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NOTES ON A C_0-GROUP GENERATED BY THE LÉVY LAPLACIAN

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ABSTRACT. In this paper we shall give some results on a C_0-group generated by the Lévy Laplacian and operators approximating that group in the space $\mathcal{L}(\mathbf{E})$ of continuous linear operators defined on a certain locally convex space $\mathbf{E}$ in $(\mathcal{S})^*$.

1. INTRODUCTION

L. Gross [4] and P. Lévy [20] introduced Laplacians on infinite dimensional abstract Wiener space and Hilbert space, respectively. Those Laplacians have been reformulated as differential operators within the framework of white noise distribution theory (see [15, 16], etc.).

Let $(\mathcal{S})^*$ be the space of generalized white noise functionals and let $(\mathcal{S})$ be the space of test white noise functionals. The *white noise differential operator* ∂_t is defined to be the Gâteaux differential operator D_{δ_t} in a direction $\delta_t \in \mathcal{S}^*$, acting on $(\mathcal{S})$. For any $\varphi \in (\mathcal{S})$, the *Gross Laplacian* Δ_G is given by

$$\Delta_G \varphi = \int_{\mathbf{R}} \partial_t^2 \varphi dt.$$

On the other hand, the *Lévy Laplacian* Δ_L is given as follows. Let T be a finite interval of $\mathbf{R}$ and let $\{\zeta_k; k \in \mathbf{N} \cup \{0\}\} \subset \mathcal{S}$ be a complete orthonormal system for $L^2(T)$ satisfying the equal density and the uniform boundedness (see [16, 17]). The Lévy Laplacian Δ_L^T *depending on* T is defined by

$$\Delta_L^T \Phi = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \widetilde{D}_{\zeta_k}^2 \Phi$$

if the limit exists in $(\mathcal{S})^*$. From now on we fix an interval T and denote Δ_L^T by Δ_L. This Laplacian acts on some domain in $(\mathcal{S})^*$ and disappears on the Hilbert space (L^2) consisting of square-integrable functionals defined on the Gaussian white noise probability space $(\mathcal{S}^*, \mu)$. However Δ_L does not act on the whole space of $(\mathcal{S})^*$. So, it is important to decide a domain of the Lévy Laplacian to consider solving a white noise differential equation associated with the Laplacian.

In previous papers [3, 27], we have introduced a domain $\mathbf{E}$ of Δ_L and discussed some relations between a group generated by Δ_G and a group generated by Δ_L in

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the strong topology of $(\mathcal{S})^*$. In [3], we also discussed the Cauchy problem associated with Δ_L in the strong topology of $(\mathcal{S})^*$ by using the generalized Gross Laplacian $\Delta_G(K)$, $K \in \mathcal{L}(\mathcal{S}_{\mathbf{C}}^*, \mathcal{S}_{\mathbf{C}})$ (see also [2, 27]).

In this article we discuss the above relations in the strong topology of $\mathcal{L}(\mathbf{E})$ by using the generalized Gross Laplacian.

The paper is organized as follows. In Section 2, we summarize some basic definitions and results in the white noise distribution theory. In Section 3, we give a relation between $\Delta_G(P_N)$ and Δ_L by the limit theorem in $\mathbf{E}$. In Section 4, we define groups generated by the Laplacian operators acting on the Hida distributions and give a result that the group generated by Δ_L is approximated by the group $e^{t\Delta_G(P_N)}$ in the strong topology of $\mathcal{L}(\mathbf{E})$. In Section 5, we investigate the fundamental solution of a differential equation associated with the Lévy Laplacian.

