INTERPOLATION IN NEST ALGEBRA MODULES

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ABSTRACT. Let $\mathcal{A}$ be a nest algebra and $\mathcal{Lat}\mathcal{A}$ its invariant projection (or subspace) lattice. In this paper, using order homomorphisms of $\mathcal{Lat}\mathcal{A}$, we give necessary and sufficient conditions on bounded linear operators X and Y on a Hilbert space to guarantee the existence of an operator A in a certain $\mathcal{A}$ -module such that $AX = Y$.

INTRODUCTION

In the study of a reflexive operator algebra $\mathcal{A}$, it is useful to know for which vectors x and y there exists $A \in \mathcal{A}$ satisfying $Ax = y$. E.C. Lance initiated the study of such problems and gave necessary and sufficient conditions on vectors x and y to guarantee the existence of an operator A in a given nest algebra $\text{Alg}\mathcal{N}$ such that $Ax = y$, where x and y are vectors in a Hilbert space H and $\mathcal{N}$ is a nest. In addition, Lance specified the minimum norm such an operator can have. A. Hopenwasser [4] extended Lance's result to the case where the nest $\mathcal{N}$ is replaced by an arbitrary commutative invariant projection lattice. Munch [8] restricted the operator A to be a Hilbert-Schmidt operator in nest algebra $\text{Alg}\mathcal{N}$. In [1] the authors considered the problem of finding A so that $Ax = y$ and A is required to be in certain ideals contained in $\text{Alg}\mathcal{N}$. All the results mentioned above are in the case of “*vector interpolation*”.

There is another problem related to the one above: for a given operator algebra $\mathcal{A}$ on a Hilbert space H , for which bounded operators X and Y on H does there exist an A in $\mathcal{A}$ satisfying $AX = Y$? In [5] the authors solved this problem in nest algebra and gave a sufficient and necessary condition. We call this case *operator interpolation*. It is clear that the “*operator interpolation*” problem includes the case of the “*vector interpolation*” problem. Indeed, if we denote by $x \otimes m^*$ the rank-one operator defined by the equation $x \otimes m^*(u) = \langle m, u \rangle x$, and if we set $X = x \otimes u^*$ and $Y = y \otimes u^*$, then the equations $AX = Y$ and $Ax = y$ represent the same restriction on A .

In this paper, we consider the operator interpolation problem in the case of a nest algebra module. We follow the suggestion hinted in [5] and use the order

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homomorphisms of the invariant projection lattice to solve the operator interpolation problem and give a sufficient and necessary condition in following Theorem 1, extending the main result in [5].

INTERPOLATION IN NEST ALGEBRA MODULES

Let H be a separable Hilbert space, $B(H)$ the set of all bounded linear operators on H and $\mathcal{A}$ an algebra in $B(H)$.

First, we give some notations and definitions.

A projection $P \in B(H)$ is *invariant* under $\mathcal{A}$ if $AP = PAP$ for any $A \in \mathcal{A}$. A closed subspace S of H is *invariant* under $\mathcal{A}$ if $AS \subseteq S$ for any $A \in \mathcal{A}$. A *nest* $\mathcal{N}$ is a family of closed subspaces of a Hilbert space H totally ordered by inclusion. $\mathcal{N}$ is *complete* if it contains (0) and H and contains the join (closed linear span) and meet (intersection) of every subfamily.

Since there is a one-to-one correspondence between closed subspaces and projections, a complete *nest* is also described as a strongly closed, linearly ordered collection of projections on H , containing 0 and the identity.

A *nest algebra* corresponding a nest $\mathcal{N}$ is the set of all bounded linear operators on H that leave each projection (or subspace) in $\mathcal{N}$ invariant.

If $\mathcal{A}$ is a nest algebra, $\mathcal{Lat}\mathcal{A}$ denotes the invariant projection (or subspace) lattice. In this paper all nests we consider are always complete.

