ON THE MAXIMAL INEQUALITIES FOR MARTINGALES INVOLVING TWO FUNCTIONS

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ABSTRACT. Let $\Phi(t)$ and $\Psi(t)$ be nonnegative convex functions, and let φ and ψ be the right continuous derivatives of Φ and Ψ , respectively. In this paper, we prove the equivalence of the following three conditions: (i) $\|f^*\|_\Phi \leq c\|f\|_\Psi$, (ii) $L^\Psi \subseteq H^\Phi$ and (iii) $\int_{s_0}^t \frac{\varphi(s)}{s} ds \leq c\psi(ct)$, $\forall t > s_0$, where L^Ψ and H^Φ are the Orlicz martingale spaces. As a corollary, we get a sufficient and necessary condition under which the extension of Doob's inequality holds. We also discuss the converse inequalities.

1. INTRODUCTION

Let Φ be a nonnegative convex continuous function on $[0, \infty)$ with $\Phi(0) = 0$ and $\lim_{t \rightarrow \infty} \Phi(t) = \infty$, and $(\Omega, \mathcal{F}, \mu)$ be a complete probability space. We denote by $\mathcal{M}$ the set of all $\mathcal{F}$ -measurable functions and put

$$L^\Phi(\Omega) = \{f \in \mathcal{M}, \exists \epsilon > 0, E\Phi(\epsilon |f|) < \infty\}$$

where E stands for the expectation with respect to μ . $L^\Phi(\Omega)$ is the so-called Orlicz space (see Rao [11]). Define $\|f\|_\Phi = \inf\{k > 0, E\Phi(\frac{|f|}{k}) \leq 1\}$. Let $\mathcal{F}_n$ ($n \geq 1$) be a nondecreasing sequence of complete sub- σ -fields and $\mathcal{F} = \bigvee_{n=0}^\infty \mathcal{F}_n$. Denote the maximal function and the Φ -norm of a submartingale $f = (f_n)_{n \geq 0}$ adapted to $\mathcal{F}_n$ ($n \geq 1$) respectively by

$$f^*(\omega) = \sup_{n \geq 0} |f_n(\omega)|, \quad \|f\|_\Phi = \sup_{n \geq 0} \|f_n\|_\Phi.$$

It is well known that if Φ is a strictly convex function on $[0, \infty)$, i.e. $q_\Phi = \inf_{t>0} \frac{t\varphi(t)}{\Phi(t)} > 1$ (where φ is the right continuous derivative of Φ), the extension of the classical Doob's inequality

$$\|f^*\|_\Phi \leq c \sup_{n \geq 0} \|f_n\|_\Phi$$

holds for every martingale or nonnegative submartingale $f = (f_n)_{n \geq 0}$; here c is a constant depending only on Φ . When Φ is not strictly convex, the situation

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is rather different. To see this, recall Doob's inequality in the case of $p = 1$ and Imkeller's inequality [4]

$$Ef^* \leq \frac{e}{e-1} (1 + \sup_{n \geq 0} E|f_n| \log^+ |f_n|), \quad \|f^*\|_1 \leq c \sup_{n \geq 0} \|f_n\|_\Psi,$$

where $\Psi = t \log^+ t$, which hold for every martingale or nonnegative submartingale f . It means that $f^* \in L^1$ if $f \in L \log^+ L$.

In [5]–[7], Kita considered some conditions about two such functions Φ and Ψ , for which the inequality

$$(1) \quad \int_{R^n} \Phi(f^*(x)) dx \leq c \int_{R^n} \Psi(c|f(x)|) dx$$

holds for every function $f \in L^\Psi$ defined in R^n , where f^* is the Hardy-Littlewood maximal function of f . In particular, he observed that condition (1) holds for every function $f \in L^\Psi$ if and only if $\int_0^t \frac{\varphi(s)}{s} ds \leq c\psi(ct)$, $\forall t > 0$, where φ, ψ stand for the right continuous derivatives of Φ and Ψ , respectively and $c > 0$ is some constant. He also considered the condition for which the converse inequality holds. Moreover, the authors noticed that D. Gilat [2] had obtained a sharp inequality that gives a comparison of the L_p norm of a martingale to the L_1 norm of its maximal function.

In this paper, we consider the Orlicz spaces of martingales and prove the equivalence of the following three conditions: (i) $\|f^*\|_\Phi \leq c \|f\|_\Psi$ for every martingale or nonnegative submartingale f , (ii) $L^\Psi \subseteq H^\Phi$, (iii) $\int_{s_0}^t \frac{\varphi(s)}{s} ds \leq c\psi(ct)$, $\forall t > s_0$, where s_0, c are some positive constants and the Orlicz martingale spaces L^Φ and H^Φ are defined as

$$L^\Phi = \{f = (f_n)_{n \geq 0} : \|f\|_\Phi < \infty\},$$

$$H^\Phi = \{f = (f_n)_{n \geq 0} : \|f\|_{H^\Phi} = \|f^*\|_\Phi < \infty\}.$$

As a corollary, a sufficient and necessary condition under which the extension of Doob's inequality holds is obtained. We also consider the converse inequalities for the regular martingales.