2. Preliminaries

In this section, we assemble some basic notation of white noise distribution theory following [9, 14, 16, 22]. White noise distribution theory is a Schwartz type distribution theory on the infinite dimensional space $(\mathcal{S}^*, \mu)$, where $\mathcal{S}^* \equiv \mathcal{S}'(\mathbf{R})$ is the space of tempered distributions and μ is the standard Gaussian measure such that

$$\int_{\mathcal{S}^*} \exp\{i\langle x, \xi\rangle\}\, d\mu(x) = \exp\left(-\frac{1}{2}|\xi|_0^2\right),\ \xi \in \mathcal{S} \equiv \mathcal{S}(\mathbf{R}),$$

where $\langle\cdot,\cdot\rangle$ is the canonical bilinear form on $\mathcal{S}^* \times \mathcal{S}$ and $|\cdot|_0$ is the $L^2(\mathbf{R})$-norm. Let $A = -(d/du)^2 + u^2 + 1$. This is a densely defined self-adjoint operator on $L^2(\mathbf{R})$ and there exists an orthonormal basis $\{e_\nu; \nu \geq 0\} \subset \mathcal{S}$ for $L^2(\mathbf{R})$ such that $Ae_\nu = 2(\nu+1)e_\nu$. We define the norm $|\cdot|_p$ by $|f|_p = |A^p f|_0$ for $f \in \mathcal{S}$ and $p \in \mathbf{R}$, and let $\mathcal{S}_p$ be the completion of $\mathcal{S}$ with respect to the norm $|\cdot|_p$. Then the dual space $\mathcal{S}_p'$ of $\mathcal{S}_p$ is the same as $\mathcal{S}_{-p}$ (see [12]). We denote the complexifications of $L^2(\mathbf{R})$, $\mathcal{S}$ and $\mathcal{S}_p$ by $L^2_{\mathbf{C}}(\mathbf{R})$, $\mathcal{S}_{\mathbf{C}}$ and $\mathcal{S}_{\mathbf{C},p}$, respectively.

The space $(L^2) = L^2(\mathcal{S}^*, \mu)$ of complex-valued square-integrable functionals defined on $\mathcal{S}^*$ admits the well-known Wiener-Itô decomposition

$$(L^2) = \bigoplus_{n=0}^{\infty} H_n,$$

where H_n is the space of multiple Wiener integrals of order $n \in \mathbf{N}$ and $H_0 = \mathbf{C}$. Let $L^2_{\mathbf{C}}(\mathbf{R})^{\hat{\otimes} n}$ denote the n-fold symmetric tensor product of $L^2_{\mathbf{C}}(\mathbf{R})$. If $\varphi \in (L^2)$ is represented by $\varphi = \sum_{n=0}^\infty \mathbf{I}_n(f_n)$, $f_n \in L^2_{\mathbf{C}}(\mathbf{R})^{\hat{\otimes} n}$, then the (L^2)-norm $\|\varphi\|_0$ is given by

$$\|\varphi\|_0 = \left(\sum_{n=0}^\infty n!|f_n|_0^2\right)^{1/2},$$

where $|\cdot|_0$ also means the norm of $L^2_{\mathbf{C}}(\mathbf{R})^{\hat{\otimes} n}$.

For $p \in \mathbf{R}$, let $\|\varphi\|_p = \|\Gamma(A)^p\varphi\|_0$, where $\Gamma(A)$ is the second quantization operator of A. If $p \geq 0$, let $(\mathcal{S})_p$ be the domain of $\Gamma(A)^p$. If $p < 0$, let $(\mathcal{S})_p$ be the completion of (L^2) with respect to the norm $\|\cdot\|_p$. Then $(\mathcal{S})_p$, $p \in \mathbf{R}$, is a Hilbert space with the norm $||\cdot||_p$. It is easy to see that for $p > 0$, the dual space $(\mathcal{S})_p^*$ of

$(\mathcal{S})_p$ is given by $(\mathcal{S})_{-p}$. Moreover, for any $p \in \mathbf{R}$, we have the decomposition

$$(\mathcal{S})_p = \bigoplus_{n=0}^{\infty} H_n^{(p)},$$

where $H_n^{(p)}$ is the completion of $\{\mathbf{I}_n(f); f \in \mathcal{S}_{\mathbf{C}}^{\hat{\otimes} n}\}$ with respect to $\|\cdot\|_p$. Here $\mathcal{S}_{\mathbf{C}}^{\hat{\otimes} n}$ is the n-fold symmetric tensor product of $\mathcal{S}_{\mathbf{C}}$. We also have $H_n^{(p)} = \{\mathbf{I}_n(f); f \in \mathcal{S}_{\mathbf{C},p}^{\hat{\otimes} n}\}$ for any $p \in \mathbf{R}$, where $\mathcal{S}_{\mathbf{C},p}^{\hat{\otimes} n}$ is also the n-fold symmetric tensor product of $\mathcal{S}_{\mathbf{C},p}$. The norm $\|\varphi\|_p$ of $\varphi = \sum_{n=0}^{\infty} \mathbf{I}_n(f_n) \in (\mathcal{S})_p$ is given by

