PROCEEDINGS OF THE AMERICAN MATHEMATICAL SOCIETY Volume 130, Number 6, Pages 1755-1764 S 0002-9939(01)06259-1 Article electronically published on October 24, 2001

ON THE DISTRIBUTION SINGULAR VALUES OF TOEPLITZ MATRICES

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(Communicated by David R. Larson)

ABSTRACT. We prove a second order formula concerning distribution of singular values of Toeplitz matrices in some cases when conditions of the H. Widom Theorem are not satisfied.

1. INTRODUCTION AND NOTATION

In 1920 G. Szegö proved a basic result concerning the distribution of the eigenvalues $\left\{\lambda_k^{(n)}\right\}_{k=1}^n$ of the Toeplitz matrix

$$T_{n}(f) = \left(\hat{f}_{i-j}\right)_{i,j=0}^{n-1} \left(\hat{f}_{k} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) e^{-ik\theta} \, d\theta\right)$$

associated with a bounded real valued function f on the interval $[-\pi,\pi]$: For any continuous function F one has

(1)
$$\lim_{n \to \infty} \frac{\sum_{k=1}^{n} F\left(\lambda_{k}^{(n)}\right)}{n} = \frac{1}{2\pi} \int_{-\pi}^{\pi} F\left(f(\theta)\right) d\theta$$

An analogous result holds for the singular values $s_1^{(n)} \ge s_2^{(n)} \ge \ldots \ge s_n^{(n)}$ of not necessarily selfadjoint Toeplitz matrices $T_n(f)$. The analogue of (1) is

(2)
$$\lim_{n \to \infty} \frac{\sum_{k=1}^{n} F\left(s_{k}^{(n)}\right)}{n} = \frac{1}{2\pi} \int_{-\pi}^{\pi} F\left(|f(\theta)|\right) d\theta.$$

Let

$$K = \left\{ f \in L^{\infty}(-\pi,\pi) : \sum_{k=-\infty}^{\infty} |k| |\hat{f}_k|^2 < \infty \right\},$$
$$M = \|f\|_{\infty}, \ m = dist \left(0, convR(f)\right)$$

where R(f) denotes the essential range of f and "conv" denotes the convex hull. For $f \in K$ we let

$$||f||_{K} = \left(\sum_{k \in \mathbb{Z}} |k| \left| \hat{f}_{k} \right|^{2} \right)^{\frac{1}{2}}.$$

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Received by the editors April 26, 2000 and, in revised form, December 15, 2000.

²⁰⁰⁰ Mathematics Subject Classification. Primary 47B10, 47B35.

Key words and phrases. Toeplitz matrices, singular values.

Then it is easy to see that each singular value of $T_n(f)$ belongs to the interval [m, M] (see Lemma 1.2 in [7]).

Let $t_k^{(n)} = \left(s_k^{(n)}\right)^2$. H. Widom [7] proved a more exact formula than (1) and (2). Namely he proved

Theorem 1. If $f \in K$ and $G \in C^3[m^2, M^2]$, then

(3)
$$\lim_{n \to \infty} \left(\sum_{k=1}^{n} G\left(t_{k}^{(n)} \right) - \frac{n}{2\pi} \int_{-\pi}^{\pi} G\left(|f(\theta)|^{2} \right) d\theta \right)$$
$$= \operatorname{tr} \left(G\left(T(\overline{f})T(f) \right) + G\left(T\left(f \right)T(\overline{f}) \right) - 2T\left(G(|f|^{2}) \right) \right).$$

We denote by T(f) the infinite Toeplitz matrix $(\hat{f}_{i-j})_{i,j=0}^{\infty}$ and by $H(f) = (\hat{f}_{i-j})_{i,j=0}^{\infty}$

 $\left(\hat{f}_{i+j+1}\right)_{i,j=0}^{\infty}$ the infinite Hankel matrix associated with the symbol f.

If $f \in K$, it is obvious that H(f) is a Hilbert-Schmidt operator. Moreover, $|H(f)|_2 \leq ||f||_K$, where $|\cdot|_2$ denotes the Hilbert-Schmidt norm of the operator H(f) acting on l^2 of the nonnegative integers. For the little bit of the theory of trace class (i.e., nuclear) and the Hilbert-Schmidt operators that will be needed we refer the reader to [3].

