# ON THE DISTRIBUTION SINGULAR VALUES OF TOEPLITZ MATRICES 

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#### Abstract

We prove a second order formula concerning distribution of singular values of Toeplitz matrices in some cases when conditions of the H. Widom Theorem are not satisfied.


## 1. Introduction and notation

In 1920 G. Szegö proved a basic result concerning the distribution of the eigenvalues $\left\{\lambda_{k}^{(n)}\right\}_{k=1}^{n}$ of the Toeplitz matrix

$$
T_{n}(f)=\left(\hat{f}_{i-j}\right)_{i, j=0}^{n-1}\left(\hat{f}_{k}=\frac{1}{2 \pi} \int_{-\pi}^{\pi} f(\theta) e^{-i k \theta} d \theta\right)
$$

associated with a bounded real valued function $f$ on the interval $[-\pi, \pi]$ : For any continuous function $F$ one has

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \frac{\sum_{k=1}^{n} F\left(\lambda_{k}^{(n)}\right)}{n}=\frac{1}{2 \pi} \int_{-\pi}^{\pi} F(f(\theta)) d \theta \tag{1}
\end{equation*}
$$

An analogous result holds for the singular values $s_{1}^{(n)} \geq s_{2}^{(n)} \geq \ldots \geq s_{n}^{(n)}$ of not necessarily selfadjoint Toeplitz matrices $T_{n}(f)$. The analogue of (1) is

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \frac{\sum_{k=1}^{n} F\left(s_{k}^{(n)}\right)}{n}=\frac{1}{2 \pi} \int_{-\pi}^{\pi} F(|f(\theta)|) d \theta \tag{2}
\end{equation*}
$$

Let

$$
\begin{gathered}
K=\left\{f \in L^{\infty}(-\pi, \pi): \sum_{k=-\infty}^{\infty}\left|k \| \hat{f}_{k}\right|^{2}<\infty\right\}, \\
M=\|f\|_{\infty}, m=\operatorname{dist}(0, \operatorname{conv} R(f))
\end{gathered}
$$

where $R(f)$ denotes the essential range of $f$ and "conv" denotes the convex hull. For $f \in K$ we let

$$
\|f\|_{K}=\left(\sum_{k \in \mathbb{Z}}|k|\left|\hat{f}_{k}\right|^{2}\right)^{\frac{1}{2}}
$$

[^0]Then it is easy to see that each singular value of $T_{n}(f)$ belongs to the interval [ $m, M$ ] (see Lemma 1.2 in [7]).

Let $t_{k}^{(n)}=\left(s_{k}^{(n)}\right)^{2}$. H. Widom [7] proved a more exact formula than (1) and (2)). Namely he proved
Theorem 1. If $f \in K$ and $G \in C^{3}\left[m^{2}, M^{2}\right]$, then

$$
\begin{align*}
& \lim _{n \rightarrow \infty}\left(\sum_{k=1}^{n} G\left(t_{k}^{(n)}\right)-\frac{n}{2 \pi} \int_{-\pi}^{\pi} G\left(|f(\theta)|^{2}\right) d \theta\right)  \tag{3}\\
& =\operatorname{tr}\left(G(T(\bar{f}) T(f))+G(T(f) T(\bar{f}))-2 T\left(G\left(|f|^{2}\right)\right)\right)
\end{align*}
$$

We denote by $T(f)$ the infinite Toeplitz matrix $\left(\hat{f}_{i-j}\right)_{i, j=0}^{\infty}$ and by $H(f)=$ $\left(\hat{f}_{i+j+1}\right)_{i, j=0}^{\infty}$ the infinite Hankel matrix associated with the symbol $f$.

If $f \in K$, it is obvious that $H(f)$ is a Hilbert-Schmidt operator. Moreover, $|H(f)|_{2} \leq\|f\|_{K}$, where $|\cdot|_{2}$ denotes the Hilbert-Schmidt norm of the operator $H(f)$ acting on $l^{2}$ of the nonnegative integers. For the little bit of the theory of trace class (i.e., nuclear) and the Hilbert-Schmidt operators that will be needed we refer the reader to [3].

It is easy to see that $T(\bar{f})=(T(f))^{*}$ and hence the operators $T(\bar{f}) T(f)$ and $T(f) T(\bar{f})$ are selfadjoint. Therefore, operators $G(T(\bar{f}) T(f))$ and $G(T(f) T(\bar{f}))$ are defined by the spectral theorem. If $f \geq 0$, the operator $T(f)$ is obviously nonnegative.

In the case of eigenvalues, formulae similar to (3) are established in [1] and [4] but under much more restrictive assumptions on $G$ (which is assumed to be analytic) and on the symbol $f$.