$$\|\varphi\|_p = \left(\sum_{n=0}^{\infty} n! |f_n|_p^2 \right)^{1/2}, \ f_n \in \mathcal{S}_{\mathbf{C},p}^{\hat{\otimes} n},$$

where the norm of $\mathcal{S}_{\mathbf{C},p}^{\hat{\otimes} n}$ is also denoted by $|\cdot|_p$.

The projective limit space $(\mathcal{S})$ of spaces $(\mathcal{S})_p$, $p \in \mathbf{R}$, is a nuclear space. The inductive limit space $(\mathcal{S})^*$ of spaces $(\mathcal{S})_p, p \in \mathbf{R}$, is nothing but the dual space of $(\mathcal{S})$. The space $(\mathcal{S})^*$ is called the space of *Hida distributions* or *generalized white noise functionals.* We denote by $\langle\langle \cdot, \cdot \rangle\rangle$ the canonical bilinear form on $(\mathcal{S})^* \times (\mathcal{S})$. Then we have

$$\langle\langle \Phi, \varphi \rangle\rangle = \sum_{n=0}^{\infty} n! \langle F_n, f_n \rangle$$

for any $\Phi = \sum_{n=0}^{\infty} \mathbf{I}_n(F_n) \in (\mathcal{S})^*$ and $\varphi = \sum_{n=0}^{\infty} \mathbf{I}_n(f_n) \in (\mathcal{S})$, where the canonical bilinear form on $(\mathcal{S}_{\mathbf{C}}^{\otimes n})^* \times (\mathcal{S}_{\mathbf{C}}^{\otimes n})$ is also denoted by $\langle \cdot, \cdot \rangle$.

Let $\phi_\xi = \exp\{-1/2\langle \xi, \xi \rangle\} \exp\langle \cdot, \xi \rangle$, $\xi \in E_{\mathbf{C}}$. Then $\{\phi_\xi; \xi \in E_{\mathbf{C}}\}$ spans a dense subspace of $(\mathcal{S})$. The *S-transform* is defined on $(\mathcal{S})^*$ by

$$S[\Phi](\xi) = \langle\langle \Phi, \phi_\xi \rangle\rangle, \qquad \xi \in \mathcal{S}_{\mathbf{C}}.$$

3. The generalized Gross Laplacian and the Lévy Laplacian

We first introduce the definitions of Laplacian operators following [16] (see also [9, 17]). For any $\varphi = \sum_{n=0}^{\infty} \mathbf{I}_n(f_n) \in (\mathcal{S})$, define the Gâteaux derivative D_y in any direction $y \in \mathcal{S}^*$ by

$$D_y \varphi = \sum_{n=0}^{\infty} n \mathbf{I}_{n-1}(\langle y, f_n \rangle).$$

The *white noise differential operator* ∂_t is defined to be the operator D_{δ_t} acting on $(\mathcal{S})$. For topological linear spaces E and F we denote the set of all continuous linear operators from E into F by $\mathcal{L}(E, F)$. For simplicity we denote $\mathcal{L}(E, E)$ by $\mathcal{L}(E)$. Then, for any fixed $y \in \mathcal{S}^*$, the operator D_y is in $\mathcal{L}((\mathcal{S}))$. Therefore the differentiation ∂_t is in $\mathcal{L}((\mathcal{S}))$ and its adjoint operator ∂_t^* is in $\mathcal{L}((\mathcal{S})^*)$. For any $\eta \in \mathcal{S}$, the differentiation D_η has a unique extension to a continuous linear operator $\widetilde{D}_\eta$ in $\mathcal{L}((\mathcal{S})^*)$ (for more details, see [16] or [22]).