Definition 1. Let $\mathcal{A}$ be a nest algebra. We call ϕ an order homomorphism of $\mathcal{Lat}\mathcal{A}$ into itself if $E \leq F$ implies $\phi(E) \leq \phi(F)$ for any $E, F \in \mathcal{Lat}\mathcal{A}$. Denote

$$\text{Hom } \mathcal{Lat}\mathcal{A} = \{\phi : \phi \text{ is an order homomorphism from } \mathcal{Lat}\mathcal{A} \text{ into } \mathcal{Lat}\mathcal{A}\}$$

and for any $\phi \in \text{Hom}\mathcal{Lat}\mathcal{A}$,

$$\mathcal{U}_\phi = \{G \in B(H) : \phi(E)^\perp GE = 0, \text{ for all } E \in \mathcal{Lat}\mathcal{A}\}, \quad \phi \in \text{Hom}\mathcal{Lat}\mathcal{A}.$$

We know that $\mathcal{U}_\phi$ is a weakly closed $\mathcal{A}$ -module ([6]).

Definition 2 (see [6]). The weakly closed $\mathcal{A}$ -module $\mathcal{U}$ is said to be determined by the order homomorphism ϕ if $\mathcal{U} = \mathcal{U}_\phi$.

We define

$$H[\mathcal{U}] = \{\phi \in \text{Hom}\mathcal{Lat}\mathcal{A} : \phi \text{ determines } \mathcal{U}\}.$$

By [6], we know that for a given weakly closed $\mathcal{A}$ -module $\mathcal{U}$, there is an element η in $H[\mathcal{U}]$ such that $\eta(0) = 0$.

First, we introduce a theorem of Douglas [3] and an extension theorem (in [5]).

Theorem D ([3]). Let Y and X be bounded operators on a Hilbert space H . The following statements are equivalent:

- (1) $r(Y^*) \subseteq r(X^*)$;
- (2) $Y^*Y \leq \lambda^2 X^*X$, for some $\lambda \geq 0$;
- (3) there exists a bounded operator A on H so that $AX = Y$.

Here $r(T)$ means the range of operator T . Moreover if the above conditions are valid, operator A is unique and:

- (a) $\|A\|^2 = \inf\{u : Y^*Y \leq uX^*X\}$;
- (b) $\ker(Y^*) = \ker(A^*)$; and
- (c) $r(A^*) \subseteq r(X)^-$.

Theorem E ([5]). *Let H , K and L be Hilbert spaces, and let N be a subspace of H . Let A and D be operators in $B(N, K)$ and in $B(H, L)$ respectively and suppose that, for every vector x in N , we have $\|Ax\| \leq k\|Dx\|$ for some fixed positive constant k . Then there exists an operator $\tilde{A}$ in $B(H, K)$, such that:*

- a) $\tilde{A}x = Ax$ for every $x \in N$;
- b) $\|\tilde{A}x\| \leq k\|Dx\|$ for every $x \in H$.

Theorem 1. *Let $\mathcal{A}$ be a nest algebra, $\mathcal{U}$ a weakly closed $\mathcal{A}$ -module, and ϕ be in $H[\mathcal{U}]$ satisfying $\phi(0) = 0$. Let X and Y be bounded operators on H . The following are equivalent:*

- (1) *there exists an operator A in $\mathcal{U}$ such that $AX = Y$.*
- (2) $\sup\{\frac{\|(\phi(E))^\perp Yf\|}{\|E^\perp Xf\|} : f \in H, E \in \mathcal{Lat}\mathcal{A}\} = k < \infty$.

[We use the convention $0/0 = 0$, when necessary.] Moreover if case (2) holds, we may choose the operator A so that $\|A\| = k$.

Remark. When ϕ is an identity homomorphism, then Theorem 1 turns into Theorem 2 in [5].

Proof. (1) $\implies$ (2): Suppose there exists the indicated operator A in $\mathcal{U}$, such that $AX = Y$. For any projections E in $\mathcal{Lat}\mathcal{A}$, and for f in H , we have

$$\begin{aligned} \|\phi(E)^\perp Yf\| &= \|\phi(E)^\perp AXf\| = \|\phi(E)^\perp A(I - E + E)Xf\| \\ &\leq \|\phi(E)^\perp AEXf\| + \|\phi(E)^\perp AE^\perp Xf\|. \end{aligned}$$

Since $\phi(E)^\perp AE = 0$, for every E in $\mathcal{Lat}\mathcal{A}$, we have

$$\begin{aligned} \|\phi(E)^\perp Yf\| &\leq \|\phi(E)^\perp AE^\perp Xf\| \leq \|\phi(E)^\perp A\| \cdot \|E^\perp Xf\| \\ &\leq \|A\| \|E^\perp Xf\|. \end{aligned}$$

Therefore $\|A\|$ may play the role of the desired constant k .