The function $G$ in Theorem 1 is given in terms of the function $F$ in relation (2) by $G(\lambda)=F(\sqrt{\lambda})$. For $G$ to belong to $C^{3}$ it is not enough that $F$ belongs to $C^{3}$ but we also must have $F^{\prime}(0)=F^{\prime \prime}(0)=F^{\prime \prime \prime}(0)=0$ (in the case when $m=0$ ).

It is conjectured in [7] that the condition $F \in C^{3}[0, M], F^{\prime}(0)=0$ (in the case $m=0$ ) is sufficient for the statement of Theorem 1.

Essentially, in the case $m=0$ it is necessary to prove Theorem 1 when $F(\lambda)=$ $\lambda^{\beta}$ and $\beta$ small enough. In this paper, we shall prove formula (3) in the case $m=0$ and when the conditions of Theorem 1 are not satisfied.

## 2. Result

Theorem 2. Let $f \in K$ and $m=0$. If $F(\lambda)=\lambda^{\alpha}(\alpha \geq 2)$, or $F \in C^{6}[0, M]$, $F^{\prime}(0)=0$, then the operator

$$
S=F(\sqrt{T(\bar{f}) T(f)})+F(\sqrt{T(f) T(\bar{f})})-2 T(F(|f|))
$$

is of the trace class and

$$
\begin{equation*}
\lim _{n \rightarrow \infty}\left(\sum_{k=1}^{n} F\left(s_{k}^{(n)}\right)-\frac{n}{2 \pi} \int_{-\pi}^{\pi} F(|f(\theta)|) d \theta\right)=\operatorname{tr} S . \tag{4}
\end{equation*}
$$

Remark 1. Theorem 2 is stated for the case $m=0$. If $m>0$, (4) holds according to Theorem (Of course, in that case the condition $F^{\prime}(0)=0$ is superficial.)

In the proof of Theorem 2 we will use the following Lemma of Lizorkin [5]:
Lemma 1. Let $\alpha \geq 1, \sigma>0$. Then

$$
(i \lambda)^{\alpha}=A \exp \left(-i \frac{\pi \alpha \lambda}{2 \cdot \sigma}\right) \int_{-\infty}^{\infty} e^{-i(\lambda-\sigma) \xi} d \varrho(\xi) \quad(|\lambda| \leq \sigma)
$$

where $\varrho$ is a nondecreasing function of bounded variation and $A=\exp (-i \pi \alpha)$, $\varrho(\xi)=\sum_{k<\frac{\xi \sigma}{\pi}} a_{k}$, where $a_{k}$ are positive numbers related to Fourier coefficients $c_{n}$ of the function $(i \lambda)^{\alpha} e^{i \lambda \frac{\pi \alpha}{2 \sigma}}$ by

$$
c_{k}=\bar{A} a_{k} e^{-i k \pi}
$$

Here the function $(i \lambda)^{\alpha} e^{i \lambda \frac{\pi \alpha}{2 \sigma}}$ is assumed to be periodically extended from $(-\sigma, \sigma)$ to the entire real line. Our convention is $z^{\gamma}=e^{\gamma \ln z}, \ln z=\ln |z|+i \arg z$, $0 \leq \arg z<2 \pi$.
Lemma 2. If $1<\alpha<2$ and $M>0$, then there exists a nondecreasing function $\varrho_{0}$ of bounded variation, such that for each $\lambda \in\left[0, M^{2}\right]$

$$
\lambda^{\alpha}=\int_{-\infty}^{\infty} e^{-i 2 \pi \alpha-i M^{2} t} e^{i \lambda t} d \varrho_{0}(t)
$$

holds and $\int_{-\infty}^{\infty} t^{2} d \varrho_{0}(t)<\infty$.
Proof. We apply Lemma 1 , with $\sigma=M^{2}$. Since $\lambda \geq 0$, we have $(i \lambda)^{\alpha}=e^{i \frac{\pi}{2} \alpha} \cdot \lambda^{\alpha}$ and thus

$$
\lambda^{\alpha} e^{-i \frac{\pi}{2} \alpha}=e^{-\pi \alpha} e^{i \lambda \frac{\pi \alpha}{2 M^{2}}} \int_{-\infty}^{\infty} e^{i\left(\lambda-M^{2}\right) \xi} d \varrho(\xi)
$$

i.e.,

$$
\lambda^{\alpha}=e^{-\frac{3 \pi i \alpha}{2}} \int_{-\infty}^{\infty} e^{-i M^{2} \xi} e^{i \lambda\left(\xi-\frac{\pi \alpha}{2 M^{2}}\right)} d \varrho(\xi)
$$

