$ can be extended to a continuous map $k_2 : H_2 \rightarrow \varphi(x_0) \setminus \psi(x_0)$, where $H = H_1 \cap H_2$. Finally, we define a continuous extension $\bar{k} : \text{Cone}(P) \rightarrow \Phi_{(\varphi, \psi)}(x_0)$ of k by $\bar{k}|_{H_1} = k_1$ and $\bar{k}|_{H_2} = k_2$.

We now proceed to the rest of the proof. By Proposition 2.4, $\Phi_{(\varphi, \psi)}$ has a continuous selection $g : X \rightarrow Y$ because, by Claim 1, it is aspherical and, by Lemma 2.1, it has an open graph. Relying once again on Lemma 2.1 and a result of [3, 4, 7], there exists a continuous function $r : X \rightarrow \mathbb{R}$ such that, for every $x \in X$,

$$(6) \quad d(g(x), \varphi(x)) \leq u_{(\varphi, \psi)}(x, g(x)) < r(x) < \ell_{(\varphi, \psi)}(x, g(x)) \leq d(g(x), \psi(x)).$$

Therefore, $\theta(x) = \overline{B_{r(x)}(g(x)) \cap \varphi(x)}$ defines an l.s.c. mapping $\theta : X \rightarrow \mathcal{F}_c(Y)$. Hence, applying again the Michael selection theorem [8, Theorem 3.2''], we get a continuous selection f for θ . According to (6), f is as required. $\square$

We finally accomplish the implication (a) $\Rightarrow$ (b) of Theorem 1.1 with the following proposition.

Proposition 3.2. *Let X be a paracompact C -space, let Y be a Banach space, and let $\varphi : X \rightarrow \mathcal{F}_c(Y)$ be an l.s.c. mapping. Then, for every sequence of mappings*

$\psi_n : X \rightarrow \mathcal{F}(Y)$ such that each ψ_n has a closed graph and $\psi_n(x) \cap \varphi(x)$ is a Z_∞ -set in $\varphi(x)$ for every $x \in X$ and $n \in \mathbb{N}$, there is a single-valued continuous map $f : X \rightarrow Y$ with $f(x) \in \varphi(x) \setminus \bigcup \{\psi_n(x) : n \in \mathbb{N}\}$ for every $x \in X$.

Proof. By Proposition 3.1, φ admits a continuous selection f_1 with $f_1(x) \notin \psi_1(x)$ for each $x \in X$. Then, as in the proof of Proposition 3.1, there exists a continuous function $\delta_1 : X \rightarrow \mathbb{R}$ with

$$u_{(\varphi, \psi_1)}(x, f_1(x)) < \delta_1(x) < \ell_{(\varphi, \psi_1)}(x, f_1(x)), \quad x \in X,$$

i.e., a continuous selection for the set-valued mapping $\Delta_{(\varphi, \psi_1)}(x, f_1(x))$, $x \in X$. Hence, $r_1(x) = \min\{\delta_1(x), 1\}$, $x \in X$, defines a continuous function $r_1 : X \rightarrow (0, 1]$ such that $r_1(x) \leq d(f_1(x), \psi_1(x))$ for every $x \in X$. As in the proof of [5, Lemma 2.1], we shall construct a sequence of continuous selections f_n for φ and a sequence of continuous functions $r_n : X \rightarrow (0, 1]$ such that, for every $x \in X$ and n ,

$$(7) \quad r_n(x) \leq \min\{d(f_k(x), \psi_k(x)) : k \leq n\},$$

$$(8) \quad d(f_{n+1}(x), f_n(x)) \leq 3^{-n}r_n(x).$$

The above conditions will be sufficient. Indeed, by (8), $\{f_n\}$ is a Cauchy sequence and the limit of this sequence defines a continuous map $f : X \rightarrow Y$ which is a selection for φ . By (7) and [12, Lemma 6.1.1], $f(x) \notin \bigcup \{\psi_n(x) : n \in \mathbb{N}\}$.