In this paper $|A|$ always means the measure of set A with respect to μ .

2. THE MAXIMAL INEQUALITIES

Lemma 1. *Let $f = (f_n)_{n \geq 0}$ be a nonnegative submartingale. Then*

$$(2) \quad |\{f_n^* > t\}| \leq \frac{2}{t} \int_{\frac{t}{2}}^\infty |\{f_n > \lambda\}| d\lambda, \quad \forall t > 0, \quad n \in N,$$

and

$$(3) \quad \int_{\frac{t}{2}}^\infty |\{f_n > \lambda\}| d\lambda \leq \int_{\{f_n > \frac{t}{2}\}} f_n d\mu.$$

Proof. For $t > 0$, let $h_n = (f_n \vee \frac{t}{2}) - \frac{t}{2}$ for every $n \in N$. It is easy to see that $h = (h_n)_{n \geq 0}$ is a nonnegative submartingale. By using Kolmogorof's inequality and Fubini's theorem we have

$$\begin{aligned} |\{f_n^* > t\}| &= |\{h_n^* > \frac{t}{2}\}| \leq \frac{2}{t} \int_\Omega h_n d\mu = \frac{2}{t} \int_\Omega (f_n \vee \frac{t}{2}) d\mu - 1 \\ &= \frac{2}{t} \int_0^\infty |\{f_n \vee \frac{t}{2} > \lambda\}| d\lambda - 1 = \frac{2}{t} \int_{\frac{t}{2}}^\infty |\{f_n > \lambda\}| d\lambda \end{aligned}$$

which implies (2). From this equation we get

$$\frac{2}{t} \int_{\frac{t}{2}}^{\infty} |\{f_n > \lambda\}| d\lambda = \frac{2}{t} \int_{\Omega} (f_n \vee \frac{t}{2}) d\mu - 1 \leq \frac{2}{t} \int_{\{f_n > \frac{t}{2}\}} f_n d\mu$$

which implies (3).

Theorem 1. Suppose that Φ, Ψ are nondecreasing convex functions defined on $[0, \infty)$ with $\Phi(0) = \Psi(0) = 0$. Then the following statements are equivalent:

(i) There exist $s_0, c_1 > 0$ such that

$$(4) \quad \int_{s_0}^t \frac{\varphi(s)}{s} ds \leq c_1 \psi(c_1 t), \quad \forall t > s_0.$$

(ii) There exists $c_4 > 0$ such that

$$(5) \quad \|f^*\|_{\Phi} \leq c_4 \|f\|_{\Psi}$$

for every martingale or nonnegative submartingale $f = (f_n)_{n \geq 0}$.

(iii) $L^{\Psi} \subseteq H^{\Phi}$.

Proof. (i) $\implies$ (ii). We only prove (5) for nonnegative submartingales. By using Lemma 1 and Fubini's theorem we get

$$\begin{aligned} (6) \quad E\Phi(f_n^*) &= \int_0^{\infty} |\{f_n^* > t\}| d\Phi(t) \\ &\leq \int_{s_0}^{\infty} |\{2f_n^* > 2t\}| d\Phi(t) + \Phi(s_0) \\ &\leq \int_{s_0}^{\infty} \frac{1}{t} \int_t^{\infty} |\{2f_n > \lambda\}| d\lambda d\Phi(t) + \Phi(s_0) \\ &= \int_{s_0}^{\infty} |\{2f_n > \lambda\}| \int_{s_0}^{\lambda} \frac{1}{t} d\Phi(t) d\lambda + \Phi(s_0) \\ &\leq E\Psi(2c_1 f_n) + \Phi(s_0). \end{aligned}$$

Without loss of generality, we assume that $\|2c_1 f\|_{\Psi} = 1$. Then $E\Phi(f^*) \leq 1 + \Phi(s_0)$ and $E\Phi(\frac{f^*}{1+\Phi(s_0)}) \leq 1$. Hence

$$\|f^*\|_{\Phi} \leq 1 + \Phi(s_0) \leq c_4 \|f\|_{\Psi},$$

where $c_4 = 2(1 + \Phi(s_0))c_1$, as desired.

(ii) $\implies$ (iii) is obvious.

(iii) $\implies$ (i). First we prove that (5) holds for every nonnegative martingale $f = (f_n)_{n \geq 0}$ with $\|f\|_{\Psi} = 1$. Assume it is not the case. Then for every n , choose a nonnegative martingale $f^{(n)} = (f_{nk})_{k \geq 0}$ defined on $(\Omega_n, \mathcal{F}_n, \mathcal{F}_{nk}, \mu_n)$ satisfying $\|f^{(n)}\|_{\Psi} = 1$, $\|f^{(n)*}\|_{\Phi} > 4^n$. Consider the product space $(\Omega, \mathcal{F}, \mathcal{F}_k, \mu) = \prod_{n=1}^{\infty} (\Omega_n, \mathcal{F}_n, \mathcal{F}_{nk}, \mu_n)$, where $\mathcal{F}_k = \prod_{n=1}^{\infty} \mathcal{F}_{nk}$ and take $h^{(n)} = (h_{nk})_{k \geq 1}$, where $h_{nk} = \chi_{\Omega_1} \times \cdots \times \chi_{\Omega_{n-1}} \times f_{nk} \times \chi_{\Omega_{n+1}} \times \cdots$. It is easy to see that $h^{(n)}$ is a martingale with $\|h^{(n)}\|_{\Psi} = 1$ and $\|h^{(n)*}\|_{\Phi} > 4^n$. Let $g_k(\omega, t) = \sum_{n=1}^{\infty} \frac{h_{nk}}{2^n}$. Then $\|g\|_{\Psi} \leq \sum_{n=1}^{\infty} \|\frac{h_{nk}}{2^n}\|_{\Psi} = 1$ and $\|g^*\|_{\Phi} \geq \|\frac{h^{(n)*}}{4^n}\|_{\Phi} > 1, \forall n > 0$. This contradicts the fact that $g \in L^{\Psi} \subseteq H^{\Phi}$.