Substituting $\xi-\frac{\pi \alpha}{2 M^{2}}=t, \varrho_{0}(t) \stackrel{\text { def }}{=} \varrho\left(t+\frac{\pi \alpha}{2 M^{2}}\right)$ in the last formula we get

$$
\lambda^{\alpha}=\int_{-\infty}^{\infty} e^{-2 \pi i \alpha} e^{-i M^{2} t} e^{i \lambda t} d \varrho_{0}(t) .
$$

The last formula holds for $\lambda \in\left[0, M^{2}\right]$ and $\alpha \geq 1$. Since $\varrho$ is a function of bounded variation, so is $\varrho_{0}$.

We will show now that for $1<\alpha<2$,

$$
\begin{equation*}
\int_{-\infty}^{\infty} t^{2} d \varrho_{0}(t)<\infty \tag{5}
\end{equation*}
$$

Since $\varrho_{0}$ is a function of bounded variation, applying the Cauchy inequality to (5) we get

$$
\int_{-\infty}^{\infty}|t| d \varrho_{0}(t)<\infty
$$

Since $\varrho_{0}(t)=\varrho\left(t+\frac{\pi \alpha}{2 M^{2}}\right)$, in order to prove (5) it is enough to show that

$$
\int_{-\infty}^{\infty} t^{2} d \varrho(t)<\infty
$$

From the way the function $\varrho$ is defined it follows that it suffices to prove

$$
\sum_{n \in \mathbb{Z}} n^{2} a_{n}<\infty
$$

where $a_{n}$ is the sequence of positive numbers from Lemma 1 (with $\sigma=M^{2}$ ), i.e., the convergence of the series

$$
\begin{equation*}
\sum_{n \in \mathbb{Z}}(-1)^{n} n^{2} c_{n} \tag{6}
\end{equation*}
$$

where $c_{n}$ are the Fourier coefficients of the function $(i \lambda)^{\alpha} e^{i \lambda \frac{\pi \alpha}{2 M^{2}}}$ on the interval $\left[-M^{2}, M^{2}\right]$. Since

$$
\begin{gathered}
\int_{-M^{2}}^{M^{2}}(i \lambda)^{\alpha} \exp \left(\frac{i \lambda \pi \alpha}{2 M^{2}}\right) \cdot \exp \left(-\frac{i \lambda n \pi}{M^{2}}\right) d \lambda \\
=\left(\frac{M^{2}}{\pi}\right)^{\alpha+1} \cdot \int_{-\pi}^{\pi}(i x)^{\alpha} e^{\frac{i \alpha x}{2}} e^{-i n x} d x
\end{gathered}
$$

in order to prove the convergence of the series (6) it is enough to prove that for $1<\alpha<2$ the series

$$
\sum_{n \in \mathbb{Z}}(-1)^{n} n^{2} \int_{-\pi}^{\pi}(i x)^{\alpha} e^{\frac{i \alpha x}{2}} e^{-i n x} d x
$$

converges.
Since $\alpha>1$, integrating by parts twice and having in mind the definition of the function $z \longmapsto z^{\gamma}$, we conclude that the convergence of the above series will be established once we prove that the series

$$
\sum_{n \in \mathbb{Z}}(-1)^{n} A_{n}
$$

converges for $1<\alpha<2$. Here $A_{n} \stackrel{\text { def }}{=} \int_{-\pi}^{\pi}(i x)^{\alpha-2} e^{\frac{i \alpha x}{2}} e^{-i n x} d x$. Consider now the behavior of $A_{n}$ as $n \rightarrow \infty$. If $n>0$, one gets

$$
\begin{aligned}
A_{n}=\left(n-\frac{\alpha}{2}\right)^{1-\alpha}\left[-e^{\frac{i \pi \alpha}{2}} \cdot \int_{0}^{\pi\left(n-\frac{\alpha}{2}\right)}\right. & t^{\alpha-2} e^{-i t} d t-e^{\frac{3 \pi i \alpha}{2}} \\
& \left.\cdot \int_{0}^{\pi\left(n-\frac{\alpha}{2}\right)} t^{\alpha-2} e^{i t} d t\right]
\end{aligned}
$$