So, it remains to define these f_n 's and r_n 's. Suppose that $f_1, \dots, f_n$ and $r_1, \dots, r_n$ have been already defined. Consider the l.s.c. mapping $\varphi_{n+1} : X \rightarrow \mathcal{F}_c(Y)$ defined by $\varphi_{n+1}(x) = \overline{B_{3^{-n}r_n(x)}(f_n(x)) \cap \varphi(x)}$, $x \in X$. Applying Lemma 2.3 for any x (with $A = \varphi(x)$, $F = \psi_{n+1}(x) \cap \varphi(x)$, $y = f_n(x)$ and $\delta = 3^{-n}r_n(x)$), we obtain that $\psi_{n+1}(x) \cap \varphi_{n+1}(x)$ is a Z_∞ -set in $\varphi_{n+1}(x)$. Hence, by Proposition 3.1 (applied with the mappings φ_{n+1} and ψ_{n+1}), there exists a continuous selection f_{n+1} for φ_{n+1} such that $f_{n+1}(x) \notin \psi_{n+1}(x)$, $x \in X$. For r_{n+1} , as before, take a continuous selection δ_{n+1} for $\Delta_{(\varphi, \psi_{n+1})}(x, f_{n+1}(x))$, $x \in X$. Then, merely set $r_{n+1}(x) = \min\{\delta_{n+1}(x), r_n(x)\}$, $x \in X$. $\square$

Recall that if (Y, d) is a metric space, X is a space and $f \in C(X, Y)$, then the *fine topology* on $C(X, Y)$ is the topology in which the family of all sets

$$V(f, \alpha) = \{g \in C(X, Y) : d(f(x), g(x)) < \alpha(x) \text{ for each } x \in X\},$$

where α is a positive continuous function on X , is a local base at f (see [13]). The next proposition is an analogue of Lemma 3.2 from [10] and demonstrates that, in Proposition 3.1, there are many selections for φ avoiding ψ . Also, it can be used (instead of Proposition 3.2) to prove the final step of implication (a) $\Rightarrow$ (b) in Theorem 1.1.

Proposition 3.3. *Let X, Y, φ and ψ be as in Proposition 3.1. Also, let $\mathcal{S}$ be the set of all continuous selections for φ , endowed with the fine topology. Then $\mathcal{S}$ is a Baire space and $\mathcal{U} = \{f \in \mathcal{S} : f \text{ avoids } \psi\}$ is open and dense in $\mathcal{S}$.*

Proof. As in the proof of [10, Lemma 3.2], one can show that $\mathcal{S}$ is a Baire space and $\mathcal{U} \subset \mathcal{S}$ is open. It remains to prove that $\mathcal{U} \subset \mathcal{S}$ is dense. So, let $f \in \mathcal{S}$ and $\alpha : X \rightarrow (0, \infty)$ be continuous, and let us show that $\mathcal{U} \cap V(f, \alpha) \neq \emptyset$. Consider the l.s.c. mapping $\theta : X \rightarrow \mathcal{F}_c(Y)$ defined by $\theta(x) = \overline{\varphi(x) \cap B_{2^{-1}\alpha(x)}(f(x))}$. By Lemma 2.3, $\psi(x) \cap \theta(x)$ is a Z_∞ -set in $\theta(x)$ for every $x \in X$. Hence, by Proposition 3.1, θ has a continuous selection g avoiding ψ . Then $g \in \mathcal{U} \cap V(f, \alpha)$. $\square$

(c) $\Rightarrow$ (a). We follow the proof of [16, Theorem 1.3, implication $S_2 \Rightarrow C$]. Take a sequence $\{\gamma_n : n = 1, 2, \dots\}$ of open covers of X . By definition, we must prove that there exists a sequence $\{\mu_n : n = 1, 2, \dots\}$ of disjoint families of open sets in X such that each μ_n refines γ_n and the union $\bigcup\{\mu_n : n = 1, 2, \dots\}$ is a cover of X . To this end, we proceed as in [16] with a few modifications. Namely, first note that, for every n , there exists a metric space (M_n, d_n) and a continuous map $f_n : X \rightarrow M_n$ such that

$$(9) \quad d_n \leq 3 \text{ (i.e., } d_n(z, t) \leq 3 \text{ for all } z, t \in M_n);$$

$$(10) \quad \{f_n^{-1}(B_2(z)) : z \in M_n\} \text{ is an open cover of } X \text{ refining } \gamma_n.$$

Let $M = \bigsqcup\{M_n : n = 1, 2, \dots\}$ be the disjoint union of our spaces M_n while d is the metric on M defined as

$$(11) \quad d|_{M_n \times M_n} = d_n, \text{ and } d(z, t) = 3 \text{ provided } z \in M_n, t \in M_k \text{ and } n \neq k.$$