It is easy to see that $T(\overline{f}) = (T(f))^*$ and hence the operators $T(\overline{f})T(f)$ and $T(f)T(\overline{f})$ are selfadjoint. Therefore, operators $G(T(\overline{f})T(f))$ and $G(T(f)T(\overline{f}))$ are defined by the spectral theorem. If $f \ge 0$, the operator T(f) is obviously nonnegative.

In the case of eigenvalues, formulae similar to (3) are established in [1] and [4] but under much more restrictive assumptions on G (which is assumed to be analytic) and on the symbol f.

The function G in Theorem 1 is given in terms of the function F in relation (2) by $G(\lambda) = F(\sqrt{\lambda})$. For G to belong to C^3 it is not enough that F belongs to C^3 but we also must have F'(0) = F''(0) = F'''(0) = 0 (in the case when m = 0).

It is conjectured in [7] that the condition $F \in C^3[0, M]$, F'(0) = 0 (in the case m = 0) is sufficient for the statement of Theorem 1.

Essentially, in the case m = 0 it is necessary to prove Theorem 1 when $F(\lambda) = \lambda^{\beta}$ and β small enough. In this paper, we shall prove formula (3) in the case m = 0 and when the conditions of Theorem 1 are not satisfied.

2. Result

Theorem 2. Let $f \in K$ and m = 0. If $F(\lambda) = \lambda^{\alpha}$ $(\alpha \ge 2)$, or $F \in C^{6}[0, M]$, F'(0) = 0, then the operator

$$S = F\left(\sqrt{T(\overline{f})T(f)}\right) + F\left(\sqrt{T(f)T(\overline{f})}\right) - 2T\left(F(|f|)\right)$$

is of the trace class and

(4)
$$\lim_{n \to \infty} \left(\sum_{k=1}^{n} F\left(s_k^{(n)}\right) - \frac{n}{2\pi} \int_{-\pi}^{\pi} F\left(|f(\theta)|\right) d\theta \right) = \operatorname{tr} S.$$

Remark 1. Theorem 2 is stated for the case m = 0. If m > 0, (4) holds according to Theorem 1. (Of course, in that case the condition F'(0) = 0 is superficial.)

In the proof of Theorem 2 we will use the following Lemma of Lizorkin [5]:

Lemma 1. Let $\alpha \geq 1$, $\sigma > 0$. Then

$$(i\lambda)^{\alpha} = A \exp\left(-i\frac{\pi\alpha\lambda}{2\cdot\sigma}\right) \int_{-\infty}^{\infty} e^{-i(\lambda-\sigma)\xi} d\varrho(\xi) \qquad (|\lambda| \le \sigma)$$

where ϱ is a nondecreasing function of bounded variation and $A = \exp(-i\pi\alpha)$, $\varrho(\xi) = \sum_{k < \frac{\xi\sigma}{\pi}} a_k$, where a_k are positive numbers related to Fourier coefficients c_n of

the function $(i\lambda)^{\alpha} e^{i\lambda \frac{\pi\alpha}{2\sigma}}$ by

$$c_k = \overline{A}a_k e^{-i\,k\pi}.$$

Here the function $(i\lambda)^{\alpha} e^{i\lambda \frac{\pi \alpha}{2\sigma}}$ is assumed to be periodically extended from $(-\sigma, \sigma)$ to the entire real line. Our convention is $z^{\gamma} = e^{\gamma \ln z}$, $\ln z = \ln |z| + i \arg z$, $0 \leq \arg z < 2\pi$.

Lemma 2. If $1 < \alpha < 2$ and M > 0, then there exists a nondecreasing function ϱ_0 of bounded variation, such that for each $\lambda \in [0, M^2]$

$$\lambda^{\alpha} = \int_{-\infty}^{\infty} e^{-i\,2\pi\,\alpha - i\,M^2 t} e^{i\,\lambda\,t} d\varrho_0(t)$$

holds and $\int_{-\infty}^{\infty} t^2 d\varrho_0(t) < \infty.$

Proof. We apply Lemma 1, with $\sigma = M^2$. Since $\lambda \ge 0$, we have $(i \ \lambda)^{\alpha} = e^{i\frac{\pi}{2}\alpha} \cdot \lambda^{\alpha}$ and thus