Since for $1<\alpha<2 \int_{0}^{\infty} t^{\alpha-2} e^{ \pm i t} d t=\Gamma(\alpha-1) e^{ \pm i \frac{\pi}{2}(\alpha-1)}$ we obtain

$$
\begin{aligned}
& A_{n}=\left(n-\frac{\alpha}{2}\right)^{1-\alpha}\left[i \Gamma(\alpha-1)\left(e^{2 \pi i \alpha}-1\right)\right.+e^{\frac{i \pi \alpha}{2}} \cdot \int_{\pi\left(n-\frac{\alpha}{2}\right)}^{\infty} t^{\alpha-2} e^{-i t} d t \\
&\left.+e^{\frac{3 \pi i \alpha}{2}} \cdot \int_{\pi\left(n-\frac{\alpha}{2}\right)}^{\infty} t^{\alpha-2} e^{i t} d t\right]
\end{aligned}
$$

Integrating by parts, we get

$$
\begin{aligned}
\int_{\pi\left(n-\frac{\alpha}{2}\right)}^{\infty} t^{\alpha-2} e^{-i t} d t & =(-1)^{n+1} i\left(\pi\left(n-\frac{\alpha}{2}\right)\right)^{\alpha-2} \cdot e^{i \frac{\pi \alpha}{2}}+O\left(n^{\alpha-3}\right) \\
\int_{\pi\left(n-\frac{\alpha}{2}\right)}^{\infty} t^{\alpha-2} e^{i t} d t & =(-1)^{n} i\left(\pi\left(n-\frac{\alpha}{2}\right)\right)^{\alpha-2} \cdot e^{-i \frac{\pi \alpha}{2}}+O\left(n^{\alpha-3}\right), \quad n \rightarrow \infty
\end{aligned}
$$

and thus,

$$
A_{n}=\left(n-\frac{\alpha}{2}\right)^{1-\alpha}\left[i \Gamma(\alpha-1)\left(e^{2 \pi i \alpha}-1\right)+O\left(n^{\alpha-3}\right)\right]
$$

Therefore, the series $\sum_{n=1}^{\infty}(-1)^{n} A_{n}$ converge. In a similar way one shows that the series

$$
\sum_{n=-\infty}^{-1}(-1)^{n} A_{n}
$$

also converge.
Remark 2. From Lemma 2 (by integrating over $\lambda$ ) we obtain the representation

$$
\begin{equation*}
\lambda^{\alpha+1}=\int_{-\infty}^{+\infty} e^{-2 \pi i \alpha-i M^{2} t} \cdot \frac{e^{i \lambda t}-1}{i t}(\alpha+1) d \rho_{0}(t) \tag{7}
\end{equation*}
$$

for $1<\alpha<2, \lambda \in\left[0, M^{2}\right]$.
Let $d \nu=\frac{\alpha+1}{i t} d \rho_{0}$. If $1<\alpha<2$, then the function $\rho_{0}$ does not have a jump at the $t=0$, hence

$$
\int_{-\infty}^{+\infty}|t|^{k} d|\nu|<\infty \quad \text { for } k=0,1,2,3
$$

( $|\nu|$ is a variation of measure $\nu$ ). From (77), putting $\beta=\alpha+1$, we get

$$
\begin{equation*}
\lambda^{\beta}=\int_{-\infty}^{+\infty} e^{-2 \pi i \alpha-i M^{2} t} e^{i \lambda t} d \nu(t)-A, \quad 2<\beta<3, \lambda \in\left[0, M^{2}\right] \tag{8}
\end{equation*}
$$

and

$$
A=\int_{-\infty}^{+\infty} e^{-2 \pi i \alpha-i M^{2} t} d \nu(t)
$$

We write $P_{n}$ for the projection operator, defined by

$$
P_{n}\left(x_{0}, x_{1} \ldots\right)=\left(x_{0}, x_{1}, x_{2}, \ldots, x_{n-1}, 0,0, \ldots\right)
$$

from $l^{2}$ to the subspace of $l^{2}$ on which $T_{n}(f)$ may be thought of as acting. We identify $T_{n}(f)$ with $P_{n} T(f) P_{n}$ in the obvious way. We define an operator $Q_{n}$ on $l^{2}$ by

$$
Q_{n}\left(x_{0}, x_{1}, \ldots\right)=\left(x_{n-1}, x_{n-2}, \ldots, x_{1}, x_{0}, 0,0, \ldots\right)
$$

For $f \in L^{\infty}(-\pi, \pi)$ we define $\tilde{f}(\theta)=f(-\theta)$.