(2) $\implies$ (1): Let X and Y be fixed bounded linear operators. We first consider a slightly simpler case. We show by induction that for an arbitrary finite subnest, $\mathcal{N}_{[n]} = \{0, E_1, E_2, \dots, E_n, I\} \subseteq \mathcal{Lat}\mathcal{A}$, under the assumption

$$\sup\{\frac{\|(\phi(E))^\perp Yf\|}{\|E^\perp Xf\|} : f \in H, E \in \mathcal{N}_{[n]}\} = k < \infty,$$

there exists an operator $A_{[n]} \in \mathcal{U}_{\phi[n]}$ satisfying $A_{[n]}X = Y$, where

$$\mathcal{U}_{\phi[n]} = \{G \in B(H) : \phi(E)^\perp GE = 0, E \in \mathcal{N}_{[n]}\}.$$

First step. If there are no projections in the subnest other than 0 and I , by assumption (2):

- if $E = I$, we have $\|\phi(I)^\perp Yf\| \leq \|I^\perp Xf\| = 0$; hence $Y = \phi(I)Y$;
- if $E = 0$, $r(Y^*) \subseteq r(X^*)$, we have

$$\|\phi(0)^\perp Yf\| \leq k\|Xf\| \implies \|Yf\|^2 \leq k^2\|Xf\|^2.$$

It is apparent that $Y^*Y \leq k^2X^*X$. By Theorem D, we have $r(Y^*) \subseteq r(X^*)$. By Douglas' theorem, we obtain an operator B such that $BX = Y$. Set $A_{[0]} = \phi(I)B$; obviously, $A_{[0]}$ belongs to

$$\mathcal{U}_{[0]} = \{G \in B(H) : \phi(E)^\perp GE = 0, E = 0, E = I\},$$

and $A_{[0]}X = \phi(I)BX = \phi(I)Y = Y$. The fact that $\|A_{[0]}\|$ can be taken equal to k is a consequence of condition (a) of Theorem D.

Second step. Now, suppose that the statement is true for arbitrary finite subnests of $\mathcal{Lat}\mathcal{A}(=\mathcal{N})$ with no more than $n-1$ nontrivial projections.

Let $\{0, E_1, E_2, \dots, E_n, I\} = \mathcal{N}_{[n]}$ be a finite subnest (of $\mathcal{Lat}\mathcal{A}$) with n nontrivial projections; correspondingly, there exists $\mathcal{U}_{\phi[n]}$.

Denote

$$X_1 = E_1^\perp X \quad \text{and} \quad Y_1 = \phi(E_1)^\perp Y.$$

We have for each $j = 2, 3, \dots, n$,

$$X_1^* E_j^\perp = (E_1^\perp X)^* E_j^\perp = X^* E_1^\perp E_j^\perp = X^* E_j^\perp$$

and

$$Y_1^* \phi(E_j)^\perp = (\phi(E_1)^\perp Y)^* \phi(E_j)^\perp = Y^* \phi(E_1)^\perp \phi(E_j)^\perp = Y^* \phi(E_j)^\perp.$$

In the last equality, we use $\phi(E_1) \leq \phi(E_j)$, since ϕ is an order homomorphism.

Consequently for each $j = 2, 3, \dots, n$,

$$\begin{aligned} \|\phi(E_j)^\perp Y_1 f\| &= \|\phi(E_1)^\perp \phi(E_j)^\perp Y_1 f\| \\ &= \|\phi(E_j)^\perp \phi(E_1)^\perp Y_1 f\| \\ &= \|\phi(E_j)^\perp Y f\| \\ &\leq k \|E_j^\perp X f\| = k \|E_j^\perp E_1^\perp X f\| \\ &= k \|E_j^\perp X_1 f\| \quad [\text{by (2)}]. \end{aligned}$$