$$\lambda^{\alpha} e^{-i\frac{\pi}{2}\alpha} = e^{-\pi\alpha} e^{i\lambda\frac{\pi\alpha}{2M^2}} \int_{-\infty}^{\infty} e^{i(\lambda-M^2)\xi} d\varrho(\xi),$$

i.e.,

$$\lambda^{\alpha} = e^{-\frac{3\pi i \,\alpha}{2}} \int_{-\infty}^{\infty} e^{-i\,M^2\xi} e^{i\,\lambda\left(\xi - \frac{\pi\,\alpha}{2M^2}\right)} \,d\varrho(\xi).$$

Substituting $\xi - \frac{\pi \alpha}{2M^2} = t$, $\varrho_0(t) \stackrel{def}{=} \varrho\left(t + \frac{\pi \alpha}{2M^2}\right)$ in the last formula we get

$$\lambda^{\alpha} = \int_{-\infty}^{\infty} e^{-2\pi i \alpha} e^{-i M^2 t} e^{i \lambda t} d\varrho_0(t).$$

The last formula holds for $\lambda \in [0, M^2]$ and $\alpha \ge 1$. Since ρ is a function of bounded variation, so is ρ_0 .

We will show now that for $1 < \alpha < 2$,

(5)
$$\int_{-\infty}^{\infty} t^2 d\varrho_0(t) < \infty.$$

Since ρ_0 is a function of bounded variation, applying the Cauchy inequality to (5) we get

$$\int_{-\infty}^{\infty} |t| \, d\varrho_0(t) < \infty.$$

Since $\rho_0(t) = \rho\left(t + \frac{\pi \alpha}{2M^2}\right)$, in order to prove (5) it is enough to show that

$$\int_{-\infty}^{\infty} t^2 \, d\varrho(t) < \infty.$$

From the way the function ρ is defined it follows that it suffices to prove

$$\sum_{n\in\mathbb{Z}}n^2a_n<\infty$$

where a_n is the sequence of positive numbers from Lemma 1 (with $\sigma = M^2$), i.e., the convergence of the series

(6)
$$\sum_{n\in\mathbb{Z}}(-1)^n n^2 c_n$$

where c_n are the Fourier coefficients of the function $(i\lambda)^{\alpha} e^{i\lambda \frac{\pi\alpha}{2M^2}}$ on the interval $[-M^2, M^2]$. Since

$$\int_{-M^2}^{M^2} (i\lambda)^{\alpha} \exp\left(\frac{i\lambda\pi\alpha}{2M^2}\right) \cdot \exp\left(-\frac{i\lambda n\pi}{M^2}\right) d\lambda$$
$$= \left(\frac{M^2}{\pi}\right)^{\alpha+1} \cdot \int_{-\pi}^{\pi} (ix)^{\alpha} e^{\frac{i\alpha\pi}{2}} e^{-inx} dx,$$

in order to prove the convergence of the series (6) it is enough to prove that for $1 < \alpha < 2$ the series

$$\sum_{n \in \mathbb{Z}} (-1)^n n^2 \int_{-\pi}^{\pi} (ix)^{\alpha} e^{\frac{i \alpha x}{2}} e^{-i nx} dx$$

converges.

Since $\alpha > 1$, integrating by parts twice and having in mind the definition of the function $z \mapsto z^{\gamma}$, we conclude that the convergence of the above series will be established once we prove that the series

$$\sum_{n\in\mathbb{Z}}(-1)^nA_n$$

converges for $1 < \alpha < 2$. Here $A_n \stackrel{def}{=} \int_{-\pi}^{\pi} (ix)^{\alpha-2} e^{\frac{i\alpha x}{2}} e^{-inx} dx$. Consider now the behavior of A_n as $n \to \infty$. If n > 0, one gets

$$A_n = \left(n - \frac{\alpha}{2}\right)^{1-\alpha} \left[-e^{\frac{i\pi\alpha}{2}} \cdot \int_0^{\pi\left(n - \frac{\alpha}{2}\right)} t^{\alpha - 2} e^{-it} dt - e^{\frac{3\pi i\alpha}{2}} \\ \cdot \int_0^{\pi\left(n - \frac{\alpha}{2}\right)} t^{\alpha - 2} e^{it} dt \right]$$