Lemma 3. 1) For any $f, g \in L^{\infty}(-\pi, \pi)$ we have

$$
\begin{aligned}
& T(f \cdot g)-T(f) \cdot T(g)=H(f) H(\tilde{g}) \\
& T_{n}(f \cdot g)-T_{n}(f) \cdot T_{n}(g)=P_{n} H(f) H(\tilde{g}) P_{n}+Q_{n} H(\tilde{f}) H(g) Q_{n}
\end{aligned}
$$

2) If $f \in K$, then we have $\tilde{f} \in K,|f|^{2} \in K$, $e^{i t f} \in K(\forall t \in \mathbb{R})$ and

$$
\begin{aligned}
& \left\|\mid f^{2}\right\| \|_{K} \leq \text { const }\|f\|_{K} \\
& \left\|e^{i t f}\right\|_{K} \leq \text { const } \cdot|t| \cdot\|f\|_{K}
\end{aligned}
$$

Proof. 1) Routine computation. (Or see [2], Propositions 2.7 and 3.6.)
2) Can be proved in a same way as Proposition 1 in [6].

Lemma 4. If $f \in K, 1<\alpha<2$, then the operator $\left(T\left(|f|^{2}\right)\right)^{\alpha}-T\left(|f|^{2 \alpha}\right)$ is nuclear.

Proof. Integrating the identity

$$
\begin{aligned}
& \frac{d}{d s}\left(T\left(e^{i s|f|^{2}}\right) e^{-i s T\left(|f|^{2}\right)}\right) \\
& \quad=\left[T\left(e^{i s|f|^{2}} \cdot i|f|^{2}\right)-T\left(e^{i s|f|^{2}}\right) T\left(i|f|^{2}\right)\right] e^{-i s T\left(|f|^{2}\right)}
\end{aligned}
$$

on the interval $[0, t]$ and multiplying the result by $e^{i t T\left(|f|^{2}\right)}$ on the right, we get

$$
\begin{equation*}
T\left(e^{i t|f|^{2}}\right)-e^{i t T\left(|f|^{2}\right)}=\int_{0}^{t} H\left(e^{i s|f|^{2}}\right) \cdot H\left(i|\tilde{f}|^{2}\right) e^{i(t-s) T\left(|f|^{2}\right)} d s \tag{9}
\end{equation*}
$$

Since $f \in K$, we have $e^{i s|f|^{2}} \in K, i|f|^{2} \in K$. Applying Lemma 3, formula (9) yields

$$
\left|T\left(e^{i t|f|^{2}}\right)-e^{i t T\left(|f|^{2}\right)}\right|_{1} \leq \int_{0}^{|t|}\left|H\left(e^{i s|f|^{2}}\right) \cdot H\left(i|\tilde{f}|^{2}\right)\right|_{1} d s
$$

for all $t \in \mathbb{R}$, since the operator $e^{i(t-s) T\left(|f|^{2}\right)}$ is unitary. Here $|\cdot|_{1}$ denotes the nuclear norm of an operator. Since the operators $H\left(e^{i s|f|^{2}}\right)$ and $H\left(i|\tilde{f}|^{2}\right)$ are HilbertSchmidt, their product is nuclear, and thus, according to Lemma 3 (statement 2)) we get

$$
\left|H\left(e^{i s|f|^{2}}\right) \cdot H\left(i|\tilde{f}|^{2}\right)\right|_{1} \leq c_{0} \cdot|s| \cdot\|f\|_{K}^{2}
$$

( $c_{0}$ is independent of $s$ ), and thus

$$
\begin{equation*}
\left|T\left(e^{i t|f|^{2}}\right)-e^{i t T\left(|f|^{2}\right)}\right|_{1} \leq c_{0}\|f\|_{K}^{2} \int_{0}^{|t|} s d s=\frac{c_{0}}{2} t^{2}\|f\|_{K}^{2} \tag{10}
\end{equation*}
$$

According to Lemma 2

$$
\begin{aligned}
\left(T\left(|f|^{2}\right)\right)^{\alpha} & =\int_{\mathbb{R}} e^{-2 \pi i \alpha-i M^{2} t} e^{i t T\left(|f|^{2}\right)} d \varrho_{0}(t) \\
|f|^{2 \alpha} & =\int_{\mathbb{R}} e^{-2 \pi i \alpha-i M^{2} t} e^{i t|f|^{2}} d \varrho_{0}(t)
\end{aligned}
$$

and thus,

$$
T\left(|f|^{2 \alpha}\right)=\int e^{-2 \pi i \alpha-i M^{2} t} T\left(e^{i t|f|^{2}}\right) d \varrho_{0}(t)
$$

Therefore,

$$
\left(\left(T\left(|f|^{2}\right)\right)^{\alpha}-T\left(|f|^{2 \alpha}\right)\right)=\int_{\mathbb{R}} e^{-2 \pi i \alpha-i M^{2} t}\left(e^{i t T\left(|f|^{2}\right)}-T\left(e^{i t|f|^{2}}\right)\right) d \varrho_{0}(t) .
$$