Thus, it follows that assumption (2) holds for the operators X_1 and Y_1 with respect to the subnest $\mathcal{N}'_{[n]} = \{0, E_2, \dots, E_n, I\}$. Using the inductive hypothesis, there exists an operator A' in $\mathcal{U}'_{\phi[n]}$ such that $\|A'\| \leq k$, and $A'X_1 = Y_1$, where

$$\mathcal{U}'_{\phi[n]} = \{G \in B(H) : \phi(E)^\perp G E = 0, E \in \mathcal{N}'_{[n]}\}.$$

Since $\phi(E_1)^\perp Y = (\phi(E_1)^\perp)^2 Y = \phi(E_1)^\perp Y_1$, we have

$$\phi(E_1)^\perp Y = Y_1 = A'X_1 = A'E_1^\perp X = \phi(E_1)^\perp A'E_1^\perp X.$$

Set $B = \phi(E_1)^\perp A'E_1^\perp$. Since the projections in $\mathcal{N}_{[n]}$ all commute with each other, it is clear that B is in

$$\mathcal{U}_{\phi[n]} = \{G \in B(H) : \phi(E)^\perp G E = 0, E \in \mathcal{N}_{[n]}\}.$$

Indeed, (i) if $E = E_1$, then

$$\phi(E_1)^\perp B E = \phi(E_1)^\perp A'E_1^\perp E_1 = 0.$$

Also (ii) if $E = E_j$ for each $j = 2, 3, \dots, n$, then

$$\begin{aligned} \phi(E_j)^\perp B E_j &= \phi(E_j)^\perp \phi(E_1)^\perp A'E_1^\perp E_j \\ &= \phi(E_1)^\perp \cdot (\phi(E_j)^\perp A'E_j) E_1^\perp = 0. \end{aligned}$$

From above, B is in $\mathcal{U}_{\phi[n]}$. Moreover,

$$BX = \phi(E_1)^\perp A'E_1^\perp X = \phi(E_1)^\perp A'X_1 = \phi(E_1)^\perp Y$$

and $\|B\| = \|\phi(E_1)^\perp A'E_1^\perp\| \leq \|A'\| \leq k$.

Next: $\mathbf{r}(Y^*\phi(E_1)) \subseteq \mathbf{r}(Y^*) \subseteq \mathbf{r}(X^*)$ follows from choosing $E = 0$ in part (2). (Since $\phi(0) = 0^\perp = I$.) Theorem D then asserts the existence of an operator A_0 for which $A_0X = \phi(E_1)Y$. For any vector f , we have

$$\begin{aligned}\|A_0Xf\|^2 + \|BXf\|^2 &= \|\phi(E_1)Yf\|^2 + \|\phi(E_1)^\perp Yf\|^2 \\ &= \|Yf\|^2 \leq k^2\|Xf\|^2;\end{aligned}$$

therefore

$$\begin{aligned}\|A_0Xf\|^2 &\leq k^2\|Xf\|^2 - \|BXf\|^2 = k^2\langle Xf, Xf \rangle - \langle BXf, BXf \rangle \\ &= \langle k^2IXf, Xf \rangle - \langle B^*BXf, Xf \rangle \\ &= \langle (k^2I - B^*B)Xf, Xf \rangle \\ &\leq \|(k^2I - B^*B)^{1/2}Xf\|^2.\end{aligned}$$

So we have the form

$$\|A_0g\|^2 \leq \|Dg\|^2, \quad g \in \mathbf{r}(X)^- \text{ and } D = (k^2I - B^*B)^{1/2}.$$

Applying Theorem E with $N = \mathbf{r}[X]^-$, $K = \mathbf{r}[\phi(E_1)]$ and $L = H$, $A_0 \in B(N, K)$, $D \in B(H, H)$, we see that there is an operator $\tilde{A}$ in $B(H, \mathbf{r}[\phi(E_1)])$ such that $\|\tilde{A}x\| \leq \|Dx\|$ for all x in H , and such that $\tilde{A}Xf = A_0Xf$ for all f in H . Finally set $A_{[n]} = \tilde{A} + B$. It is the $A_{[n]}$ that we are looking for.