Since for $1 < \alpha < 2 \int_0^\infty t^{\alpha - 2} e^{\pm it} dt = \Gamma(\alpha - 1) e^{\pm i \frac{\pi}{2}(\alpha - 1)}$ we obtain

$$A_n = \left(n - \frac{\alpha}{2}\right)^{1-\alpha} \left[i\Gamma(\alpha - 1)(e^{2\pi i\alpha} - 1) + e^{\frac{i\pi\alpha}{2}} \cdot \int_{\pi\left(n - \frac{\alpha}{2}\right)}^{\infty} t^{\alpha - 2}e^{-it} dt + e^{\frac{3\pi i\alpha}{2}} \cdot \int_{\pi\left(n - \frac{\alpha}{2}\right)}^{\infty} t^{\alpha - 2}e^{it} dt\right].$$

Integrating by parts, we get

$$\begin{split} &\int_{\pi\left(n-\frac{\alpha}{2}\right)}^{\infty} t^{\alpha-2} e^{-it} dt = (-1)^{n+1} i \left(\pi \left(n-\frac{\alpha}{2}\right)\right)^{\alpha-2} \cdot e^{i\frac{\pi\alpha}{2}} + O\left(n^{\alpha-3}\right), \\ &\int_{\pi\left(n-\frac{\alpha}{2}\right)}^{\infty} t^{\alpha-2} e^{it} dt = (-1)^{n} i \left(\pi \left(n-\frac{\alpha}{2}\right)\right)^{\alpha-2} \cdot e^{-i\frac{\pi\alpha}{2}} + O\left(n^{\alpha-3}\right), \quad n \to \infty, \end{split}$$

and thus,

$$A_n = \left(n - \frac{\alpha}{2}\right)^{1-\alpha} \left[i\Gamma(\alpha - 1)\left(e^{2\pi i\alpha} - 1\right) + O\left(n^{\alpha-3}\right)\right].$$

Therefore, the series $\sum_{n=1}^{\infty} (-1)^n A_n$ converge. In a similar way one shows that the series

$$\sum_{n=-\infty}^{-1} (-1)^n A_n$$

also converge.

Remark 2. From Lemma 2 (by integrating over λ) we obtain the representation

(7)
$$\lambda^{\alpha+1} = \int_{-\infty}^{+\infty} e^{-2\pi i\alpha - iM^2 t} \cdot \frac{e^{i\lambda t} - 1}{it} (\alpha + 1) d\rho_0 (t),$$

for $1 < \alpha < 2, \lambda \in \left[0, M^2\right]$.

Let $d\nu = \frac{\alpha+1}{it} d\rho_0$. If $1 < \alpha < 2$, then the function ρ_0 does not have a jump at the t = 0, hence

$$\int_{-\infty}^{+\infty} |t|^k \, d \, |\nu| < \infty \quad \text{for } k = 0, 1, 2, 3$$

 $(|\nu|$ is a variation of measure ν). From (7), putting $\beta = \alpha + 1$, we get

(8)
$$\lambda^{\beta} = \int_{-\infty}^{+\infty} e^{-2\pi i\alpha - iM^2 t} e^{i\lambda t} d\nu(t) - A, \quad 2 < \beta < 3, \lambda \in [0, M^2],$$

and

$$A = \int_{-\infty}^{+\infty} e^{-2\pi i \alpha - i M^2 t} \, d\nu \left(t \right).$$

We write P_n for the projection operator, defined by

$$P_n(x_0, x_1, \dots) = (x_0, x_1, x_2, \dots, x_{n-1}, 0, 0, \dots)$$

from l^2 to the subspace of l^2 on which $T_n(f)$ may be thought of as acting. We identify $T_n(f)$ with $P_nT(f)P_n$ in the obvious way. We define an operator Q_n on l^2 by

$$Q_n(x_0, x_1, \dots) = (x_{n-1}, x_{n-2}, \dots, x_1, x_0, 0, 0, \dots).$$

For $f \in L^{\infty}(-\pi, \pi)$ we define $\tilde{f}(\theta) = f(-\theta)$.