To prove (i), let $A_k = (0, \frac{1}{2^k}]$, $\mathcal{F}_k = \sigma\{A_0, A_1, A_2, \dots, A_k\}$, $f = t\chi_{A_n}$, $f = (E(f | \mathcal{F}_k))_{k \geq 0}$, where $k, n \geq 0, t = \Psi^{-1}(2^n)$. Then f is a finite dyadic martingale

on $(0, 1]$ with $\frac{t}{2^n} \leq f^*(\omega) \leq t$. It is clear that $\|f\|_\Psi = 1$ and then $\|f^*\|_\Phi \leq c_4 \|f\|_\Psi = c_4$. Thus $E\Phi(\frac{f^*}{c_4}) \leq 1$, i.e.

$$(7) \quad \int_0^\infty |\{f^* > c_4 s\}| \varphi(s) ds \leq 1.$$

Notice that $\frac{1}{s} \leq \frac{c_4 2^{n-k}}{t}$ and $|\{f^* > c_4 s\}| = \frac{1}{2^{k+1}}$ when $s \in (\frac{t}{c_4 2^{n-k}}, \frac{t}{c_4 2^{n-k-1}}]$ ($0 \leq k \leq n-1$). We have

$$\frac{1}{s} \leq \frac{c_4 2^{n+1}}{t} \cdot \frac{1}{2^{k+1}} = \frac{c_4 2^{n+1}}{t} \cdot |\{f^* > c_4 s\}|, \quad \forall s \in (\frac{t}{c_4 2^n}, \frac{t}{c_4}].$$

From (7) we get

$$\int_{\frac{t}{c_4 2^n}}^{\frac{t}{c_4}} \frac{\varphi(s)}{s} ds \leq \frac{c_4 2^{n+1}}{t} \int_0^\infty |\{f^* > c_4 s\}| \varphi(s) ds \leq \frac{c_4 2^{n+1}}{t} \leq \frac{2c_4 \Psi(t)}{t} \leq 2c_4 \psi(t).$$

By the convexity of Ψ , we have $\frac{t}{c_4 2^n} \leq \frac{\Psi^{-1}(1)}{c_4} = s_0$, $\forall n \geq 0$, and then $\int_{s_0}^{\frac{t}{c_4}} \frac{\varphi(s)}{s} ds \leq 2c_4 \psi(t)$,

$$\begin{aligned} \int_{s_0}^u \frac{\varphi(s)}{s} ds &\leq 2c_4 \psi(\Psi^{-1}(2^{n+1})) \\ &\leq 2c_4 \psi(2c_4 u), \quad \forall u \in (\frac{1}{c_4} \Psi^{-1}(2^n), \frac{1}{c_4} \Psi^{-1}(2^{n+1})], \quad n \geq 0. \end{aligned}$$

That is,

$$\int_{s_0}^t \frac{\varphi(s)}{s} ds \leq c_1 \psi(c_1 t), \quad \forall t \geq s_0,$$

where $c_1 = 2c_4$. This completes the proof.

Letting $\Phi = \Psi$ in Theorem 1, we get

Corollary 1. *Suppose that Φ is a nondecreasing convex function defined on $[0, \infty)$ with $\Phi(0) = 0$. Then*

$$\|f^*\|_\Phi \leq c \sup_n \|f_n\|_\Phi$$

holds for every martingale or nonnegative submartingale $f = (f_n)_{n \geq 0}$ if and only if there exist $s_0, c_1 > 0$ such that

$$(8) \quad \int_{s_0}^t \frac{\varphi(s)}{s} ds \leq c_1 \varphi(c_1 t), \quad \forall t > s_0.$$

We will prove in Section 3 that (8) is equivalent to the condition $\lim_{t \rightarrow \infty} \frac{t\varphi(t)}{\Phi(t)} > 1$ when $\Phi \in \Delta_2$ (i.e. there exist $c, t_0 > 0$ such that $\Phi(2t) \leq c\Phi(t), \forall t > t_0$).

Stein [12] proved that if f is a function supported in a ball $B \subseteq R^n$, then $f^* \in L^1(B)$ implies $f \in L \log^+ L$. Kita [7] proved some similar results on the Orlicz function spaces. The following theorem shows that the situation for martingales is different, i.e. for martingale space, it can happen that $H_+^1 \not\subseteq L \log^+ L$.