Inequality (10) shows that the integral on the right side in the formula above, converges in nuclear norm and thus $\left(\left(T\left(|f|^{2}\right)\right)^{\alpha}-T\left(|f|^{2 \alpha}\right)\right)$ is a nuclear operator. Moreover,

$$
\left|\left(\left(T\left(|f|^{2}\right)\right)^{\alpha}-T\left(|f|^{2 \alpha}\right)\right)\right|_{1} \leq \frac{c_{0}}{2}\|f\|_{K}^{2} \int_{\mathbb{R}} t^{2} d \varrho_{0}(t)<+\infty \quad \text { (by Lemma 2). }
$$

Lemma 5. If $1<\alpha<2$ and $f \in K$, then

$$
\lim _{n \rightarrow \infty} \operatorname{tr}\left(\left(T_{n}\left(|f|^{2}\right)\right)^{\alpha}-T_{n}\left(|f|^{2 \alpha}\right)\right)=2 \operatorname{tr}\left(\left(T\left(|f|^{2}\right)\right)^{\alpha}-T\left(|f|^{2 \alpha}\right)\right) .
$$

Proof. In a same way as we proved (9) we get

$$
\begin{aligned}
& T_{n}\left(e^{i t|f|^{2}}\right)-e^{i t T_{n}\left(|f|^{2}\right)} \\
& \quad=\int_{0}^{t}\left[T_{n}\left(e^{i s|f|^{2}} i|f|^{2}\right)-T_{n}\left(e^{i s|f|^{2}}\right) T_{n}\left(i|f|^{2}\right)\right] e^{i(t-s) T_{n}\left(|f|^{2}\right)} d s
\end{aligned}
$$

and thus by Lemma 3

$$
\begin{align*}
T_{n}\left(e^{i t|f|^{2}}\right) & -e^{i t T_{n}\left(|f|^{2}\right)}=\int_{0}^{t}\left[P_{n} H\left(e^{i s|f|^{2}}\right) H\left(i|\tilde{f}|^{2}\right) P_{n}\right.  \tag{11}\\
& \left.+Q_{n} H\left(e^{i s|\tilde{f}|^{2}}\right) H\left(i|f|^{2}\right) Q_{n}\right] e^{i(t-s) T_{n}\left(|f|^{2}\right)} d s .
\end{align*}
$$

From Lemma 2 we obtain

$$
\begin{align*}
& \left(\left(T_{n}\left(|f|^{2}\right)\right)^{\alpha}-T_{n}\left(|f|^{2 \alpha}\right)\right) \\
& \quad=\int_{\mathbb{R}} e^{-2 \pi i \alpha-i M^{2} t}\left(e^{i t T_{n}\left(|f|^{2}\right)}-T_{n}\left(e^{i t|f|^{2}}\right)\right) d \varrho_{0}(t) \quad(1<\alpha<2) . \tag{12}
\end{align*}
$$

It follows from (11) that

$$
\left|T_{n}\left(e^{i t|f|^{2}}\right)-e^{i t T_{n}\left(|f|^{2}\right)}\right|_{1} \leq \text { const }|t|^{2}, \quad \forall t \in \mathbb{R}
$$

(const does not depend on $t$ and $n$ ) and thus, since $\int_{\mathbb{R}}|t|^{2} d \varrho_{0}(t)<\infty$, by the same arguments as in the proof of (14) in [7] and by the Lebesgue theorem on dominant convergence, (111) and (12) give

$$
\lim _{n \rightarrow \infty} \operatorname{tr}\left(\left(T_{n}\left(|f|^{2}\right)\right)^{\alpha}-T_{n}\left(|f|^{2 \alpha}\right)\right)=2 \operatorname{tr}\left(\left(T\left(|f|^{2}\right)\right)^{\alpha}-T\left(|f|^{2 \alpha}\right)\right) .
$$

Lemma 6. If $f \in K, 1<\alpha<2$, then the operator $(T(\bar{f}) T(f))^{\alpha}+(T(f) T(\bar{f}))^{\alpha}-$ $2 T\left(|f|^{2 \alpha}\right)$ is nuclear and

$$
\begin{aligned}
& \lim _{n \rightarrow \infty} \operatorname{tr}\left[\left(T_{n}(\bar{f}) T_{n}(f)\right)^{\alpha}-T_{n}\left(|f|^{2 \alpha}\right)\right] \\
& \quad=\operatorname{tr}\left[(T(\bar{f}) T(f))^{\alpha}+(T(f) T(\bar{f}))^{\alpha}-2 T\left(|f|^{2 \alpha}\right)\right] .
\end{aligned}
$$

