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THE STRONG RADICAL AND FINITE-DIMENSIONAL IDEALS

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Abstract. Let A be a semi-prime Banach algebra with strong radical $\mathfrak{R}$ (intersection of its two-sided modular maximal ideals). A minimal left or right ideal K of A is infinite-dimensional if and only if $K \subset \mathfrak{R}$. Thus all minimal one-sided ideals in A are finite-dimensional if A is strongly semi-simple.

1. Introduction

All ideals considered in this paper are two-sided unless otherwise specified. In his pioneering paper [9, p. 74] Segal called an algebra strongly semi-simple if the intersection $\mathfrak{R}$ of its modular maximal ideals is (0). That intersection $\mathfrak{R}$ is called the strong radical of A. For a discussion of $\mathfrak{R}$ see [8, p. 59] and [7, pp. 490–495].

Let A be a semi-prime algebra over a field Φ. In the special case where A is a Banach algebra Smythe [10] showed that, for $x \in A$, xA is finite-dimensional if and only if Ax is finite-dimensional. We show this to be valid for all A. Let $\mathfrak{F}$ be the set of $x \in A$ where xA and Ax are finite-dimensional. It is shown that $\mathfrak{F}$ is the direct sum of those minimal ideals of A which are finite-dimensional. Then it is shown that $\mathfrak{F}\mathfrak{R} = (0)$ so that xA and Ax are infinite-dimensional for all $x \neq 0$ in $\mathfrak{R}$.

We provide more detail in the case of a semi-prime Banach algebra A. Let $\mathfrak{S}$ denote the socle of A. To say that $\mathfrak{F} = \mathfrak{S}$ is to say that every minimal one-sided ideal in A is finite-dimensional. We find that $\mathfrak{F} = \mathfrak{S}$ if and only if $\mathfrak{R}$ has zero socle. Consequently $\mathfrak{F} = \mathfrak{S}$ if A is strongly semi-simple. More specifically a minimal one-sided ideal K is infinite-dimensional if and only if $K \subset \mathfrak{R}$.

2. On finite-dimensionality

Let A be a semi-prime algebra over a field Φ. We extend a result of Smythe [9] to show that, for $x \in A$, xA is finite-dimensional if and only if Ax is also finite-dimensional.

2.1. Lemma. *Let $W \neq (0)$ be a finite-dimensional right ideal in A. There exists an idempotent $p \in W$ where $pv = v$ for all $v \in W$.*

Proof. We refer to the proof [5, Th. 1.4.2] of the classical theorem that if $K \neq (0)$ is a right ideal in a semi-prime artinian ring R there exists an idempotent p so that $K = pR$. For Lemma 2.1 finite-dimensionality replaces the descending chain condition in that elegant argument. Only a few minor changes in [5, Th. 1.4.2] provide the proof of this lemma.

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2.2. Theorem. *For $x \in A$ we have xA finite-dimensional if and only if Ax is finite-dimensional.*

Proof. Let $p \neq 0$ be an idempotent in A. We show first that pA is finite-dimensional simultaneously with Ap. Suppose that Ap is finite-dimensional. Then so is pAp. For each $x \in A$ we define T_{px} as a mapping of Ap into pA by $T_{px}(yp) = pxyp$. T_{px} is a linear mapping. The set Γ of all linear mappings of Ap into pAp is finite-dimensional. The mapping $px \to T_{px}$ is a linear mapping of pAp into Γ. We show that this mapping is injective so that pA is finite-dimensional.

Suppose that $T_{pv} = T_{pw}$. Then $p(v-w)yp = 0$ for all $y \in A$. Hence $[p(v-w)A]^2 = (0)$. As A is a semi-prime algebra we have $p(v-w)A = (0)$ so that, also, $pv = pw$.