For any $K \in \mathcal{L}(\mathcal{S}_{\mathbf{C}}^*, \mathcal{S}_{\mathbf{C}})$, the *generalized Gross Laplacian* $\Delta_G(K)$ is defined by

$$\Delta_G(K)\Phi = \sum_{n=0}^{\infty} \mathbf{I}_n \left((n+2)(n+1)\tau(K) \hat{\otimes}_2 F_{n+2} \right),$$

for $\Phi = \sum_{n=0}^{\infty} \mathbf{I}_n(F_n) \in (\mathcal{S})^*$, where $\tau(K)$ is defined by a generalized function in $(\mathcal{S}_{\mathbf{C}} \otimes \mathcal{S}_{\mathbf{C}})^*$ such that

$$\langle \tau(K), \xi \otimes \eta \rangle = \langle K\xi, \eta \rangle,$$

and $\tau(K)\hat{\otimes}_2 F_{n+2}$ is the right contraction (see [22]). The Laplacian $\Delta_G(K)$ is a continuous linear operator in $\mathcal{L}((\mathcal{S})^*)$.

Let T be a finite interval of $\mathbf{R}$ and let $\{\zeta_k; k \in \mathbf{N} \cup \{0\}\} \subset \mathcal{S}$ be a complete orthonormal system for $L^2(T)$ satisfying the equal density and the uniform boundedness (see [16, 17]). The *Lévy Laplacian* Δ_L^T *depending on* T is defined by

$$\Delta_L^T \Phi = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \widetilde{D}_{\zeta_k}^2 \Phi$$

if the limit exists in $(\mathcal{S})^*$. We denote the set of all functionals Φ such that $\Delta_L^T \Phi$ exists in $(\mathcal{S})^*$ and $S[\Phi](\eta) = 0$ for any $\eta \in \mathcal{S}_{\mathbf{C}}$ with *supp* $\eta \subset T^c$ by $\mathcal{D}_L^T$. From now on, we fix an interval T and use the notation Δ_L instead of Δ_L^T.

For $p \geq 1$ and $\Phi \in \mathcal{D}_L^T$, we define a $(-p)$-norm $|||\cdot|||_{-p}$ by

$$|||\Phi|||_{-p} = \left(\sum_{k=0}^{\infty} \|\Delta_L^k \Phi\|_{-p}^2 \right)^{1/2} \in [0, \infty]$$

and let $\mathcal{K}_{T,-p}$ denote the set of Hida distributions Φ in $(\mathcal{S})_{-p}$ such that $|||\Phi|||_{-p} < \infty$. Denote the completion of $\mathcal{K}_{T,-p}$ with respect to the norm $|||\cdot|||_{-p}$ by $\mathbf{D}_{-p}$. Then $\mathbf{D}_{-p}$ is a Hilbert space with the norm $|||\cdot|||_{-p}$. Since $\mathcal{K}_{T,-p}$ is dense in $\mathbf{D}_{-p}$ and $|||\Delta_L \Phi|||_{-p} \leq |||\Phi|||_{-p}$ for all $\Phi \in \mathcal{K}_{T,-p}$, the Lévy Laplacian can be extended to an operator $\overline{\Delta}_L$ in $\mathcal{L}(\mathbf{D}_{-p})$ such that $|||\overline{\Delta}_L \Phi|||_{-p} \leq |||\Phi|||_{-p}$ for all $\Phi \in \mathbf{D}_{-p}$. We denote $\overline{\Delta}_L$ by the same notation Δ_L from now on. We put $\mathbf{D} = \bigcup_{p=1}^{\infty} \mathbf{D}_{-p}$ with the inductive limit topology. Then Δ_L is a continuous linear operator in $\mathcal{L}(\mathbf{D})$.