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Lemma 3. 1) For any $f, g \in L^{\infty}(-\pi, \pi)$ we have

$$T(f \cdot g) - T(f) \cdot T(g) = H(f)H(\tilde{g}),$$

$$T_n(f \cdot g) - T_n(f) \cdot T_n(g) = P_nH(f)H(\tilde{g})P_n + Q_nH(\tilde{f})H(g)Q_n.$$
2) If $f \in K$, then we have $\tilde{f} \in K$, $|f|^2 \in K$, $e^{itf} \in K \; (\forall t \in \mathbb{R})$ and

$$\||f^2|\|_K \leq \operatorname{const} \|f\|_K,$$

$$\|e^{itf}\|_K \leq \operatorname{const} \cdot |t| \cdot \|f\|_K.$$

Proof. 1) Routine computation. (Or see [2], Propositions 2.7 and 3.6.)

2) Can be proved in a same way as Proposition 1 in [6].

Lemma 4. If $f \in K$, $1 < \alpha < 2$, then the operator $(T(|f|^2))^{\alpha} - T(|f|^{2\alpha})$ is nuclear.

Proof. Integrating the identity

$$\frac{d}{ds} \left(T\left(e^{i\,s|f|^2}\right) e^{-i\,sT\left(|f|^2\right)} \right)$$
$$= \left[T\left(e^{i\,s|f|^2} \cdot i|f|^2\right) - T\left(e^{i\,s|f|^2}\right) T\left(i\,|f|^2\right) \right] e^{-i\,sT\left(|f|^2\right)}$$

on the interval [0, t] and multiplying the result by $e^{i t T(|f|^2)}$ on the right, we get

(9)
$$T\left(e^{it|f|^2}\right) - e^{itT\left(|f|^2\right)} = \int_0^t H\left(e^{is|f|^2}\right) \cdot H\left(i\left|\tilde{f}\right|^2\right) e^{i(t-s)T\left(|f|^2\right)} ds.$$

Since $f \in K$, we have $e^{i s |f|^2} \in K$, $i |f|^2 \in K$. Applying Lemma 3, formula (9) yields

$$\left| T\left(e^{i\,t|f|^2}\right) - e^{i\,tT\left(|f|^2\right)} \right|_1 \le \int_0^{|t|} \left| H\left(e^{i\,s|f|^2}\right) \cdot H\left(i|\tilde{f}|^2\right) \right|_1 \, ds$$

for all $t \in \mathbb{R}$, since the operator $e^{i(t-s)T(|f|^2)}$ is unitary. Here $|\cdot|_1$ denotes the nuclear norm of an operator. Since the operators $H\left(e^{i\,s|f|^2}\right)$ and $H\left(i\left|\tilde{f}\right|^2\right)$ are Hilbert-Schmidt, their product is nuclear, and thus, according to Lemma 3 (statement 2)) we get

$$\left| H\left(e^{i\,s|f|^2}\right) \cdot H\left(i|\tilde{f}|^2\right) \right|_1 \le c_0 \cdot |s| \cdot \|f\|_K^2$$

 $(c_0 \text{ is independent of } s)$, and thus

(10)
$$\left| T\left(e^{it|f|^2}\right) - e^{itT\left(|f|^2\right)} \right|_1 \le c_0 \|f\|_K^2 \int_0^{|t|} s \, ds = \frac{c_0}{2} t^2 \|f\|_K^2.$$

According to Lemma 2

$$\left(T\left(|f|^2\right)\right)^{\alpha} = \int_{\mathbb{R}} e^{-2\pi i \,\alpha - iM^2 t} e^{itT\left(|f|^2\right)} \, d\varrho_0(t),$$
$$|f|^{2\alpha} = \int_{\mathbb{R}} e^{-2\pi i \,\alpha - iM^2 t} e^{it|f|^2} \, d\varrho_0(t),$$

and thus,

$$T\left(|f|^{2\alpha}\right) = \int e^{-2\pi i \alpha - iM^2 t} T\left(e^{it|f|^2}\right) d\varrho_0(t).$$

Therefore,

$$\left(\left(T\left(|f|^2\right)\right)^{\alpha} - T\left(|f|^{2\alpha}\right)\right) = \int_{\mathbb{R}} e^{-2\pi i \,\alpha - iM^2 t} \left(e^{itT\left(|f|^2\right)} - T\left(e^{it|f|^2}\right)\right) \, d\varrho_0(t).$$

