DOMAIN FUNCTIONALS AND EXIT TIMES FOR BROWNIAN MOTION

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ABSTRACT. Two domain functionals describing the averaged expectation of exit times and averaged variance of exit times of Brownian motion from a domain, respectively, are studied. We establish the variational formulas for maximizing the functionals over C^k domains with a volume constraint, and characterize the stationary points and maximizers.

1. INTRODUCTION

Let W_t be a standard n -dimensional Brownian motion satisfying $W_0 = x \in R^n$. For any bounded domain Ω , denote by $\tau_\Omega(\omega)$ the (first) exit time of the Brownian motion from Ω , i.e., $\tau_\Omega(\omega) = \inf \{t \geq 0 : W_t(\omega) \in \partial\Omega\}$, where $\partial\Omega$ is the boundary of Ω . Consider the following domain functionals:

$$\mathfrak{F}_E(\Omega) = \frac{1}{|\Omega|} \int_{\Omega} E_x(\tau_\Omega) dx,$$

$$\mathfrak{F}_V(\Omega) = \frac{1}{|\Omega|} \int_{\Omega} Var_x(\tau_\Omega) dx,$$

where $E_x(\tau_\Omega)$ and $Var_x(\tau_\Omega)$ are the expectation and variance of τ_Ω , respectively. The values $\mathfrak{F}_E(\Omega)$ and $\mathfrak{F}_V(\Omega)$ may be understood as the averaged expected exit time and averaged variance of exit time from Ω for the Brownian motion initiated in Ω .

We are interested in characterizing stationary points for the following optimization problems (P_E) and (P_V) :

$$\begin{aligned} (P_E) \quad & \text{find } \Omega \text{ that maximizes } \mathfrak{F}_E(\Omega) \text{ over } \mathfrak{D}^k, \\ (P_V) \quad & \text{find } \Omega \text{ that maximizes } \mathfrak{F}_V(\Omega) \text{ over } \mathfrak{D}^k \end{aligned}$$

where $\mathfrak{D}^k$ is the collection of bounded C^k ($k \geq 2$) domains $\Omega \subset R^n$ satisfying the volume constraint

$$(1.1) \quad \text{volume of } \Omega = A_0, \quad A_0 \text{ is a positive constant.}$$

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A domain Ω in $\mathfrak{D}^k$ is called a stationary point of any domain functional $\mathfrak{F}$ over $\mathfrak{D}^k$ if for any $\{\Omega_\varepsilon\}_{|\varepsilon| \leq \varepsilon_0} \subset \mathfrak{D}^k$ with $\Omega_0 = \Omega$, for some $\varepsilon_0 > 0$, we have $d\mathfrak{F}(\Omega_\varepsilon)/d\varepsilon = 0$ at $\varepsilon = 0$ if it exists. We show the following results.

Theorem 1.1. *Let $\Omega \in \mathfrak{D}^k$ be a stationary point of functional $\mathfrak{F}_E$ over $\mathfrak{D}^k$. Then there exists a solution to the overdetermined boundary value problem*

$$(1.2) \quad \Delta u(x) = -2 \text{ in } \Omega,$$

$$(1.3) \quad u(x) = 0 \text{ on } \partial\Omega,$$

$$(1.4) \quad \frac{\partial u(x)}{\partial v} \text{ is constant on } \partial\Omega,$$

where v is the exterior unit normal vector. In particular, $\Omega \in \mathfrak{D}^k$ is a maximizer of $\mathfrak{F}_E$ over $\mathfrak{D}^k$ if and only if Ω is a ball.

Theorem 1.2. *Let $\Omega \in \mathfrak{D}^k$ be a stationary point of functional $\mathfrak{F}_V$ over $\mathfrak{D}^k$. Then there exists a solution to the overdetermined boundary value problem*

$$(1.5) \quad \Delta u(x) = -2 \text{ in } \Omega,$$

$$(1.6) \quad \Delta w(x) = -2u(x) \text{ in } \Omega,$$

$$(1.7) \quad u(x) = w(x) = 0 \text{ on } \partial\Omega,$$

$$(1.8) \quad \frac{\partial u(x)}{\partial v} \frac{\partial w(x)}{\partial v} \text{ is constant on } \partial\Omega,$$

where v is the exterior unit normal vector. In particular, $\Omega \in \mathfrak{D}^k$ is a maximizer of $\mathfrak{F}_V$ over $\mathfrak{D}^k$ if and only if Ω is a ball.