Let $\mathfrak{F}_L$ ($\mathfrak{F}_R$) denote the set of all $x \in A$ for which xA (Ax) is finite-dimensional. These are ideals in A. Let $x \in \mathfrak{F}_L$, $x \neq 0$, and set $V = \{\lambda x : \lambda \in \Phi\}$. Then $K = V + xA$ is a finite-dimensional right ideal. By Lemma 2.1 there is an idempotent $p \neq 0$ so that $K = pA$. Then Ap is also finite-dimensional as is Apx. But $x \in K$ so that $px = x$ and $x \in \mathfrak{F}_R$.

Henceforth we set $\mathfrak{F} = \mathfrak{F}_L = \mathfrak{F}_R$. We say that an ideal K is *unital* if it possesses an identity element e so that $K = eA$. A simple argument shows that e lies in the center of A ([1, Th. 4.4A]). It follows from Lemma 2.1 that any finite-dimensional ideal is unital.

3. On minimal ideals in algebras

Let A be a semi-prime algebra over a field Φ. For a subset K of A let $L(K) = \{x \in A : xK = (0)\}$ and $R(K) = \{x \in A : Kx = (0)\}$. If K is an ideal in A, then $L(K) = R(K)$ by [2, p. 162] and their common value is denoted by K^a.

3.1. Lemma. *Let W be a minimal ideal in A. There exists a unique prime ideal P in A such that $P \cap W = (0)$. Also $P = W^a$.*

Proof. As (0) is the intersection of all the prime ideals in A, then there exists a prime ideal P where $P \not\supset W$. Then $P \cap W = PW = (0)$ and $P \subset W^a$. Now $WW^a \subset P$ so, as P is a prime ideal, $W^a \subset P$. Thus $P = W^a$. This enforces the uniqueness of P.

3.2. Lemma. *If the minimal ideal W contains a non-zero idempotent p, then the prime ideal P of Lemma* 3.1 *is a primitive ideal.*

Proof. As p fails to be in the radical of A, there exists a primitive ideal P which does not contain p.

The idempotent p of Lemma 3.2 has the property that, given an ideal K in A, either $p \in K$ or $p \in K^a$. For if $K \not\supset W$, then $KW = (0)$ and $p \in K^a$. Consider an idempotent e such that eA is a minimal right ideal. Given an ideal K, either $e \in K$ or $e \in K^a$ [12, Lemma 5.1]. It follows that AeA is a minimal ideal of A.

3.3. Lemma. *The prime ideal P of Lemma* 3.1 *is a modular maximal ideal if and only if the minimal ideal W is unital.*

Proof. Suppose that W is unital with identity element q. Then $W = qA$ and q is in the center of A. As $A = qA \oplus (1-q)A$ we have $(1-q)A = W^a$ and W^a is *a* modular maximal ideal.

Suppose that W^a is a modular maximal ideal. Then $W \oplus W^a = A$. As A/W^a has an identity, we see that W is unital.

As in [8] by the *spectrum* Σ of A we mean the set of its modular maximal ideals with the hull-kernel topology. The $M \in \Sigma$ which are of the form W^a for a minimal ideal W are precisely the $M \in \Sigma$ for which $M^a \neq (0)$.

3.4. Lemma. *Any $M \in \Sigma$ for which $M^a \neq (0)$ is an isolated point of Σ. If A is strongly semi-simple $M^a \neq (0)$ for any isolated point M of Σ.*

Proof. Suppose $M_0^a \neq (0)$ for $M_0 \in \Sigma$. Then, by Lemma 3.1, M_0 is the unique prime ideal of A not containing M_0^a. Hence every $M \in \Sigma$, $M \neq M_0$ contains M_0^a. Thus M_0 is not in the closure of $\{M \in \Sigma : M \neq M_0\}$.

Suppose that A is strongly semi-simple and that M_0 is an isolated point of Σ. Let $Z = \{M \in \Sigma : M \neq M_0\}$; we cannot have $Z = (0)$ for otherwise M_0 is not an isolated point. However $M_0 Z = (0)$ so that $M_0^a \neq (0)$.