Proof. By Lemma 2 we have

$$
\begin{align*}
& (T(\bar{f}) T(f))^{\alpha}-T_{n}\left(|f|^{2}\right)^{\alpha} \\
& =\int_{-\infty}^{\infty} e^{-2 \pi i \alpha-i M^{2} t} \cdot\left(e^{i t T(\bar{f}) T(f)}-e^{i t T\left(|f|^{2}\right)}\right) d \varrho_{0}(t) \tag{13}
\end{align*}
$$

Since $\left|e^{i t T(\bar{f}) T(f)}-e^{i t T\left(|f|^{2}\right)}\right|_{1} \leq$ const $\cdot|t|(t \in \mathbb{R})$, and $\int_{\mathbb{R}}|t| d \varrho_{0}(t)<+\infty$, the integral in (13) converges (in nuclear norm) and thus the operator $(T(\bar{f}) T(f))^{\alpha}$ $T\left(|f|^{2}\right)^{\alpha}$ is nuclear. In a similar way we prove that the operator $(T(f) T(\bar{f}))^{\alpha}$ $T_{n}\left(|f|^{2}\right)^{\alpha}$ is nuclear. Thus $(T(\bar{f}) T(f))^{\alpha}+(T(f) T(\bar{f}))^{\alpha}-2 T\left(|f|^{2}\right)^{\alpha}$ is also nuclear. Therefore, according to Lemma4 the operator $(T(\bar{f}) T(f))^{\alpha}+(T(f) T(\bar{f}))^{\alpha}-$ $2 T\left(|f|^{2 \alpha}\right)$ is nuclear. Since $\left|e^{i t T_{n}(\bar{f}) T(f)}-e^{i t T_{n}\left(|f|^{2}\right)}\right|_{1} \leq d_{0} \cdot|t|\left(t \in \mathbb{R}, d_{0}\right.$ is independent of $n$ and $t$ ) and $\int_{-\infty}^{\infty}|t| d \varrho_{0}(t)<+\infty$, it follows from

$$
\begin{aligned}
& \left(T_{n}(\bar{f}) T_{n}(f)\right)^{\alpha}-T_{n}\left(|f|^{2}\right)^{\alpha} \\
& \quad=\int_{-\infty}^{\infty} e^{-2 \pi i \alpha-i M^{2} t}\left(e^{i t T_{n}(\bar{f}) T(f)}-e^{i t T_{n}\left(|f|^{2}\right)}\right) d \varrho_{0}(t)
\end{aligned}
$$

that

$$
\begin{aligned}
& \lim _{n \rightarrow \infty} \operatorname{tr}\left(\left(T_{n}(\bar{f}) T_{n}(f)\right)^{\alpha}-T_{n}\left(|f|^{2}\right)^{\alpha}\right) \\
& \quad=\int_{-\infty}^{\infty} e^{-2 \pi i \alpha-i M^{2} t} \lim _{n \rightarrow \infty} \operatorname{tr}\left(e^{i t T_{n}(\bar{f}) T(f)}-e^{i t T_{n}\left(|f|^{2}\right)}\right) d \varrho_{0}(t)
\end{aligned}
$$

From relation (14) in [7] and from Lemma 2, the last equality becomes

$$
\begin{align*}
& \lim _{n \rightarrow \infty} \operatorname{tr}\left((T(\bar{f}) T(f))^{\alpha}-T_{n}\left(|f|^{2}\right)^{\alpha}\right)  \tag{14}\\
& \quad=\operatorname{tr}\left((T(\bar{f}) T(f))^{\alpha}+(T(f) T(\bar{f}))^{\alpha}-2 T\left(|f|^{2}\right)^{\alpha}\right)
\end{align*}
$$

From (14) and Lemma 5 adding, we obtain

$$
\begin{aligned}
& \lim _{n \rightarrow \infty} \operatorname{tr}\left[\left(T_{n}(\bar{f}) T_{n}(f)\right)^{\alpha}-T_{n}\left(|f|^{2 \alpha}\right)\right] \\
& \quad=\operatorname{tr}\left[(T(\bar{f}) T(f))^{\alpha}+(T(f) T(\bar{f}))^{\alpha}-2 T\left(|f|^{2 \alpha}\right)\right]
\end{aligned}
$$

Remark 3. By using representation (8) by the same method as the one used for proving Lemmas 4, 5, 6 one can show that:
Lemma 7. If $f \in K$ and $2<\beta<3$, then the operator $T\left(|f|^{2}\right)^{\beta}-T\left(|f|^{2 \beta}\right)$ is the trace class and the following holds:

$$
\lim _{n \longrightarrow \infty} \operatorname{tr}\left[T_{n}\left(|f|^{2}\right)^{\beta}-T_{n}\left(|f|^{2 \beta}\right)\right]=2 \operatorname{tr}\left[T\left(|f|^{2}\right)^{\beta}-T\left(|f|^{2 \beta}\right)\right] .
$$