Theorem 2. *Suppose that Φ, Ψ are nondecreasing convex functions defined on $[0, \infty)$ with $\Phi(0) = \Psi(0) = 0$. Then the following statements are equivalent:*

- (i) *There exists $c > 0$ such that $\lim_{t \rightarrow \infty} \frac{\Phi(ct)}{\Psi(t)} > 0$.*
- (ii) $H^\Phi \subseteq L^\Psi$.

(iii) $H_+^\Phi \subseteq L_+^\Psi$, where H_+^Φ and L_+^Ψ are the sets of all positive members of H^Φ , L^Ψ , respectively.

Proof. (i) $\implies$ (ii). The condition $\lim_{t \rightarrow \infty} \frac{\Phi(ct)}{\Psi(t)} > 0$ means that there exist $c_1 > 0$ and $n_0 \in N$ such that $\Phi(c_1 t) > \Psi(t)$, $\forall t > n_0$. For $f \in H^\Phi$, there exists $k > 0$ such that $E\Phi(\frac{f}{k}) < M < \infty$, and then

$$E\Psi(\frac{f}{c_1 k}) < \Psi(t_0) + E_{\{\frac{f}{c_1 k} > n_0\}} \Phi(\frac{f}{k}) < \Psi(t_0) + M < \infty;$$

hence $f \in L^\Psi$. It shows that $H^\Phi \subseteq L^\Psi$.

(ii) $\implies$ (iii) is obvious.

(iii) $\implies$ (i). When $H_+^\Phi \subseteq L_+^\Psi$, it follows from the proof of (iii) $\implies$ (i) in Theorem 1 that there exists $c > 0$ such that

$$\|f\|_\Psi \leq c \|f^*\|_\Phi$$

for every nonnegative martingale $f = (f_n)_{n \geq 0} \in H^\Phi$ defined on $(\Omega, \mathcal{F}, \mathcal{F}_n, \mu)$. Now, for every such a martingale f with $E\Psi(f) \geq 1$, define $g = (g_m)_{m \geq 0}$ on the product space $(\Omega \times (0, 1], \mathcal{F}_n \times \sigma\{(0, \alpha]\}, \mu \times \nu)$:

$$g_m(\omega, t) := f_m(\omega) \chi_{(0, \alpha]}(t), \quad \forall t \in (0, 1],$$

where ν is Lebesgue measure on $(0, 1]$ and $\alpha = \frac{1}{E\Psi(f)}$. Denote by $\tilde{E}$ the expectation with respect to $\Omega \times (0, 1]$. Then $\tilde{E}\Psi(g) = \alpha E\Psi(f) = 1$ and $\|cg^*\|_\Phi \geq \|g\|_\Psi = 1$. It follows that $\tilde{E}\Phi(cg^*) \geq 1$ and then

$$(9) \quad E\Phi(cf^*) \geq E\Psi(f).$$

Let $t \geq \Psi^{-1}(1)$ and $f \equiv t$. Then $E\Psi(f) \geq 1$. From (9) we have $\Phi(ct) \geq \Psi(t)$, $\forall t \geq \Psi^{-1}(1)$. This implies that $\lim_{t \rightarrow \infty} \frac{\Phi(ct)}{\Psi(t)} > 0$. $\square$

Corollary 2. Suppose that Φ, Ψ are nondecreasing convex functions on $[0, \infty)$ with $\Phi(0) = \Psi(0) = 0$. Then $H^\Phi = L^\Psi$ if and only if $H^\Psi = L^\Phi$.

Proof. We only prove the sufficiency, and the necessity can be obtained from the "symmetry". Indeed, from $H^\Phi = L^\Psi$ and Theorems 1 and 2, there exist $s_0, c_1 > 0$ such that

$$\int_{s_0}^t \frac{\varphi(s)}{s} ds \leq c_1 \psi(c_1 t), \quad \Phi(c_1 t) > \Psi(t), \quad \forall t > s_0.$$

A simple computation shows that $c_1 \varphi(c_1 t) > \frac{t}{2} \psi(\frac{t}{2})$, $\forall t > s_0$, and

$$\int_{c_1 s_0}^t \frac{\psi(\frac{s}{2c_1})}{s} ds \leq \int_{c_1 s_0}^t \frac{\varphi(s)}{s} ds \leq c_1 \psi(c_1 t) \leq 2c_1^2 \varphi(2c_1^2 t), \quad \forall t > c_1 s_0;$$

therefore

$$\int_{s_1}^t \frac{\psi(s)}{s} ds \leq c \varphi(ct), \quad \forall t > s_1,$$

where $s_1 = \frac{s_0}{2}$, $c = 2c_1^2$. Thus $H^\Psi \supseteq L^\Phi$. The converse relation comes from $H^\Phi = L^\Psi$ directly, and hence $H^\Psi = L^\Phi$ is proved.