In case A is not strongly semi-simple one can have an isolated point M_0 of Σ where $M_0^a = (0)$. Consider the algebra A of all bounded linear operators on a separable infinite-dimensional Hilbert space H. As shown by Calkin [3] the sole modular maximal ideal M_0 of A is the set of all compact linear operators on H. Clearly $M_0^a = (0)$. We say that the spectrum Σ of A is *granular* if the set of isolated points of Σ is dense in Σ.

3.5. Theorem. *Let $\mathfrak{Q}$ be the direct sum of the unital minimal ideals of A. A is strongly semi-simple and Σ is granular if and only if $\mathfrak{Q}^a = (0)$.*

Proof. By our earlier remarks $\mathfrak{Q}^a = \cap\{M \in \Sigma : M^a \neq (0)\}$. Suppose $\mathfrak{Q}^a = (0)$. Clearly $\mathfrak{R}$ as the intersection of all $M \in \Sigma$ is (0). By Lemma 3.4 the set of isolated points of Σ is $\{M \in \Sigma : M^a \neq (0)\}$. By the definition of closure in Σ we see that Σ is granular.

Suppose that $R = (0)$ and Σ is granular. Then, by Lemma 3.4, the closure of the set of isolated points of Σ is the set of $M \in \Sigma$ containing $\mathfrak{Q}^a$. As Σ is granular $M \supset \mathfrak{Q}^a$ for all $M \in \Sigma$. As $\mathfrak{R} = (0)$ we have $\mathfrak{Q}^a = (0)$.

We turn to a discussion of $\mathfrak{F}$. It is readily shown [13, Lemma 1] that $\mathfrak{F}$ is the union of all finite-dimensional ideals of A. Each such ideal and therefore $\mathfrak{F}$ is the direct sum of unital finite-dimensional minimal ideals of A. In terms of the set $\mathfrak{Q}$ of Theorem 3.5 we have $\mathfrak{F} \subset \mathfrak{Q}$. As $\mathfrak{Q}^a$ is the intersection of some $M \in \Sigma$ we see that $\mathfrak{Q}^a \supset \mathfrak{R}$.

3.6. Theorem. $\mathfrak{F}\mathfrak{R} = (0)$.

Proof. As $\mathfrak{Q}\mathfrak{Q}^a = (0)$ we have $\mathfrak{F}\mathfrak{R} = (0)$.

3.7. Theorem. *For every $x \neq 0$ in $\mathfrak{R}$ we have xA and Ax infinite-dimensional.*

Proof. Let $x \neq 0$ be in $\mathfrak{R}$. As $(\mathfrak{F} \cap \mathfrak{R})^2 = (0)$ we have $\mathfrak{F} \cap \mathfrak{R} = (0)$. Thus $x \notin \mathfrak{F}$.

As a consequence of Theorem 3.7 every minimal one-sided ideal of A which lies in $\mathfrak{R}$ is infinite-dimensional. In §4 we show, under the additional requirement that A is a Banach algebra, that every such ideal must be contained in $\mathfrak{R}$.

3.8. Theorem. *Let A be a primitive algebra. Then either $\mathfrak{F} = (0)$ or A is finite-dimensional and simple.*

Proof. Suppose that $\mathfrak{F} \neq (0)$. Then $\mathfrak{F}$ contains a unital minimal ideal K with identity v. Then $A = vA \oplus (1-v)A$. By primitivity $(1-v)A = (0)$ so that $A = vA = K$.

For a particular case let A be the algebra of all bounded linear operators on an infinite-dimensional Banach space. Here $\mathfrak{F} = (0)$.

We let $\mathfrak{S}$ denote the socle of A. As pointed out in [13, Lemma 3] we have $\mathfrak{F} \subset \mathfrak{S}$ and $\mathfrak{S} = \mathfrak{F} \oplus (\mathfrak{S} \cap \mathfrak{F}^a)$. If $\mathfrak{F}^a = (0)$, then $\mathfrak{S} = \mathfrak{F}$. Also $\mathfrak{R} = (0)$ by Theorem 3.6. This situation occurs in the case of $L^1(G)$ for a compact group G.