Inequality (10) shows that the integral on the right side in the formula above, converges in nuclear norm and thus $((T(|f|^2))^{\alpha} - T(|f|^{2\alpha}))$ is a nuclear operator. Moreover,

$$\left| \left(\left(T\left(|f|^2 \right) \right)^{\alpha} - T\left(|f|^{2\alpha} \right) \right) \right|_1 \le \frac{c_0}{2} \|f\|_K^2 \int_{\mathbb{R}} t^2 \, d\varrho_0(t) < +\infty \quad \text{(by Lemma 2).}$$

Lemma 5. If $1 < \alpha < 2$ and $f \in K$, then

$$\lim_{n \to \infty} \operatorname{tr}\left(\left(T_n(|f|^2)\right)^{\alpha} - T_n(|f|^{2\alpha})\right) = 2\operatorname{tr}\left(\left(T(|f|^2)\right)^{\alpha} - T(|f|^{2\alpha})\right).$$

Proof. In a same way as we proved (9) we get

$$T_n\left(e^{i\,t|f|^2}\right) - e^{i\,tT_n\left(|f|^2\right)} = \int_0^t \left[T_n\left(e^{i\,s|f|^2}i\,|f|^2\right) - T_n\left(e^{i\,s|f|^2}\right)T_n\left(i\,|f|^2\right)\right]e^{i(t-s)T_n\left(|f|^2\right)}\,ds$$

and thus by Lemma 3

(11)
$$T_n\left(e^{i\,t|f|^2}\right) - e^{i\,tT_n\left(|f|^2\right)} = \int_0^t \left[P_nH\left(e^{i\,s|f|^2}\right)H\left(i\,|\tilde{f}|^2\right)P_n + Q_nH\left(e^{i\,s|\tilde{f}|^2}\right)H\left(i\,|f|^2\right)Q_n\right]e^{i(t-s)T_n\left(|f|^2\right)}\,ds.$$

From Lemma 2 we obtain

(12)
$$\begin{pmatrix} \left(T_n \left(|f|^2 \right) \right)^{\alpha} - T_n \left(|f|^{2\alpha} \right) \\ = \int_{\mathbb{R}} e^{-2\pi i \, \alpha - iM^2 t} \left(e^{itT_n \left(|f|^2 \right)} - T_n \left(e^{it|f|^2} \right) \right) \, d\varrho_0(t) \qquad (1 < \alpha < 2).$$

It follows from (11) that

$$\left|T_n\left(e^{i\,t|f|^2}\right) - e^{i\,tT_n\left(|f|^2\right)}\right|_1 \le \text{const}\ |t|^2, \quad \forall t \in \mathbb{R}.$$

(const does not depend on t and n) and thus, since $\int_{\mathbb{R}} |t|^2 d\varrho_0(t) < \infty$, by the same arguments as in the proof of (14) in [7] and by the Lebesgue theorem on dominant convergence, (11) and (12) give

$$\lim_{n \to \infty} \operatorname{tr}\left(\left(T_n(|f|^2)\right)^{\alpha} - T_n(|f|^{2\alpha})\right) = 2\operatorname{tr}\left(\left(T(|f|^2)\right)^{\alpha} - T(|f|^{2\alpha})\right).$$

Lemma 6. If $f \in K$, $1 < \alpha < 2$, then the operator $(T(\overline{f})T(f))^{\alpha} + (T(f)T(\overline{f}))^{\alpha} - 2T(|f|^{2\alpha})$ is nuclear and

$$\lim_{n \to \infty} \operatorname{tr} \left[\left(T_n(\overline{f}) T_n(f) \right)^{\alpha} - T_n(|f|^{2\alpha}) \right] \\ = \operatorname{tr} \left[\left(T(\overline{f}) T(f) \right)^{\alpha} + \left(T(f) T(\overline{f}) \right)^{\alpha} - 2T(|f|^{2\alpha}) \right].$$