In the case $k = \infty$, the optimization problems (P_E) and (P_V) have been studied in [4, 5], where the same results with a different definition of stationary points (called critical points there) were established. See also [1, 2, 6, 8] for more general domain functionals. However, the notion of “critical point” adopted in these papers is with respect to infinitesimal volume preservation. More precisely, a domain Ω is called a critical point in [4, 5, 6] if $d\mathfrak{F}_E(\Omega_\varepsilon)/d\varepsilon = 0$ at $\varepsilon = 0$ for any normal domain variations Ω_ε ($\Omega_0 = \Omega$) satisfying $d|\Omega_\varepsilon|/d\varepsilon = 0$ at $\varepsilon = 0$. This same definition was also employed in [1, 2] for interior domain variations. In general, this domain variation Ω_ε may not satisfy constraint (1.1), i.e., $\Omega_\varepsilon \notin \mathfrak{D}^k$. Strictly speaking, a maximizer of $\mathfrak{F}_E$ or $\mathfrak{F}_V$ over $\mathfrak{D}^k$ may not be a critical point in this sense.

The novelty of this note is the introduction of a family of volume-preserving domain variations Ω_ε , i.e., $\Omega_\varepsilon \in \mathfrak{D}^k$ for all ε . This guarantees that a maximizer of $\mathfrak{F}_E$ or $\mathfrak{F}_V$ over $\mathfrak{D}^k$ is necessarily a stationary point. In addition, we weaken the regularity requirement on Ω which cannot be achieved by using the method of normal domain variations.

2. VOLUME-PRESERVING DOMAIN VARIATIONS

For simplicity, throughout the paper, we assume that in (1.1), $A_0 = 1$. Let Ω be a bounded C^k domain. For any $V \in C^k(\Omega', R^n)$, where Ω' is a neighborhood of $\bar{\Omega}$, we denote by $\Phi(x, \varepsilon)$ the solution of the initial value problem, for fixed $x \in \bar{\Omega}$,

$$(2.1) \quad \frac{d\Phi(x, \varepsilon)}{d\varepsilon} = V(\Phi(x, \varepsilon)), \quad \Phi(x, 0) = x.$$

A standard ODE theorem shows that $\Phi(x, \varepsilon)$ is a C^k diffeomorphism for $|\varepsilon| < \varepsilon_0$ for some $\varepsilon_0 > 0$. We define a variation $\Omega_{\varepsilon, V}$ of the domain Ω by the vector field V , by

$$\Omega_{\varepsilon, V} = \Phi(\Omega, \varepsilon) = \{\Phi(x, \varepsilon) : x \in \Omega\}.$$

When no confusion arises, we shall omit the subscript V : $\Omega_{\varepsilon} = \Omega_{\varepsilon, V}$. Obviously, $\Omega_0 = \Omega$, and $\Omega_{\varepsilon} \subset \Omega'$ (for small ε_0) is a C^k domain. Also, $\Omega_{\varepsilon} \rightarrow \Omega$ as $\varepsilon \rightarrow 0$, in the sense of the C^k topology.

Lemma 2.1. *Suppose $\Omega \in \mathfrak{D}^k$. Let $\Omega' \supset \bar{\Omega}$ be a neighborhood of $\bar{\Omega}$, and let $V \in C^k(\Omega', \mathbb{R}^n)$ be a divergence free vector field, i.e., it satisfies $\nabla \cdot V(x) = 0$ for $x \in \Omega'$. Let $\Omega_{\varepsilon} = \Phi(\Omega, \varepsilon)$ be the variation of Ω by the vector field V . Then $\Omega_{\varepsilon} \in \mathfrak{D}^k$.*

Proof. By the ODE theory and the fact that $\nabla \cdot V(x) = 0$, its Jacobian $J(\Phi(x, \varepsilon))$ satisfies

$$\frac{dJ(\Phi(x, \varepsilon))}{d\varepsilon} = (\nabla \cdot V)(\Phi(x, \varepsilon)) J(\Phi(x, \varepsilon)) = 0.$$

Hence

$$(2.2) \quad J(\Phi(x, \varepsilon)) = J(\Phi(x, 0)) = 1.$$

This immediately leads to $|\Omega_{\varepsilon}| = |\Omega| = 1$. □

Let V be such a divergence free C^k vector field and let Ω_{ε} be the variation of Ω by V . Consider the Dirichlet problem

$$\Delta u_{\varepsilon}(x) = -2 \text{ in } \Omega_{\varepsilon}, \quad u_{\varepsilon}(x) = 0 \text{ on } \partial\Omega_{\varepsilon}.$$

Set $u = u_0$. We define the first variation $\delta_V(u)$ of u along the vector field V by

$$(2.3) \quad \delta_V(u)(x_0) = \left. \frac{d}{d\varepsilon} u_{\varepsilon}(\Phi(x_0, \varepsilon)) \right|_{\varepsilon=0},$$

for $x_0 \in \bar{\Omega}$.