First: we have $A_{[n]} \in \mathcal{U}_{\phi[n]}$. We only need to check that $\tilde{A}$ is in $\mathcal{U}_{\phi[n]}$.

For (i) if $E = E_1$, since $\tilde{A}E_1H \subseteq \mathbf{r}[\phi(E_1)]$, then $\phi(E_1)^\perp \tilde{A}E_1 = 0$.

For (ii) if $E = E_j, j = 2, 3, \dots, n$, then we have

$$\begin{aligned}\phi(E_i)^\perp \tilde{A}E_i &= \phi(E_1)^\perp \phi(E_i)^\perp \tilde{A}E_i \quad (\text{since } E_i > E_1 \text{ so } \phi(E_i)^\perp \leq \phi(E_1)^\perp) \\ &= \phi(E_i)^\perp \phi(E_1)^\perp \tilde{A}E_i.\end{aligned}$$

Since $\mathbf{r}[\tilde{A}E_i] \subseteq \mathbf{r}[\phi(E_1)]$, we have $\phi(E_1)\tilde{A}E_i = 0$ and $\phi(E_i)^\perp \tilde{A}E_i = 0$, from which we have $\tilde{A}$ in $\mathcal{U}_{\phi[n]}$.

Second: we have

$$\begin{aligned}A_{[n]}X &= \tilde{A}X + BX = A_0X + BX \\ &= \phi(E_1)^\perp Y + \phi(E_1)Y = Y.\end{aligned}$$

Third: the facts $\mathbf{r}[\tilde{A}] \subseteq \mathbf{r}[\phi(E_1)]$ and $\mathbf{r}[B] \subseteq \mathbf{r}[\phi(E_1)^\perp]$ mean that for any vector x ,

$$\begin{aligned}\|A_{[n]}x\|^2 &= \|\tilde{A}x + Bx\|^2 = \|\tilde{A}x\|^2 + \|Bx\|^2 \leq \|Dx\|^2 + \|Bx\|^2 \\ &= \langle (k^2I - B^*B)^{1/2}x, (k^2I - B^*B)^{1/2}x \rangle + \|Bx\|^2 \\ &= \langle (k^2I - B^*B)x, x \rangle + \|Bx\|^2 \\ &= k^2\|x\|^2 - \|Bx\|^2 + \|Bx\|^2 = k^2\|x\|^2,\end{aligned}$$

so

$$(*) \quad \|A_{[n]}\| \leq k.$$

This completes the proof for any subnest of $\mathcal{Lat}\mathcal{A}$ with n nontrivial projections and by induction for all finite subnests.

The remainder of the proof is the same as [5].

Consider a maximal chain $\mathcal{M}$ of finite subnests of $\mathcal{Lat}\mathcal{A}$, ordered by inclusion. The net $\{A_{\mathcal{F}} \in \mathcal{U}_{\phi(\mathcal{F})} : \mathcal{F} \in \mathcal{M}\}$ is bounded (by $(*)$) and will therefore have a weak

limit point, say A . Clearly, $AX = Y$. Furthermore, because, for any projection E in $\mathcal{Lat}A$, E will eventually lie in some finite subnest in the chain $\mathcal{M}$, we have $A \in \mathcal{U}$. This completes the proof that (2) implies (1).

Example. For any ϕ in $H[u]$, ϕ satisfies condition (2) of the above theorem but $\phi(0) \neq 0$. There need not exist an A in $\mathcal{U}$ such that $AX = Y$. For example, let $\mathcal{N}$ be a nest given by $\mathcal{N} = \{0, E_1, E_2, \dots, E_n, I\}$ and $n > 1$. Set $\mathcal{A} = \text{Alg}\mathcal{N}$; by [2] we know that $\mathcal{Lat}\mathcal{A} = \mathcal{N}$. Let $\phi(E) = E_1$ for all E in $\mathcal{N}$, $Y = E_1$ and $X = 0$. We have

$$\sup\left\{\frac{\|\phi(E)^\perp Y f\|}{\|E^\perp X f\|}; E \in \mathcal{N}, f \in H\right\} = 0;$$

therefore, ϕ satisfies condition (2), but for any G in $B(H)$, $GX = 0 \neq E_1$, $GX \neq Y$.

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