Proof. By Lemma 2 we have

(13)
$$(T(\overline{f})T(f))^{\alpha} - T_n (|f|^2)^{\alpha}$$
$$= \int_{-\infty}^{\infty} e^{-2\pi i \ \alpha - i \ M^2 t} \cdot \left(e^{i \ t T(\overline{f})T(f)} - e^{i \ t T(|f|^2)} \right) d\varrho_0(t).$$

Since $\left|e^{i\,tT(\overline{f})T(f)} - e^{i\,tT(|f|^2)}\right|_1 \leq \text{const} \cdot |t| \ (t \in \mathbb{R}), \text{ and } \int_{\mathbb{R}} |t| \, d\varrho_0(t) < +\infty, \text{ the integral in (13) converges (in nuclear norm) and thus the operator <math>\left(T(\overline{f})T(f)\right)^{\alpha} - T\left(|f|^2\right)^{\alpha}$ is nuclear. In a similar way we prove that the operator $\left(T(f)T(\overline{f})\right)^{\alpha} - T_n\left(|f|^2\right)^{\alpha}$ is nuclear. Thus $\left(T(\overline{f})T(f)\right)^{\alpha} + \left(T(f)T(\overline{f})\right)^{\alpha} - 2T\left(|f|^2\right)^{\alpha}$ is also nuclear. Therefore, according to Lemma 4 the operator $\left(T(\overline{f})T(f)\right)^{\alpha} + \left(T(f)T(\overline{f})\right)^{\alpha} - 2T\left(|f|^2\alpha\right)$ is nuclear. Since $\left|e^{i\,tT_n(\overline{f})T(f)} - e^{i\,tT_n(|f|^2)}\right|_1 \leq d_0 \cdot |t| \ (t \in \mathbb{R}, \ d_0 \text{ is independent of } n \text{ and } t)$ and $\int_{-\infty}^{\infty} |t| \, d\varrho_0(t) < +\infty$, it follows from

$$\left(T_n(\overline{f})T_n(f) \right)^{\alpha} - T_n \left(|f|^2 \right)^{\alpha}$$

= $\int_{-\infty}^{\infty} e^{-2\pi i \, \alpha - i \, M^2 t} \left(e^{i \, t T_n(\overline{f})T(f)} - e^{i \, t T_n(|f|^2)} \right) \, d\varrho_0(t)$

that

$$\lim_{n \to \infty} \operatorname{tr} \left(\left(T_n(\overline{f}) T_n(f) \right)^{\alpha} - T_n \left(|f|^2 \right)^{\alpha} \right)$$
$$= \int_{-\infty}^{\infty} e^{-2\pi i \, \alpha - i \, M^2 t} \lim_{n \to \infty} \operatorname{tr} \left(e^{i \, t T_n(\overline{f}) T(f)} - e^{i \, t T_n(|f|^2)} \right) \, d\varrho_0(t).$$

From relation (14) in [7] and from Lemma 2, the last equality becomes

(14)
$$\lim_{n \to \infty} \operatorname{tr} \left(\left(T(\overline{f})T(f) \right)^{\alpha} - T_n \left(|f|^2 \right)^{\alpha} \right) \\ = \operatorname{tr} \left(\left(T(\overline{f})T(f) \right)^{\alpha} + \left(T(f)T(\overline{f}) \right)^{\alpha} - 2T \left(|f|^2 \right)^{\alpha} \right).$$

From (14) and Lemma 5, adding, we obtain

$$\lim_{n \to \infty} \operatorname{tr} \left[\left(T_n(\overline{f}) T_n(f) \right)^{\alpha} - T_n \left(|f|^{2\alpha} \right) \right]$$

= $\operatorname{tr} \left[\left(T(\overline{f}) T(f) \right)^{\alpha} + \left(T(f) T(\overline{f}) \right)^{\alpha} - 2T \left(|f|^{2\alpha} \right) \right].$

Remark 3. By using representation (8) by the same method as the one used for proving Lemmas 4, 5, 6 one can show that:

Lemma 7. If $f \in K$ and $2 < \beta < 3$, then the operator $T\left(|f|^2\right)^{\beta} - T\left(|f|^{2\beta}\right)$ is the trace class and the following holds:

$$\lim_{n \to \infty} \operatorname{tr} \left[T_n \left(|f|^2 \right)^{\beta} - T_n \left(|f|^{2\beta} \right) \right] = 2 \operatorname{tr} \left[T \left(|f|^2 \right)^{\beta} - T \left(|f|^{2\beta} \right) \right]$$