Lemma 2.2. *Let u be the solution of (1.2) and (1.3) and $G_0(x, y)$ be the Green's function for the Laplacian Δ for Ω . Then for any divergence free C^k vector field V , defined in a neighborhood of $\bar{\Omega}$, and $x_0 \in \bar{\Omega}$, we have*

$$(2.4) \quad \begin{aligned} \delta_V(u)(x_0) = & - \int_{\partial\Omega} (V(\sigma) \cdot \nu(\sigma)) \frac{\partial u(\sigma)}{\partial \nu} \frac{\partial G_0(x_0, \sigma)}{\partial \nu} d\sigma \\ & + V(x_0) \cdot \nabla u(x_0), \text{ for } x_0 \text{ in } \bar{\Omega}. \end{aligned}$$

Proof. Let $G_{\varepsilon}(x, y)$ be the Green's function for the Laplacian Δ for Ω_{ε} . By Green's formula and (2.2), we have

$$\begin{aligned} u_{\varepsilon}(\Phi(x_0, \varepsilon)) &= -2 \int_{\Omega_{\varepsilon}} G_{\varepsilon}(\Phi(x_0, \varepsilon), x) dx \\ &= -2 \int_{\Omega} G_{\varepsilon}(\Phi(x_0, \varepsilon), \Phi(x, \varepsilon)) dx. \end{aligned}$$

By [3, 2.4.28] (an early argument of this type was given by Hadamard) and direct computations, we have the formula

$$\begin{aligned} & \left. \frac{\partial}{\partial \varepsilon} G_\varepsilon(\Phi(x_0, \varepsilon), \Phi(x, \varepsilon)) \right|_{\varepsilon=0} \\ &= - \int_{\Omega} \nabla_y G_0(x_0, y) \cdot \left(\nabla V(y) + (\nabla V(y))^T \right) \nabla_y G_0(x, y) dy. \end{aligned}$$

Using the fact that

$$\nabla u(y) = -2 \int_{\Omega} \nabla_y G_0(x, y) dx,$$

it follows from (2.3) that

$$\begin{aligned} \delta_V(u)(x_0) &= -2 \int_{\Omega} \left. \frac{\partial}{\partial \varepsilon} G_\varepsilon(\Phi(x_0, \varepsilon), \Phi(x, \varepsilon)) \right|_{\varepsilon=0} dx \\ &= - \int_{\Omega} (\nabla_y G_0(x_0, y)) \cdot \left(\nabla V(y) + (\nabla V(y))^T \right) \nabla u(y) dy. \end{aligned}$$

Since $\nabla^2 u$ and $\nabla^2 G_0$ are symmetric matrices and since V is divergence free, a direct computation leads to

$$\begin{aligned} & \nabla_y G_0(x_0, y) \cdot \left(\nabla V(y) + (\nabla V(y))^T \right) \nabla u(y) \\ &= \nabla G_0 \cdot \nabla(V \cdot \nabla u) + \nabla u \cdot \nabla(V \cdot \nabla G_0) - \nabla \cdot ((\nabla u \cdot \nabla G_0) V), \end{aligned}$$

for $y \neq x_0$. Since u solves (1.2) and (1.3), $\Delta G_0(x_0, y)$ is the Dirac measure at $y = x_0$, $\nabla \cdot V(x) = 0$, and $G_0(x_0, \sigma) = 0$ for $\sigma \in \partial\Omega$, the divergence theorem yields

$$\begin{aligned} \delta_V(u)(x_0) &= - \int_{\partial\Omega} (V(\sigma) \cdot \nabla u) \frac{\partial G_0(x_0, \sigma)}{\partial v} d\sigma + V(x_0) \cdot \nabla u(x_0) \\ &\quad - \int_{\partial\Omega} (V(\sigma) \cdot \nabla G_0(x_0, \sigma)) \frac{\partial u(\sigma)}{\partial v} d\sigma - 2 \int_{\Omega} V(y) \cdot \nabla G_0(x_0, y) dy \\ &\quad + \int_{\partial\Omega} (\nabla u(\sigma) \cdot \nabla G_0(x_0, \sigma)) V(\sigma) \cdot v(\sigma) d\sigma \\ &= - \int_{\partial\Omega} (V(\sigma) \cdot v(\sigma)) \frac{\partial u(\sigma)}{\partial v} \frac{\partial G_0(x_0, \sigma)}{\partial v} d\sigma + V(x_0) \cdot \nabla u(x_0). \end{aligned}$$

In deriving the last equality, we have used the fact that for $\sigma \in \partial\Omega$,

$$\nabla u(\sigma) = \frac{\partial u(\sigma)}{\partial v} v(\sigma), \quad \nabla G_0(x_0, \sigma) = \frac{\partial G_0(x_0, \sigma)}{\partial v} v(\sigma),$$

and the fact that V is divergence free. $\square$

We point out that the mappings defining interior variations used in [2, 3] are actually the first order approximation to $\Phi(x, \varepsilon)$.