Now we consider regular martingales. Recall that an increasing sequence $(\mathcal{F}_n)_{n \geq 0}$ of sub- σ -fields $(\mathcal{F}_0 = \{\emptyset, \Omega\})$ is said to be regular if there exists a $d_{(\mathcal{F}_n)} > 0$ such that

$$\chi(F) \leq d_{(\mathcal{F}_n)} E(\chi(F) | \mathcal{F}_{n-1}), \quad \forall F \in \mathcal{F}_n, \quad n \in N.$$

We say that a probability space $(\Omega, \mathcal{F}, \mathcal{F}_n, \mu)$ (a submartingale $f = (f_n, \mathcal{F}_n)_{n \geq 0}$) is regular if $(\mathcal{F}_n)_{n \geq 0}$ is regular.

Lemma 2 ([8]). *Let $f = (f_n)_{n \geq 0}$ be a regular nonnegative martingale with the constant $d_{(\mathcal{F}_n)}$. Then*

$$\int_{\{f_n^* > \lambda\}} f_n d\mu \leq d_{(\mathcal{F}_n)} \lambda |\{f_n^* > \lambda\}|, \quad \forall \lambda \geq \|f_0\|_\infty.$$

Theorem 3. *Suppose that Φ, Ψ are nondecreasing convex functions defined on $[0, \infty)$ with $\Phi(0) = \Psi(0) = 0$. Then the following statements are equivalent:*

(i) *There exist $s_0, c_1 > 0$ such that*

$$\int_1^t \frac{\varphi(s)}{s} ds \geq c_1 \psi(c_1 t), \quad \forall t > s_0.$$

(ii) *$f \in H^\Phi$ implies that $f \in L^\Psi$ for every nonnegative regular martingale $f = (f_n)_{n \geq 0}$.*

(iii) *There exists $c_2 > 0$ such that*

$$c_2 \|f^*\|_\Phi \geq \sup_n \|f_n\|_\Psi$$

for every nonnegative regular martingale $f = (f_n)_{n \geq 0}$, where c_2 depends only on $d_{(\mathcal{F}_n)}$.

Proof. (i) $\implies$ (ii). For a nonnegative regular martingale $f \in H^\Phi$, choose $c > s_0$ such that $\|f\|_1 \leq c$. By using Fubini's theorem and Lemmas 1 and 2 we get

$$\begin{aligned} E\Psi(c_1|f|) &= \int_0^\infty |\{|f| > s\}| c_1 \psi(c_1 s) ds \\ &\leq \int_{s_0}^\infty |\{|f| > s\}| \left(\int_1^s \frac{\varphi(t)}{t} dt \right) ds + \Psi(c_1 c) \\ &\leq \int_1^\infty \frac{\varphi(t)}{t} \left(\int_t^\infty |\{|f| > s\}| ds \right) dt + \Psi(c_1 c) \\ &\leq \int_1^\infty \frac{\varphi(t)}{t} \left(\int_{|f| > t} |\{|f| > s\}| ds \right) dt + \Psi(c_1 c) \\ &\leq \int_1^\infty \frac{\varphi(t)}{t} d_{(\mathcal{F}_n)} t \{f^* > t\} dt + \Psi(c_1 c) \\ &\leq d_{(\mathcal{F}_n)} E\Phi(f^*) + \Psi(c_1 c) < \infty. \end{aligned}$$

Hence $f \in L^\Psi$.

(ii) $\implies$ (iii). Without loss of generality, we assume that $\|f^*\|_\Phi = 1$. If the assertion of (iii) is not true, we can choose a sequence of nonnegative martingales $f^{(n)} = (f_{nm})_{m \geq 0}$ defined on $(\Omega, \mathcal{F}, \mathcal{F}_n, \mu)$ such that $4^n \|f^{(n)*}\|_\Phi < \|f^{(n)}\|_\Psi$. Let $g_k(\omega, t) = \sum_{n=1}^\infty \frac{f_{nk}}{2^n}$. Then $\|g^*\|_\Phi \leq \sum_{n=1}^\infty \left\| \frac{f^{(n)*}}{2^n} \right\|_\Phi = 1$ and $\left\| \frac{g}{2^n} \right\|_\Psi \geq \left\| \frac{f^{(n)}}{4^n} \right\|_\Psi > 1, \forall n > 0$, a contradiction with $g \in H^\Phi \subseteq L^\Psi$.

(iii) $\implies$ (i). We consider the martingale $f = (E(f | \mathcal{F}_k))_{k \geq 0}$ as in the proof of Theorem 1, where $f = t\chi_{(0, \frac{1}{2^n}]}$, $t = \Psi^{-1}(2^n)$. From (iii) we have that $c_2 \|f^*\|_\Phi > \|f\|_\Psi = 1$ and $E\Phi(c_2 f^*) > 1 = E\Psi(f)$. Let $c_4 = (2\Phi(1) \vee 2)c_2$. By the convexity of Φ we have that $E\Phi(c_4 f^*) > 2\Phi(1) \vee 2$. Notice that $\int_0^1 |\{c_4 f^* > s\}| \varphi(s) ds \leq \Phi(1)$, and then

$$(10) \quad \int_1^\infty |\{c_4 f^* > s\}| \varphi(s) ds > \frac{1}{2} \int_0^\infty |\{c_4 f^* > s\}| \varphi(s) ds > 1.$$