Also, The operator $(T(\overline{f})T(f))^{\beta} + (T(f)T(\overline{f}))^{\beta} - 2T(|f|^{2\beta})$ is the trace class and the following holds:

$$\lim_{n \to \infty} \operatorname{tr} \left[\left(T_n(\overline{f}) T_n(f) \right)^{\beta} - T_n \left(|f|^{2\beta} \right) \right] \\ = \operatorname{tr} \left[\left(T(\overline{f}) T(f) \right)^{\beta} + \left(T(f) T(\overline{f}) \right)^{\beta} - 2T \left(|f|^{2\beta} \right) \right].$$

3. Proof of Theorem 2

Note that Theorem 2 holds for the functions $\lambda \mapsto \lambda^2$, $\lambda \mapsto \lambda^4$ and $\lambda \mapsto \lambda^{\alpha}$ $(\alpha \ge 6)$ as a consequence of Theorem 1. In other words,

(15)
$$\lim_{n \to \infty} \left(\sum_{k=1}^{n} \left(s_{k}^{(n)} \right)^{\alpha} - \frac{n}{2\pi} \int_{-\pi}^{\pi} |f(\theta)|^{\alpha} d\theta \right) \\ = \operatorname{tr} \left[\left(T(\overline{f})T(f) \right)^{\frac{\alpha}{2}} + \left(T(f)T(\overline{f}) \right)^{\frac{\alpha}{2}} - 2T\left(|f|^{\alpha}\right) \right],$$

for $\alpha = 2, 4$ and $\alpha \ge 6$.

From Lemma 6 and Lemma 7 we obtain that (15) also holds if $2 < \alpha < 4$ and $4 < \alpha < 6$. Therefore, formula (4) holds if $F(\lambda) = \lambda^{\alpha}$ and $\alpha \ge 2$.

Now let $F \in C^6[0, M^2]$ and F'(0) = 0. Then, for the function $F_0(\lambda) = \sum_{k=2}^{6} \frac{F^{(k)}(0)}{k!} \lambda^k$ we have

(16)
$$\lim_{n \to \infty} \left(\sum_{k=1}^{n} F_0\left(s_k^{(n)}\right) - \frac{n}{2\pi} \int_{-\pi}^{\pi} F_0\left(|f(\theta)|\right) d\theta \right)$$
$$= \operatorname{tr} \left[F_0\left(\sqrt{T(\overline{f})T(f)}\right) + F_0\left(\sqrt{T(f)T(\overline{f})}\right) - 2T\left(F_0\left(|f|\right)\right) \right].$$

Let $R(\lambda) = F(\lambda) - F_0(\lambda)$. The function $\lambda \mapsto R(\sqrt{\lambda})$ satisfies the conditions of Theorem 1 and hence

(17)
$$\lim_{n \to \infty} \left(\sum_{k=1}^{n} R\left(s_{k}^{(n)}\right) - \frac{n}{2\pi} \int_{-\pi}^{\pi} R\left(|f(\theta)|\right) d\theta \right)$$
$$= \operatorname{tr} \left[R\left(\sqrt{T(\overline{f})T(f)}\right) + R\left(\sqrt{T(f)T(\overline{f})}\right) - 2T\left(R\left(|f|\right)\right) \right].$$

Adding (16) and (17) one gets (4). (The operators on the right-hand side of (16) and (17) are nuclear and so is their sum, i.e., the operator S is nuclear.)

Remark 4. The question of whether the condition F'(0) = 0 in Theorem 2 is necessary remains open. To answer it affirmatively it is enough to find the example of a function $f \in K$ such that m = dist(0, convR(f)) = 0 and

$$\lim_{n \to \infty} \left(\sum_{k=1}^{n} s_k^{(n)} - \frac{n}{2\pi} \int_{-\pi}^{\pi} |f(\theta)| \, d\theta - \operatorname{tr} \left[\sqrt{T(\overline{f})T(f)} + \sqrt{T(f)T(\overline{f})} - 2T(|f|) \right] \right) \neq 0.$$

Acknowledgment

The author is grateful to the referee for the suggestion which helped to improve this article.

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