For $p \geq 1$, let $\mathbf{E}_{-p}$ denote the normed linear space consisting of Hida distributions $\Phi = \sum_{n=0}^{\infty} \Phi_n$ in $\mathbf{D}_{-p}$ such that $\Phi_n \in \mathcal{N}_T \cap H_n^{(-p)}$ (see description 1 below for the definition of $\mathcal{N}_T$) for every $n \in \mathbf{N} \cup \{0\}$ and $\sum_{n=0}^{\infty} |||\Phi_n|||_{-p} < \infty$ with the norm $|||\cdot|||_{*,-p} = \sum_{n=0}^{\infty} |||(\cdot)_n|||_{-p}$. Then Δ_L is a bounded linear operator in $\mathcal{L}(\mathbf{E}_{-p})$ such that $|||\Delta_L \Phi|||_{*,-p} \leq |||\Phi|||_{*,-p}$ for $\Phi \in \mathbf{E}_{-p}$. We put $\mathbf{E} = \bigcup_{p=1}^{\infty} \mathbf{E}_{-p}$. Then Δ_L is also a continuous linear operator in $\mathcal{L}(\mathbf{E})$.

For any $p \geq 1$, the space $\mathbf{E}_{-p}$ includes the following functionals:

1) *Normal functionals*: A Hida distribution Φ is said to be *normal* if its S-transform $S[\Phi]$ is given by a finite linear combination of

$$\int_{T^k} f(u_1, \ldots, u_k) \xi(u_1)^{p_1} \cdots \xi(u_k)^{p_k} \, du_1 \cdots du_k,$$

where $f \in L^1(T^k)$ and $p_1, \ldots, p_k \in \mathbf{N} \cup \{0\}, k \in \mathbf{N}$. These functionals play an important role in the study of polynomials in the infinite dimensional analysis. Let $\mathcal{N}_T$ denote the set of all normal functionals in $\mathcal{D}_L^T$.

2) *Exponential functionals*: A Hida distribution

$$\Phi_c \equiv \mathcal{N} \exp\left[\frac{c}{2} \int_T x(u)^2 du \right], \mathcal{N}: \text{ normalizing factor, for any } c < 1/2,$$

is an important example as an eigen-functional of Δ_L. The S-transform of Φ_c is given by

$$S[\Phi_c](\xi) = \exp\left[\frac{c}{2(1-c)} \int_T \xi(u)^2 du \right], \ \xi \in \mathcal{S}_{\mathbf{C}}.$$

We introduce an operator $I_M \in \mathcal{L}((\mathcal{S})^*)$ by

$$I_M \Phi = \sum_{n=0}^{M} \Phi_n \text{ for } \Phi = \sum_{n=0}^{\infty} \Phi_n.$$

As in [3], the following result holds.

Theorem 1. *For any $\Phi \in \mathbf{E}$ we have*

$$\Delta_L \Phi = \lim_{M\to\infty} \lim_{N\to\infty} \frac{1}{N|T|} \Delta_G(P_N) I_M \Phi \textit{ in } (\mathcal{S})^*.$$

However, the limit in Theorem 1 converges in a stronger sense. In fact, we have the following.

Theorem 2. *For any $\Phi \in \mathbf{E}$ we have*

$$\Delta_L \Phi = \lim_{M\to\infty} \lim_{N\to\infty} \frac{1}{N|T|} \Delta_G(P_N) I_M \Phi \textit{ in } \mathbf{E}.$$

Proof. Let $\Phi \in \mathbf{E}$ be given with $\Phi = \sum_{n=0}^{\infty} \Phi_n$. Then there exists $p \geq 1$ such that for each $n \in \mathbf{N} \cup \{0\}$, $\Phi_n \in \mathcal{N}_T \cap H_n^{(-p)}$. Now, we shall show that for each $n \in \mathbf{N} \cup \{0\}$

$$\Delta_L \Phi_n = \lim_{N\to\infty} \left(\frac{1}{N|T|} \Delta_G(P_N) \right) \Phi_n \text{ in } \mathbf{E}.$$

Note that

$$\begin{aligned}
&\left|\left|\left| \Delta_L \Phi_n - \frac{1}{N|T|} \Delta_G(P_N) \Phi_n \right|\right|\right|_{*,-p}^2 \\
&= \left|\left|\left| \Delta_L \Phi_n - \frac{1}{N|T|} \Delta_G(P_N) \Phi_n \right|\right|\right|_{-p}^2 \\
&= \sum_{k=0}^{[(n-2)/2]} \left\| \Delta_L^k \left(\Delta_L \Phi_n - \frac{1}{N|T|} \Delta_G(P_N) \Phi_n \right) \right\|_{-p}^2 .
\end{aligned}$$