When $s \in (\frac{c_4 t}{2^{n-k}}, \frac{c_4 t}{2^{n-k-1}}]$ ($0 \leq k \leq n-1$), we have that

$$\frac{1}{s} \geq \frac{2^n}{c_4 t} \cdot \frac{1}{2^{k+1}} = \frac{2^n}{c_4 t} \cdot |\{c_4 f^* > s\}|;$$

and when $s \in (0, \frac{c_4 t}{2^n}]$, we have $\frac{1}{s} \geq \frac{2^n}{c_4 t}$. Thus $\frac{1}{s} \geq \frac{2^n}{c_4 t} \cdot |\{c_4 f^* > s\}|$, $\forall s < c_4 t$. From (10), the convexity of Ψ and the fact $f^* \leq t$ we get

$$\int_1^{c_4 t} \frac{\varphi(s)}{s} ds \geq \frac{2^n}{c_4 t} \int_1^{c_4 t} |\{c_4 f^* > s\}| \varphi(s) ds \geq \frac{2^n}{c_4 t} \geq \frac{1}{2c_4} \psi\left(\frac{t}{2}\right).$$

Thus for every $n \in \mathbb{N}$, when $u \in (c_4 \Psi^{-1}(2^n), c_4 \Psi^{-1}(2^{n+1})]$ we have

$$\int_1^u \frac{\varphi(s)}{s} ds \geq \int_1^{c_4 \Psi^{-1}(2^n)} \frac{\varphi(s)}{s} ds \geq \frac{1}{2c_4} \psi\left(\frac{\Psi^{-1}(2^n)}{2}\right) \geq \frac{1}{4c_4} \psi\left(\frac{u}{4c_4}\right);$$

hence

$$\int_1^t \frac{\varphi(s)}{s} ds \geq c_1 \psi(c_1 t), \quad \forall t \geq c_4 \Psi^{-1}(2)$$

where $c_1 = \frac{1}{4c_4}$, which implies (i) and the proof is complete.

Remark. There is an example which shows that Theorem 3 is not true if we replace the nonnegative regular martingale in (ii) and (iii) by a regular martingale. To see this, let $\Phi(t) = t$, $\Psi(t) = t \log^+ t$, then Φ, Ψ satisfy (i). Consider the martingale $f = (E(4^n \chi_{(0, \frac{1}{2^{n+1}}]} - 4^n \chi_{(\frac{1}{2^{n+1}}, \frac{1}{2^n}]}) | \mathcal{F}_k))_{k \geq 0}$ on $(0, 1]$, where $\mathcal{F}_k = \sigma\{(0, \frac{1}{2^k}], 1 \leq k \leq n\}$. It is easy to check that $\|f\|_\Psi > \sqrt{n} \|f^*\|_\Phi$ when n is big enough. The proof of (ii) $\implies$ (iii) shows that $H^\Phi \not\subseteq L^\Psi$.

Corollary 3. Suppose that Φ, Ψ are nondecreasing convex functions defined on $[0, \infty)$ with $\Phi(0) = \Psi(0) = 0$. Then the following statements are equivalent:

- (i) There exist $s_0, c_1 > 0$ such that $\int_1^t \frac{\varphi(s)}{s} ds \geq c_1 \psi(c_1 t)$, $\forall t > s_0$.
- (ii) There exists $c > 0$ such that

$$E\Psi(f) \leq c + cd_{(\mathcal{F}_n)}(\|f\|_1 \vee c)E\Phi(cf^*)$$

for every nonnegative regular martingale $f = (f_n)_{n \geq 0}$.

Proof. We only prove (i) $\implies$ (ii). (ii) $\implies$ (i) can be obtained from Theorem 3. From the proof of Theorem 3, we have $E\Psi(c_1 f) \leq c + d_{(\mathcal{F}_n)}E\Phi(f^*)$ for every nonnegative regular martingale f with $\|f\|_1 \leq 1$, where $c = \Psi(c_1)$. Now for a nonnegative martingale $f = (f_n)_{n \geq 0}$ adapted to $(\mathcal{F}_n)_{n \geq 0}$ with $\|f\|_1 = 2^m$, $m \in \mathbb{N}$, consider the martingale $g = (g_n)_{n \geq 0}$ defined on $(\tilde{\Omega}, \tilde{\mathcal{F}}_n, \tilde{\mu}) = (\Omega \times (0, 1], \mathcal{F}_n \times \sigma\{(0, \frac{1}{2^k}], 1 \leq k \leq n\}, \mu \times \nu) : g_n(\omega, t) = E(f\chi_{(0, \frac{1}{2^m}]} | \tilde{\mathcal{F}}_n)$, where ν is Lebesgue measure on $(0, 1]$. Then $\|g\|_1 = 1$ and

$$E\Psi(c_1 g) \leq c + d_{(\tilde{\mathcal{F}}_n)}E\Phi(g^*) = c + 2d_{(\mathcal{F}_n)}E\Phi(g^*).$$

Notice that $E\Phi(g^*) \leq \frac{1}{2^m} E\Phi(f^*) + \sum_1^m (\frac{1}{2^k}) E\Phi(f_k^*) \leq E\Phi(f^*)$; we get

$$(11) \quad \frac{1}{2^m} E\Psi(c_1 f) \leq c + 2d_{(\mathcal{F}_n)} E\Phi(f^*).$$