4. On Banach algebras

Henceforth A is a semi-prime Banach algebra. We obtain more detailed information in this case by employing the result [4, Th. 11] that A is finite-dimensional if $A = \mathfrak{S}$.

4.1. Lemma. *Let $M \in \Sigma$. Then $M \not\supset \mathfrak{S}$ if and only if $M = K^a$ where K is a finite-dimensional minimal ideal of A. Also $M \supset \mathfrak{S}$ if and only if $M \supset \mathfrak{F}$.*

Proof. Let $M \in \Sigma$, $M \not\supset \mathfrak{S}$. There exists a minimal idempotent $e \notin M$. By [12, Lemma 5.1] we get $e \in M^a$. Then M^a is a simple Banach algebra equal to its socle. Therefore, by [4, Th. 11], $M^a = K$ is finite-dimensional and $M = K^a$. This argument also shows that if $M \not\supset \mathfrak{S}$, then $M \not\supset \mathfrak{F}$.

Let W be a finite-dimensional minimal ideal. Then W is a unital ideal and also $W^a \in \Sigma$ by Lemma 3.3.

4.2. Theorem. $\mathfrak{F}^a = \mathfrak{R}$ *if and only if* $\Gamma = \{M \in \Sigma : M \not\supset \mathfrak{S}\}$ *is dense in* Σ.

Proof. By Lemma 4.1 we see that $\Gamma = \{M \in \Sigma : M \not\supset \mathfrak{F}\}$. Now $\mathfrak{F}$ is the direct sum of unital minimal finite-dimensional minimal ideals and $K^a \in \Gamma$ for each such minimal ideal by Lemma 3.3. Therefore $\mathfrak{F}^a = \bigcap\{M : M \in \Gamma\}$. Clearly $\mathfrak{F}^a \supset \mathfrak{R}$.

Suppose that $\mathfrak{F}^a = \mathfrak{R}$. Then any $M \in \Sigma$ contains $\bigcap\{M : M \in \Gamma\}$ so that Γ is dense in Σ. Suppose, conversely, that Γ is dense in Σ. Then our formula above for $\mathfrak{F}^a$ shows that every $M \in \Sigma$ contains $\mathfrak{F}^a$ so that $\mathfrak{R} \supset \mathfrak{F}^a$ and $\mathfrak{R} = \mathfrak{F}^a$.

If A is a modular annihilator algebra as defined in [11] or [7, p. 683], then $M \not\supset \mathfrak{S}$ for every $M \in \Sigma$ so that $\mathfrak{F}^a = \mathfrak{R}$ in that case.

4.3. Theorem. $\mathfrak{F}^a \cap \mathfrak{S}$ *is the socle of* $\mathfrak{R}$.

Proof. As noted above $\mathfrak{S} = \mathfrak{F} \oplus (\mathfrak{F}^a \cap \mathfrak{S})$ and $\mathfrak{F} \cap \mathfrak{R} = (0)$. Therefore, as $\mathfrak{R} \cap \mathfrak{S}$ is the socle of $\mathfrak{R}$ by [11, Lemma 3.10], we have $\mathfrak{R} \cap \mathfrak{S} \subset F^a \cap \mathfrak{S}$. Suppose that $\mathfrak{F}^a \cap \mathfrak{S} \not\subset \mathfrak{R}$. Then there exists $M \in \Sigma$ where $\mathfrak{F}^a \cap \mathfrak{S} \not\subset M$. Let p be a minimal idempotent in $\mathfrak{F}^a \cap \mathfrak{S}$ where $p \notin M$. As in Lemma 4.1 we have $p \in \mathfrak{F}$ which is impossible. Then $\mathfrak{R} \cap \mathfrak{S} = \mathfrak{F}^a \cap \mathfrak{S}$.

We see that $\mathfrak{S}$ is the direct sum of two ideals $\mathfrak{F}$ and $\mathfrak{F}^a \cap \mathfrak{S}$ where xA is finite-dimensional for all $x \in \mathfrak{F}$ and xA is infinite-dimensional for all $x \neq 0$ in $\mathfrak{F}^a \cap \mathfrak{S}$.