Since $\Delta_G(P_N)\Phi_n \in \mathcal{N}_T$, we can check that

$$\Delta_L^k \Delta_G(P_N) \Phi_n = \Delta_G(P_N) \Delta_L^k \Phi_n$$

for each $0 \leq k \leq [(n-2)/2]$ and $n \in \mathbf{N} \cup \{0\}$ by the direct calculation. Hence we have

$$\begin{aligned}
&\left|\left|\left| \Delta_L \Phi_n - \frac{1}{N|T|} \Delta_G(P_N) \Phi_n \right|\right|\right|_{*,-p}^2 \\
&= \sum_{k=0}^{[(n-2)/2]} \left\| \Delta_L(\Delta_L^k \Phi_n) - \frac{1}{N|T|} \Delta_G(P_N)(\Delta_L^k \Phi_n) \right\|_{-p}^2 .
\end{aligned}$$

On the other hand, by Theorem 1, for any $\Psi \in \mathcal{N}_T$ there exists $p \geq 1$ such that

$$\lim_{N\to\infty} \left\| \Delta_L \Psi - \frac{1}{N|T|} \Delta_G(P_N) \Psi \right\|_{-p} = 0.$$

Since $\Delta_L^k \Phi_n \in \mathcal{N}_T$, for any $0 \leq k \leq [(n-2)/2]$ there exists $p_k \geq 1$ such that

$$\lim_{N\to\infty} \left\| \Delta_L(\Delta_L^k \Phi_n) - \frac{1}{N|T|} \Delta_G(P_N)(\Delta_L^k \Phi_n) \right\|_{-p_k} = 0.$$

Take $p = \max\{p_k : 0 \le k \le [(n-2)/2]\}$. Then we have

$$\lim_{N\to\infty} \left|\left|\left| \Delta_L \Phi_n - \frac{1}{N|T|}\Delta_G(P_N)\Phi_n \right|\right|\right|_{*,-p} = 0.$$

Since $I_M\Phi = \sum_{n=0}^{M} \Phi_n$ for any $M \in \mathbf{N}\cup\{0\}$, we obtain that

$$\begin{aligned} \Delta_L(I_M\Phi) &= \sum_{n=0}^{M} \lim_{N\to\infty} \left(\frac{1}{N|T|}\Delta_G(P_N)\right)(\Phi_n) \\ &= \lim_{N\to\infty} \left(\frac{1}{N|T|}\Delta_G(P_N)\right)(I_M\Phi) \text{ in } \mathbf{E}. \end{aligned}$$

Hence, by $\Delta_L \in \mathcal{L}(\mathbf{E})$, we have

$$\lim_{M\to\infty}\lim_{N\to\infty} \left(\frac{1}{N|T|}\Delta_G(P_N)\right)(I_M\Phi) = \lim_{M\to\infty} \Delta_L(I_M\Phi) = \Delta_L\Phi \text{ in } \mathbf{E}.$$

Thus the proof is completed. □

By induction, Theorem 2 implies the following:

Corollary 3. *For any $\Phi \in \mathbf{E}$ and $\ell \in \mathbf{N}\cup\{0\}$, we have*

$$\Delta_L^\ell \Phi = \lim_{M\to\infty}\lim_{N\to\infty} \left(\frac{1}{N|T|}\right)^\ell (\Delta_G(P_N))^\ell (I_M\Phi) \quad \textit{in } \mathbf{E}.$$

4. A relation between groups generated by infinite dimensional Laplacians

For $K \in \mathcal{L}(\mathcal{S}_{\mathbf{C}}^*, \mathcal{S}_{\mathbf{C}})$, we define a group $e^{z\Delta_G(K)}$, $z \in \mathbf{C}$, by

$$e^{z\Delta_G(K)} = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{z^k}{k!}(\Delta_G(K))^k \in \mathcal{L}((\mathcal{S})^*).$$

We also define a C_0-group $e^{z\Delta_L}$, $z \in \mathbf{C}$, by

$$e^{z\Delta_L} = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{z^k}{k!}\Delta_L^k \in \mathcal{L}(\mathbf{D}).$$

It is easily checked that $e^{z\Delta_L}$ is in $\mathcal{L}(\mathbf{E})$ and for any $\Phi \in \mathbf{E}$ and $z \in \mathbf{C}$ there exists $p \ge 1$ such that $|||e^{z\Delta_L}\Phi|||_{*,-p} \le e^{|z|}|||\Phi|||_{*,-p}$. Then we get the following.