For any f with $\|f\|_1 > 1$, choose c_0, m such that $2^{m-1} < \|f\|_1 \leq 2^m, c_0 = \frac{2^m}{\|f\|_1} < 2$. From (11) we have

$$\begin{aligned} E\Psi(c_1 f) &\leq E\Psi(c_1 c_0 f) \leq c \|c_0 f\|_1 + 2d_{(\mathcal{F}_n)} \|c_0 f\|_1 E\Phi(c_0 f^*) \\ &\leq 2c \|f\|_1 + 4d_{(\mathcal{F}_n)} \|f\|_1 E\Phi(2f^*). \end{aligned}$$

Thus for every nonnegative regular martingale f , we get

$$E\Psi(c_1 f) \leq 2c(\|f\|_1 \vee 1) + 4d_{(\mathcal{F}_n)}(\|f\|_1 \vee 1) E\Phi(2f^*)$$

and

$$E\Psi(f) \leq \frac{2c}{c_1}(\|f\|_1 \vee c_1) + \frac{4}{c_1} d_{(\mathcal{F}_n)}(\|f\|_1 \vee c_1) E\Phi\left(\frac{2f^*}{c_1}\right).$$

Choose $c_2 > c_1$ such that $\frac{4}{c_1} \Phi\left(\frac{2}{c_1} t\right) > \frac{2c}{c_1}, \forall t \geq c_2$; we obtain

$$E\Psi(f) \leq c_3 + c_4 d_{(\mathcal{F}_n)}(\|f\|_1 \vee c_2) E\Phi(c_5 f^*)$$

where $c_3 = \frac{2c}{c_1} c_2, c_4 = \frac{8}{c_1}, c_5 = \frac{2}{c_1}$. Hence the assertion in (ii) is true.

3. SOME EQUIVALENT CONDITIONS AND EXAMPLES

Corollary 1 shows that condition (8) is sufficient and necessary for the extension of Doob's inequality. In the following, we discuss the relationship between (8) and the condition $q_\Phi > 1$ when $\Phi \in \Delta_2$ (i.e. there exist $c, t_0 > 0$ such that $\Phi(2t) \leq c\Phi(t), \forall t > t_0$).

Lemma 3. *If Φ is a nondecreasing convex function on $[0, \infty)$ with $\Phi(0) = 0$, and there exists $c > 0$ such that $\Phi(2t) \leq c\Phi(t), \forall \Phi(t) > 0$, then $q_\Phi > 1$ if and only if there exists $c_1 > 0$ such that $\int_0^t \frac{\varphi(s)}{s} ds \leq c_1 \varphi(c_1 t), \forall t > 0$.*

Proof. First we prove the sufficiency. Notice that $\frac{\Phi(s)}{s^2} \leq \frac{1}{q_\Phi} \frac{\varphi(s)}{s}$; we get

$$\int_0^t \frac{\varphi(s)}{s} ds \leq \frac{q_\Phi}{q_\Phi - 1} \int_0^t \frac{\varphi(s)}{s} - \frac{\Phi(s)}{s^2} ds \leq \frac{q_\Phi}{q_\Phi - 1} \frac{\Phi(t)}{t} \leq \frac{q_\Phi}{q_\Phi - 1} \varphi(t).$$

Next we prove the necessity. Notice that the condition $\int_0^t \frac{\varphi(s)}{s} ds \leq c_1 \varphi(c_1 t)$ implies $\varphi(s) \downarrow 0$ (as $s \rightarrow 0$) and $\int_0^t \frac{\varphi(s)}{s} ds < \infty, \forall t > 0$. Denote

$$(12) \quad a_k = 2^k \varphi(2^k) - \Phi(2^k) - [2^{k-1} \varphi(2^{k-1}) - \Phi(2^{k-1})] \quad (-\infty < k < +\infty);$$

we have

$$(13) \quad 2^{k-1} [\varphi(2^k) - \varphi(2^{k-1})] \leq a_k \leq 2^k [\varphi(2^k) - \varphi(2^{k-1})]$$

and

$$(14) \quad \sum_{k=-\infty}^m 2^{-k} a_k \leq \varphi(2^m) - \varphi(0) \leq \sum_{k=-\infty}^m 2^{-k+1} a_k.$$

Therefore,

$$(15) \quad \Phi(2^m) \leq \sum_{k=-\infty}^{m-1} 2^k \varphi(2^{k+1}) \leq \sum_{k=-\infty}^{m-1} 2^k \sum_{i=-\infty}^{k+1} 2^{-i+1} a_i \leq \sum_{i=-\infty}^m 2^{m-i+1} a_i.$$

On the other hand, we have

(16)

$$\int_0^{2^m} \frac{\varphi(s)}{s} ds \geq \sum_{k=-\infty}^{m-1} 2^k \frac{\varphi(2^k)}{2^{k+1}} \geq \frac{1}{2} \sum_{k=-\infty}^{m-1} \sum_{i=-\infty}^k 2^{-i} a_i = \frac{1}{2} \sum_{i=-\infty}^{m-1} 2^{-i} (m-i) a_i.$$