Also, The operator $(T(\bar{f}) T(f))^{\beta}+(T(f) T(\bar{f}))^{\beta}-2 T\left(|f|^{2 \beta}\right)$ is the trace class and the following holds:

$$
\begin{aligned}
& \lim _{n \rightarrow \infty} \operatorname{tr}\left[\left(T_{n}(\bar{f}) T_{n}(f)\right)^{\beta}-T_{n}\left(|f|^{2 \beta}\right)\right] \\
& \quad=\operatorname{tr}\left[(T(\bar{f}) T(f))^{\beta}+(T(f) T(\bar{f}))^{\beta}-2 T\left(|f|^{2 \beta}\right)\right] .
\end{aligned}
$$

## 3. Proof of Theorem 2

Note that Theorem 2 holds for the functions $\lambda \longmapsto \lambda^{2}, \lambda \longmapsto \lambda^{4}$ and $\lambda \longmapsto \lambda^{\alpha}$ $(\alpha \geq 6)$ as a consequence of Theorem 1 . In other words,

$$
\begin{align*}
& \lim _{n \rightarrow \infty}\left(\sum_{k=1}^{n}\left(s_{k}^{(n)}\right)^{\alpha}-\frac{n}{2 \pi} \int_{-\pi}^{\pi}|f(\theta)|^{\alpha} d \theta\right)  \tag{15}\\
& \quad=\operatorname{tr}\left[(T(\bar{f}) T(f))^{\frac{\alpha}{2}}+(T(f) T(\bar{f}))^{\frac{\alpha}{2}}-2 T\left(|f|^{\alpha}\right)\right]
\end{align*}
$$

for $\alpha=2,4$ and $\alpha \geq 6$.
From Lemma 6 and Lemma 7 we obtain that (15) also holds if $2<\alpha<4$ and $4<\alpha<6$. Therefore, formula (44) holds if $F(\lambda)=\lambda^{\alpha}$ and $\alpha \geq 2$.

Now let $F \in C^{6}\left[0, M^{2}\right]$ and $F^{\prime}(0)=0$. Then, for the function $F_{0}(\lambda)=$ $\sum_{k=2}^{6} \frac{F^{(k)}(0)}{k!} \lambda^{k}$ we have

$$
\begin{align*}
& \lim _{n \rightarrow \infty}\left(\sum_{k=1}^{n} F_{0}\left(s_{k}^{(n)}\right)-\frac{n}{2 \pi} \int_{-\pi}^{\pi} F_{0}(|f(\theta)|) d \theta\right)  \tag{16}\\
& \quad=\operatorname{tr}\left[F_{0}(\sqrt{T(\bar{f}) T(f)})+F_{0}(\sqrt{T(f) T(\bar{f})})-2 T\left(F_{0}(|f|)\right)\right]
\end{align*}
$$

Let $R(\lambda)=F(\lambda)-F_{0}(\lambda)$. The function $\lambda \longmapsto R(\sqrt{\lambda})$ satisfies the conditions of Theorem 1 and hence

$$
\begin{align*}
& \lim _{n \rightarrow \infty}\left(\sum_{k=1}^{n} R\left(s_{k}^{(n)}\right)-\frac{n}{2 \pi} \int_{-\pi}^{\pi} R(|f(\theta)|) d \theta\right)  \tag{17}\\
& \quad=\operatorname{tr}[R(\sqrt{T(\bar{f}) T(f)})+R(\sqrt{T(f) T(\bar{f})})-2 T(R(|f|))]
\end{align*}
$$

Adding (16) and (17) one gets (4). (The operators on the right-hand side of (16) and (17) are nuclear and so is their sum, i.e., the operator $S$ is nuclear.)
Remark 4. The question of whether the condition $F^{\prime}(0)=0$ in Theorem 2 is necessary remains open. To answer it affirmatively it is enough to find the example of a function $f \in K$ such that $m=\operatorname{dist}(0, \operatorname{conv} R(f))=0$ and

$$
\begin{aligned}
\lim _{n \rightarrow \infty}( & \sum_{k=1}^{n} s_{k}^{(n)}-\frac{n}{2 \pi} \int_{-\pi}^{\pi}|f(\theta)| d \theta \\
& -\operatorname{tr}[\sqrt{T(\bar{f}) T(f)}+\sqrt{T(f) T(\bar{f})}-2 T(|f|)]) \neq 0
\end{aligned}
$$

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