Theorem 4. *For any $\Phi \in \mathbf{E}$ and $z \in \mathbf{C}$, we have*

$$e^{z\Delta_L}\Phi = \lim_{M\to\infty}\lim_{N\to\infty} e^{z/(N|T|)\Delta_G(P_N)} I_M\Phi \textit{ in } \mathbf{E}.$$

Proof. Let $\Phi = \sum_{n=0}^{\infty} \Phi_n \in \mathbf{E}$. Then there exists some $p \geq 1$ such that Φ is in $\mathbf{E}_{-p}$. We have the following estimation:

$$\begin{aligned}
&\left|\left|\left| e^{z/(N|T|)\Delta_G(P_N)} I_M \Phi - e^{z\Delta_L}\Phi \right|\right|\right|_{-p} \\
&\leq \sum_{\ell=0}^{\infty} \frac{|z|^\ell}{\ell!} \left|\left|\left| \left(\frac{1}{N|T|}\right)^\ell (\Delta_G(P_N))^\ell I_M \Phi - \Delta_L^\ell \Phi \right|\right|\right|_{-p} \\
&\leq \sum_{\ell=0}^{\infty} \frac{|z|^\ell}{\ell!} \left[\sum_{k=0}^{M} \left|\left|\left| \left(\frac{1}{N|T|}\right)^\ell (\Delta_G(P_N))^\ell \Phi_k - \Delta_L^\ell \Phi_k \right|\right|\right|_{-p} \right. \\
&\qquad \left. + \left|\left|\left| \Delta_L^\ell \sum_{k=M+1}^{\infty} \Phi_k \right|\right|\right|_{-p} \right] \\
&= \sum_{\ell=0}^{[M/2]} \frac{|z|^\ell}{\ell!} \sum_{k=0}^{M} \left|\left|\left| \left(\frac{1}{N|T|}\right)^\ell (\Delta_G(P_N))^\ell \Phi_k - \Delta_L^\ell \Phi_k \right|\right|\right|_{-p} \\
&\qquad + \sum_{\ell=0}^{\infty} \frac{|z|^\ell}{\ell!} \left|\left|\left| \Delta_L^\ell \sum_{k=M+1}^{\infty} \Phi_k \right|\right|\right|_{-p}.
\end{aligned}$$

By Corollary 3, for any ℓ and $k \in \mathbf{N} \cup \{0\}$

$$\lim_{N\to\infty} \left|\left|\left| \left(\frac{1}{N|T|}\right)^\ell (\Delta_G(P_N))^\ell \Phi_k - \Delta_L^\ell \Phi_k \right|\right|\right|_{-p} = 0.$$

Hence we have

$$\lim_{M\to\infty} \lim_{N\to\infty} \sum_{\ell=0}^{[M/2]} \frac{|z|^\ell}{\ell!} \sum_{k=0}^{M} \left|\left|\left| \left(\frac{1}{N|T|}\right)^\ell (\Delta_G(P_N))^\ell \Phi_k - \Delta_L^\ell \Phi_k \right|\right|\right|_{-p} = 0.$$

Since

$$\left|\left|\left| \Delta_L^\ell \sum_{k=M+1}^{\infty} \Phi_k \right|\right|\right|_{-p} \leq \sum_{k=M+1}^{\infty} |||\Phi_k|||_{-p} \leq \sum_{k=0}^{\infty} |||\Phi_k|||_{-p} < \infty,$$

we have

$$\lim_{M\to\infty} \sum_{\ell=0}^{\infty} \frac{|z|^\ell}{\ell!} \left|\left|\left| \Delta_L^\ell \sum_{k=M+1}^{\infty} \Phi_k \right|\right|\right|_{-p} = 0.$$

Therefore we obtain

$$\lim_{M\to\infty} \lim_{N\to\infty} \left|\left|\left| e^{(z/(N|T|))\Delta_G(P_N)} I_M \Phi - e^{z\Delta_L}\Phi \right|\right|\right|_{-p} = 0.$$