Notice that $\inf_{t>0} \frac{t\varphi(t)-\Phi(t)}{\Phi(\frac{t}{2})} = 0$ if $q_\Phi = 1$. From (12) and (15) we get that $\forall j > 1$,

$\exists t_j \in (2^{n_j}, 2^{n_j+1}]$ such that $\Phi(\frac{t_j}{2}) > 0$ and

$$\left(\sum_{k=-\infty}^{n_j} 2^{n_j-k+1} a_k \right)^{-1} \sum_{k=-\infty}^{n_j} a_k \leq \frac{2^{n_j} \varphi(2^{n_j}) - \Phi(2^{n_j})}{\Phi(2^{n_j})} \leq \frac{t_j \varphi(t_j) - \Phi(t_j)}{\Phi(\frac{t_j}{2})} \leq \frac{1}{2^j}.$$

Then for $k_0 = n_j - j + 1$, we have

$$\left(\sum_{k=k_0+1}^{n_j} 2^{n_j-k+1} a_k \right)^{-1} \sum_{k=-\infty}^{n_j} a_k \geq \frac{1}{2^{j-1}}, \quad \left(\sum_{k=-\infty}^{k_0} 2^{n_j-k+1} a_k \right)^{-1} \sum_{k=-\infty}^{n_j} a_k \leq \frac{1}{2^{j-1}}.$$

Thus $\sum_{k=-\infty}^{n_j} 2^{-k} a_k \leq 2 \sum_{k=-\infty}^{k_0} 2^{-k} a_k$ and

$$\frac{\varphi(\frac{t_j}{2})}{\int_0^{2t_j} \frac{\varphi(s)}{s} ds} \leq \left(\frac{1}{2} \sum_{k=-\infty}^{n_j} (n_j - k + 1) 2^{-k} a_k \right)^{-1} \sum_{k=-\infty}^{n_j} 2^{-k+1} a_k \leq \frac{8}{(n_j - k_0 + 1)} = \frac{8}{j}.$$

Therefore

$$j\varphi(\frac{t_j}{2}) \leq 8 \int_0^{2t_j} \frac{\varphi(s)}{s} ds \leq 8c_1 \varphi(2c_1 t_j)$$

and

$$j\Phi(\frac{t_j}{2}) \leq j \frac{t_j}{2} \varphi(\frac{t_j}{2}) \leq 4c_1 t_j \varphi(2c_1 t_j) \leq \Phi(6c_1 t_j), \forall j > 0,$$

a contradiction that proves the necessity.

Theorem 4. Let Φ be a nondecreasing convex function on $[0, \infty)$ with $\Phi(0) = 0$ and $\Phi \in \Delta_2$. Then $\lim_{t \rightarrow \infty} \frac{t\varphi(t)}{\Phi(t)} > 1$ if and only if there exist some $s_0, c_1 > 0$ such that (8) holds.

Proof. Without loss of generality, we can assume that there exists $t_0 > 0$ such that $t\psi(t) > \Psi(t)$, $\Psi(2t) \leq c\Psi(t)$, $\forall t > t_0$. Now, for any $s_0 > t_0$, consider the function $\tilde{\Psi}(t) = \Psi(t)\chi_{[s_0, \infty)} + \Psi(s_0)(\frac{t}{s_0})^\alpha \chi_{[0, s_0)}$, where $\alpha = \frac{s_0 \psi(s_0)}{\Psi(s_0)} > 1$. Applying Lemma 3 to $\tilde{\Psi}(t)$ we get

$$\inf_{t>s_0} \frac{t\psi(t)}{\Psi(t)} > 1 \text{ if and only if } \exists c > 0 \text{ such that } \int_{s_0}^t \frac{\psi(s)}{s} ds < c\psi(ct), \forall t > s_0.$$

The proof is finished.

The following are some examples which make the inequalities in this paper hold:

Example 1. For $1 < p < \infty$, let

$$\Phi(t) = \Psi(t) = \begin{cases} t, & t < 1, \\ t^p, & t \geq 1. \end{cases}$$

In this case,

$$\varphi(t) = \psi(t) = \begin{cases} 1, & t < 1, \\ pt^{p-1}, & t \geq 1. \end{cases}$$

Example 2. Let

$$\Phi(t) = t, \Psi(t) = t \log^+ t.$$

In this case

$$\varphi(t) = 1, \psi(t) = \begin{cases} 0, & t < 1, \\ 1 + \log^+ t, & t \geq 1. \end{cases}$$

Example 3. Suppose that $n \geq 1$ and

$$\Phi(t) = t(\log^n t)^+, \quad \Psi(t) = t(\log^{n+1} t)^+.$$

In this case

$$\begin{aligned} \varphi(t) &= \begin{cases} 0, & 0 \leq t < 1, \\ n \log^{n-1} t + \log^n t, & t \geq 1, \end{cases} \\ \psi(t) &= \begin{cases} 0, & 0 \leq t < 1, \\ (n+1) \log^n t + \log^{n+1} t, & t \geq 1. \end{cases} \end{aligned}$$

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