3. VARIATIONAL FORMULATIONS FOR FUNCTIONALS

In this section we develop variational formulations for problems (P_E) and (P_V) , utilizing the results established in the previous section. Define the first variation of any domain functional $\mathfrak{F}$ along a divergence free vector field V by

$$\delta_V(\mathfrak{F})(\Omega) = \left. \frac{d\mathfrak{F}(\Omega_\varepsilon)}{d\varepsilon} \right|_{\varepsilon=0} = \lim_{\varepsilon \rightarrow 0} \frac{\mathfrak{F}(\Omega_\varepsilon) - \mathfrak{F}(\Omega)}{\varepsilon},$$

where Ω_ε is the variation of Ω along V .

Lemma 3.1. *Let $\Omega \in \mathfrak{D}^k$ and V be a divergence free C^k vector field defined in a neighborhood of $\bar{\Omega}$. Then*

$$(3.1) \quad \delta_V (\mathfrak{F}_E) (\Omega) = \frac{1}{2} \int_{\partial\Omega} (V(\sigma) \cdot v) \left(\frac{\partial u(\sigma)}{\partial v} \right)^2 d\sigma$$

where u is the solution of (1.2) and (1.3), and

$$(3.2) \quad \delta_V (\mathfrak{F}_V) (\Omega) = \int_{\partial\Omega} (V(\sigma) \cdot v) \frac{\partial u(\sigma)}{\partial v} \frac{\partial w(\sigma)}{\partial v} d\sigma$$

where u and w satisfy (1.5)-(1.7).

The formulas (3.1) and (3.2) were previously derived in [6] for normal variations of smooth domains, and in [2] for interior variations of Lipschitz domains. By the same techniques used in [6], together with Lemma 2.2, one can show that (3.1) and (3.2) remain true for our domain variations by any divergence free vector field V . We omit the details of the proof to Lemma 3.1.

We are now in a position to prove Theorem 1.1 and Theorem 1.2.

Proof of Theorem 1.1. Suppose that the functional $\mathfrak{F}_E$ has a stationary point $\Omega \in \mathfrak{D}^k$. Then for any divergence free C^k vector field V , $\delta_V (\mathfrak{F}_E) (\Omega) = 0$. Let u solve (1.2), (1.3). By Lemma 3.1, it follows that

$$(3.3) \quad \int_{\partial\Omega} (V(\sigma) \cdot v(\sigma)) \left(\frac{\partial u(\sigma)}{\partial v} \right)^2 d\sigma = 0.$$

To show (1.4), it suffices to show that for any smooth function φ defined in a neighborhood of $\partial\Omega$ and satisfying

$$(3.4) \quad \int_{\partial\Omega} \varphi(\sigma) d\sigma = 0,$$

the following equality holds:

$$(3.5) \quad \int_{\partial\Omega} \varphi(\sigma) \left(\frac{\partial u(\sigma)}{\partial v} \right)^2 d\sigma = 0.$$

To verify (3.5), we select a sequence of decreasing smooth domains $\{\Omega^{(m)}\}_{m=0}^{\infty}$ with $\Omega^{(0)} = \Omega$ such that $\Omega^{(m)} \supset \bar{\Omega}^{(m+1)}$, $\Omega^{(m)} \supset \bar{\Omega}$ for $m \geq 1$, and as $m \rightarrow \infty$, $\Omega^{(m)} \rightarrow \Omega$ in the C^k topology. Let φ be a fixed smooth function satisfying (3.4). We extend φ smoothly to R^n . Define a sequence of constants θ_m by

$$\theta_m = -\frac{1}{\int_{\partial\Omega^{(m)}} d\sigma} \int_{\partial\Omega^{(m)}} \varphi(\sigma) d\sigma.$$

It is easy to see that $\theta_m \rightarrow 0$ (by (3.4)) as $m \rightarrow \infty$ and that

$$(3.6) \quad \int_{\partial\Omega^{(m)}} (\varphi(\sigma) + \theta_m) d\sigma = 0.$$