Thus, in fact, we may regard each f_n as a continuous map from X into the metric space (M, d) . Embed (M, d) isometrically into a Banach space (E, d) , where $E = C(K)$ is the space of all continuous functions on a zero-dimensional compact space K and d is the metric on E generated by the norm $\|\cdot\|$ of E . Next, as in [16], set $A = \{h \in C(K) : h(x) \in \mathbb{Z} \text{ for some } x \in K\}$, where $\mathbb{Z}$ is the set of all integers. Then, A is a closed and nowhere dense subset of E which is the union of hyperplanes in E of the form $\{h \in E : h(x) = n\}$, $x \in K$ and $n \in \mathbb{Z}$. As a result, these hyperplanes provide a partition of E into open cells of diameter ≤ 1 . It now suffices to construct a sequence $\{g_n : n = 1, 2, \dots\}$ of continuous maps $g_n : X \rightarrow E$ such that

$$(12) \quad d(f_n(x), g_n(x)) \leq 1 \text{ for all } x \in X \text{ and } n;$$

$$(13) \quad \text{For every } x \in X \text{ there exists an } n \text{ with } g_n(x) \notin A.$$

Indeed, if λ is the family of all components of $E \setminus A$, then $\mu_n = g_n^{-1}(\lambda)$, $n = 1, 2, \dots$, is a sequence of disjoint families of open subsets of X . By (10) and (12), each μ_n refines γ_n while, by (13), the union $\bigcup\{\mu_n : n = 1, 2, \dots\}$ is a cover of X . That is, $\{\mu_n : n = 1, 2, \dots\}$ is the required one.

Turning to the construction of these g_n 's, for every n , we consider the l.s.c. mapping $\varphi_n : X \rightarrow \mathcal{F}_c(E)$ defined by $\varphi_n(x) = \overline{B_1(f_n(x))}$, $x \in X$. Also, let Z be the linear subspace of E^ω defined as

$$Z = \left\{ (y_n) \in E^\omega : \sum_{n=1}^{\infty} 2^{-n} \|y_n\| < +\infty \right\}.$$

Note that Z is a normed space because $\|(y_n)\|_\omega = \sum_{n=1}^{\infty} 2^{-n} \|y_n\|$ defines a norm on Z , and the topology of $(Z, \|\cdot\|_\omega)$ coincides with the topology of Z as a subspace of E^ω . Let ρ be the metric on Z generated by the norm $\|\cdot\|_\omega$, and let $\varphi : X \rightarrow \mathcal{F}_c(E^\omega)$ be the mapping defined by

$$\varphi(x) = \prod \{\varphi_n(x) : n = 1, 2, \dots\}, \quad x \in X.$$

It is easily seen that φ is l.s.c.

Claim 2. Each $\varphi(x)$ is a ρ -complete subset of Z .

First, note that by (9) and (11), the diameter of M as a subset of (E, d) is finite. Therefore, according to the definition of the mappings φ_n , there exists a closed ball B over the origin 0 of E such that $\varphi_n(x) \subset B$ for all $x \in X$ and $n \in \mathbb{N}$. Then, the above claim follows from the fact that $B^\omega \subset Z$.

We complete the proof as follows. Let Y be the completion of Z . Thus, we get a Banach space and, by Claim 2, an l.s.c. mapping $\varphi : X \rightarrow \mathcal{F}_c(Y)$. Further, let F be the closure in Y of the set $A^\omega \cap Z$. Then $F \cap \varphi(x) = A^\omega \cap \varphi(x)$ for every $x \in X$. Since each $A^\omega \cap \varphi(x)$ is the product $\prod \{A \cap \overline{B_1(f_n(x))} : n \in \mathbb{N}\}$, and $\overline{B_1(f_n(x))} \cap A$ is a proper subset of $\overline{B_1(f_n(x))}$, we finally obtain that each $F \cap \varphi(x)$ is a Z_∞ -set in $\varphi(x)$ (we use the well known fact that, if A_n is a proper closed subset of B_n , then $\prod \{A_n : n \in \mathbb{N}\}$ is a Z_∞ -set in $\prod \{B_n : n \in \mathbb{N}\}$; see for instance [16, Proposition 6.2]). Therefore, according to (c), there exists a continuous selection $g : X \rightarrow Z$ for φ avoiding the set A^ω . Let $g_n : X \rightarrow E$ be such that the corresponding diagonal map $\Delta\{g_n : n = 1, 2, \dots\}$ coincides with g . These g_n 's satisfy both (12) and (13).