This implies the assertion. □

5. Fundamental solution associated with the Lévy Laplacian

The Kubo-Yokoi delta function depending on T is a generalized white noise functional $\widetilde{\delta}_x$, $x \in \mathcal{S}^*$ whose S-transform is given by

$$S[\widetilde{\delta}_x](\xi) = \exp\left[\int_T x(u)\xi(u)du - \frac{1}{2}\int_T \xi(u)^2 du\right],\ \xi \in E.$$

The function $\widetilde{\delta} \equiv \widetilde{\delta}_0$ is also given by the limit of Φ_c (in §3, description 2) in $(\mathcal{S})^*$ as $c \to \infty$.

Put $L = \partial/\partial t - \alpha\Delta_L$. Consider the fundamental solution of L, i.e. a solution of

$$L\Phi(t,x) = \delta(t) \otimes \widetilde{\delta}(x). \tag{5.1}$$

Lemma 5. *For any $t \geq 0$, there exists $\Phi(t,x) \in \mathcal{S}^*_{\mathbf{C}} \otimes (\mathcal{S})^*$ such that*

$$\langle \Phi(t,\cdot), \eta \otimes \phi_\xi \rangle = e^{-\alpha t} \int_0^t \eta(u) e^{\alpha u} du S[\widetilde{\delta}](\xi), \;\; \eta, \xi \in \mathcal{S}_{\mathbf{C}}. \tag{5.2}$$

Proof. Let $F_t(u) = 1_{[0,t]}(u) e^{-\alpha(t-u)}$ for $t \geq 0$ and $u \in \mathbf{R}$, and set $\Phi(t,x) = F_t \otimes \widetilde{\delta}(x)$. Then $\Phi(t,x) \in \mathcal{S}^*_{\mathbf{C}} \otimes (\mathcal{S})^*$ and also

$$\langle \Phi(t,x), \eta \otimes \phi_\xi \rangle = \langle F_t, \eta \rangle \langle \widetilde{\delta}, \phi_\xi \rangle.$$

This implies (5.2). □

Theorem 6. *For any $t \geq 0$, $\Phi(t,x)$ given as in Lemma 5 is a solution of* (5.1).

Proof. By direct calculation, we have

$$\begin{aligned} \left\langle \left(\frac{\partial}{\partial t} - \alpha \Delta_L \right) \Phi(t,x), \eta \otimes \phi_\xi \right\rangle &= \left(\frac{\partial}{\partial t} - \alpha \widetilde{\Delta}_L \right) e^{-\alpha t} \int_0^t \eta(u) e^{\alpha u} du S[\widetilde{\delta}](\xi) \\ &= \eta(t) S[\widetilde{\delta}](\xi) \\ &= \langle \delta(t) \otimes \widetilde{\delta}, \eta \otimes \phi_\xi \rangle, \end{aligned}$$

where $\widetilde{\Delta}_L$ is defined by

$$\widetilde{\Delta}_L S[\Phi](\xi) = \lim_{N \to \infty} \frac{1}{N} \sum_{k=0}^{N-1} S[\Phi]''(\xi)(\zeta_k, \zeta_k), \qquad \Phi \in (\mathcal{S})^*, \xi \in \mathcal{S}_{\mathbf{C}},$$

if the limit exists in $S[(\mathcal{S})^*]$ (see [9, 16]). □

Concluding remarks

Let $\overline{\mathbf{E}}_{-p}$ be the completion of $\mathbf{E}_{-p}$ with respect to the norm $|||\cdot|||_{*,-p}$ for every $p \geq 1$. Then this becomes a Banach space with norm $|||\cdot|||_{*,-p}$. Since for any $p \geq 1$ the Laplacian Δ_L is in $\mathcal{L}(\mathbf{E}_{-p})$, it can be extended to a continuous linear operator in $\mathcal{L}(\overline{\mathbf{E}}_{-p})$. Our next problem is to discuss the operator in $\mathcal{L}(\overline{\mathbf{E}}_{-p})$ with the stochastic process associated with the operator. The results of this problem will appear elsewhere.

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