Consider the Neumann boundary value problem

$$\begin{aligned} \Delta g_m &= 0 \text{ in } \Omega^{(m)}, \\ \frac{\partial g_m}{\partial v} &= \varphi + \theta_m \text{ on } \partial\Omega^{(m)}. \end{aligned}$$

Note that (3.6) is the solvability condition for the Neumann problem. Hence, the above boundary value problem admits a smooth solution g_m (which is unique up to a constant). Furthermore, the $H^1(\Omega)$ norm of the function g_m is bounded uniformly in m . Set $V_m = \nabla g_m$. Obviously V_m is a divergence free vector field. Let $\eta(x)$ be a C^k extension of $\partial u(\sigma)/\partial v$. Then, by (3.3) and the divergence theorem (in $\Omega_m \setminus \Omega$), we have

$$\begin{aligned} 0 &= \int_{\partial\Omega} \frac{\partial g_m(\sigma)}{\partial v} \left(\frac{\partial u(\sigma)}{\partial v} \right)^2 d\sigma \\ &= \int_{\Omega_m \setminus \Omega} \nabla \cdot (\eta^2(x) \nabla g_m(x)) dx + \int_{\partial\Omega_m} \frac{\partial g_m(\sigma)}{\partial v} \eta^2(\sigma) d\sigma \\ &= \int_{\Omega_m \setminus \Omega} \nabla \eta^2(x) \cdot \nabla g_m(x) dx + \int_{\partial\Omega_m} (\varphi(\sigma) + \theta_m) \eta^2(\sigma) d\sigma \\ &\rightarrow \int_{\partial\Omega} \varphi(\sigma) \left(\frac{\partial u(\sigma)}{\partial v} \right)^2 d\sigma \quad (\text{as } m \rightarrow \infty). \end{aligned}$$

Our previous assertion (3.5) follows.

The rest of the assertions follow from the fact that any solution of (1.2)-(1.4) is radially symmetric [6] (see [7] for the original results) and that any ball is a maximizer [4]. $\square$

Proof of Theorem 1.2. Suppose that the functional $\mathfrak{F}_V$ has a stationary point $\Omega \in \mathfrak{D}^k$. Then for any divergence free C^k vector field V , $\delta_V(\mathfrak{F}_V)(\Omega) = 0$. Let u and w be the solutions of (1.5)-(1.7). By Lemma 3.1, it follows that

$$\int_{\partial\Omega} (V(\sigma) \cdot v(\sigma)) \frac{\partial u(\sigma)}{\partial v} \frac{\partial w(\sigma)}{\partial v} d\sigma = 0.$$

Using exactly the same argument as in the proof of Theorem 1.1, we may show that for any smooth function φ defined in a neighborhood of $\bar{\Omega}$ and satisfying $\int_{\partial\Omega} \varphi(\sigma) d\sigma = 0$, we have

$$\int_{\partial\Omega} \varphi(\sigma) \frac{\partial u(\sigma)}{\partial v} \frac{\partial w(\sigma)}{\partial v} d\sigma = 0.$$

This immediately leads to (1.8). The rest of the assertions follow from [6]. $\square$

REFERENCES

- [1] N. V. Banichuk, Optimization of elastic bars in torsion, *Int. J. Solids and Structures* 12 (1976), 275-286. MR **53**:15062
- [2] A. R. Elcrat and K. G. Miller, *Variational formulas on Lipschitz domains*, *Trans. Amer. Math. Soc.* 347 (1995), 2669-2678. MR **95i**:35068
- [3] P. R. Garabedian and M. Schiffer, *Convexity of domain functionals*, *J. Analyse Math.* 2 (1953), 281-368. MR **15**:627a
- [4] K. Kinader and P. McDonald, *Brownian functionals on hypersurfaces in Euclidean space*, *Proc. Amer. Math. Soc.* 125 (1997), 1815-1822. MR **97g**:60109
- [5] K. Kinader and P. McDonald, *Hypersurfaces in R^d and the variance of exit times for Brownian motion*, *Proc. Amer. Math. Soc.* 125 (1997), 2453-2462. MR **97k**:60220
- [6] K. Kinader, P. McDonald and D. Miller, *Exit time moments, boundary value problems, and the geometry of domains in Euclidean space*, *Probability Theory* 111 (1998), 469-487. MR **99h**:60151

- [7] J. Serrin, *A symmetry problem in potential theory*, Arch. Rat. Mech. Anal. 43 (1971), 304-318. MR **48**:11545
- [8] J. Simon, *Differentiation with respect to the domain in boundary value problems*, Numer. Funct. Anal. Optim. 2 (1980), 649-687. MR **83m**:49032

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