4.4. Corollary. *A minimal one-sided ideal K in A is infinite-dimensional if and only if $K \subset \mathfrak{R}$.*

Proof. If $K \subset \mathfrak{R}$, then K is infinite-dimensional by Theorem 3.7. Let pA, $p^2 = p$ be an infinite-dimensional right ideal. Then $p \notin F$ so $p \in \mathfrak{F}^a \cap \mathfrak{S}$ by [12, Lemma 5.1]. Therefore $pA \subset \mathfrak{R}$ by Theorem 4.3.

If A is strongly semi-simple, then any minimal one-sided ideal is finite-dimensional. Since Segal's day much progress has been made in determining when $L^1(G)$ is strongly semisimple. If the closure of the subgroup of inner automorphisms is compact in the Braconnier topology, then $\mathfrak{R} = (0)$. Similarly $\mathfrak{R} = (0)$ if all the continuous, topologically irreducible, unitary representations of G are finite-dimensional. These two classes are denoted by $[FIA]^-$ and [Moore]. We refer to the work of T. W. Palmer [6].

Corollary 4.4 shows that $\mathfrak{F} = (0)$ if and only if $\mathfrak{R} \supset \mathfrak{S}$. By [4, Th. 11] any simple infinite-dimensional Banach algebra A has $\mathfrak{F} = (0)$. If that Banach algebra A has no identity, then also Σ is the empty set and $A = \mathfrak{R}$. An example with these properties is an infinite-dimensional simple H^*-algebra [8, p. 275]. Another example with $\mathfrak{F} = (0)$ and Σ empty is the Banach algebra of all compact linear operators on an infinite-dimensional Banach space. Also (see Theorem 4.2) $\mathfrak{F} = (0)$ if and only if Σ is empty in the case of a modular annihilator Banach algebra A.

References

1. E. Artin, C. J. Nesbitt and R. M. Thrall, *Rings with minimum condition*, Univ. of Michigan Press, Ann Arbor, Mich., 1944. MR **6:**33e
2. F. F. Bonsall and J. Duncan, *Complete normed algebras*, Springer, New York, 1973. MR **54:**11013
3. J. W. Calkin, *Two-sided ideals and congruences in the ring of bounded linear operators in Hilbert space*, Ann. of Math. **42** (1941), 839–873. MR **3:**208c
4. L. Dalla, S. Giotopoulos and N. Katseli, *The socle and finite-dimensionality of a semi-prime Banach algebra*, Studia Math. **92** (1989), 201–204. MR **90f:**46079
5. I. N. Herstein, *Noncommutative rings*, Carus Math. Monographs 15, Math. Assoc. America, 1968. MR **37:**2790
6. T. W. Palmer, *Classes of non-abelian, non-compact, locally compact groups*, Rocky Mountain J. Math. 8 (1978), 683–741. MR **81j:**22003
7. T. W. Palmer, *Banach algebras and the general theory of ∗-algebras*, vol. I, Cambridge LL. Press, 1994. MR **95c:**46002
8. C. E. Rickart, *General theory of Banach algebras*, Van Nostrand, Princeton, 1960. MR 22:5903
9. I. E. Segal, *The group algebra of a locally compact group*, Trans. Amer. Math. Soc. **61** (1947), 69–105. MR **8:**438c
10. M. R. F. Smythe, *On problems of Olubummo and Alexander*, Proc. Royal Irish Acad. 80A (1980), 69–74.
11. B. Yood, *Ideals in topological rings*, Can. J. Math. **16** (1964), 28–45. MR **28:**1505
12. ———, *Closed prime ideals in topological rings*, Proc. London Math. Soc. (3) **24** (1972), 307–323. MR **45:**7475
13. ———, *Finite-dimensional ideals in Banach algebras*, Colloq. Math. **63** (1992), 295–301. MR **93m:**46052

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