4. SOME APPLICATIONS

Theorem 4.1 (Countable sum theorem). *If a paracompact space X is a countable union of closed C -subsets, then X is itself a C -space.*

Proof. Let $X = \bigcup \{X_n : n \in \mathbb{N}\}$ with each $X_n \subset X$ a closed C -space. By Theorem 1.1, it suffices to show that X satisfies condition (c). To this end, let Y , φ and F be as in (c) of Theorem 1.1. To show that φ has a continuous selection avoiding F , we construct by induction a sequence $\{f_n\}$ of continuous selections for φ and open sets $U_n \subset X$ such that

$$(14) \quad \bigcup \{X_i : 1 \leq i \leq n\} \subset U_n, \quad n \in \mathbb{N},$$

$$(15) \quad f_n(x) = f_{n+1}(x) \notin F \quad \text{for all } x \in \overline{U_n} \text{ and } n \in \mathbb{N}.$$

Suppose that f_n and U_n have been already defined. Consider the l.s.c. mapping $\varphi_{n+1} : X_{n+1} \rightarrow \mathcal{F}_c(Y)$ defined by $\varphi_{n+1}(x) = \{f_n(x)\}$ if $x \in X_{n+1} \cap \overline{U_n}$ and $\varphi_{n+1}(x) = \varphi(x)$ otherwise. Since X_{n+1} is a paracompact C -space, by (c) of Theorem 1.1, there is a continuous selection $g_{n+1} : X_{n+1} \rightarrow Y$ of φ_{n+1} avoiding the set F . Define another l.s.c. mapping $\theta_{n+1} : X \rightarrow \mathcal{F}_c(Y)$ by $\theta_{n+1}(x) = \{f_n(x)\}$ if $x \in \overline{U_n}$, $\theta_{n+1}(x) = \{g_{n+1}(x)\}$ if $x \in X_{n+1}$ and $\theta_{n+1}(x) = \varphi(x)$ if $x \notin X_{n+1} \cup \overline{U_n}$. According to the Michael's selection theorem [8, Theorem 3.2''], θ_{n+1} admits a continuous selection f_{n+1} . Obviously, $V = \{x \in X : f_{n+1}(x) \notin F\}$ is an open subset of X containing $X_{n+1} \cup \overline{U_n}$. Choose the open set $U_{n+1} \subset X$ such that $X_{n+1} \cup \overline{U_n} \subset U_{n+1} \subset \overline{U_{n+1}} \subset V$. This completes the construction.

Finally, by (14) and (15), setting $f(x) = f_n(x)$ if $x \in X_n$, $n \in \mathbb{N}$, we obtain a continuous selection f of φ avoiding F . $\square$

Having already established Theorem 4.1, we can prove that property C is a local property. The next theorem was proved in [1, Theorem 2.16] under the additional requirement on X to be hereditarily collectionwise normal. The same proof remains valid in the present situation.

Theorem 4.2. *A paracompact space X has the property C if and only if every point of X has a neighbourhood which has property C .*

To prepare for our next application, we need the following lemma.

Lemma 4.3. *Let E be a Banach space and let $Y \subset E$ be a convex G_δ -subset of E . If $H \subset E$ is closed and $H \cap Y \in \mathcal{Z}_\infty(Y)$, then*

$$H \cap \overline{Y} \in \mathcal{Z}_\infty(\overline{Y}).$$

Proof. It suffices to show that, for a given $\delta > 0$ and $f \in C(K, \overline{Y})$, where K is a finite-dimensional metrizable compactum, there exists $g \in C(K, Y)$ such that $d(f(x), g(x)) \leq \delta$ for each $x \in K$. To this end consider the set-valued mapping $\varphi : K \rightarrow \mathcal{F}_c(Y)$ defined by $\varphi(x) = \overline{Y \cap B_\delta(f(x))}^Y$, $x \in K$. Since f is continuous, φ is l.s.c. Obviously, $\varphi(x)$, $x \in K$, are closed convex subsets of Y . Then, by [5, Corollary 1.2] (or [9, Theorem 1.2 and Example 2.5]), φ admits a continuous selection g . $\square$

We are now ready to prove the following result providing a partial answer to the Michael's problem [11, Problem 396].

Theorem 4.4. *Let X be a paracompact C -space, let E be a Banach space, and let Y be a G_δ -subset of E . Then every l.s.c. mapping $\varphi : X \rightarrow \mathcal{F}_c(Y)$ admits a continuous selection.*

Proof. Let $E \setminus Y = \bigcup \{F_n : n \in \mathbb{N}\}$ with each $F_n \subset E$ closed. Consider the l.s.c. mapping $\overline{\varphi} : X \rightarrow \mathcal{F}_c(E)$ defined by $\overline{\varphi}(x) = \overline{\varphi(x)}$, $x \in X$. Since, by Lemma 4.3, $\overline{\varphi}(x) \cap F_n$ is a $\mathcal{Z}_\infty$ -set in $\overline{\varphi}(x)$ for all $x \in X$ and $n \in \mathbb{N}$, we can apply (b) of Theorem 1.1 to obtain a continuous selection f of $\overline{\varphi}$ avoiding $\bigcup \{F_n : n \in \mathbb{N}\}$. Then f is a selection of φ . $\square$

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