LOGARITHMIC CONVEXITY OF EXTENDED MEAN VALUES

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ABSTRACT. In this article, the logarithmic convexity of the extended mean values are proved and an inequality of mean values is presented. As by-products, two analytic inequalities are derived. Two open problems are proposed.

1. INTRODUCTION

The so-called extended mean values $E(r, s; x, y)$ were first defined by Professor K. B. Stolarsky in [21] as follows:

$$(1) \quad E(r, s; x, y) = \left[\frac{r}{s} \cdot \frac{y^s - x^s}{y^r - x^r} \right]^{1/(s-r)}, \quad rs(r-s)(x-y) \neq 0,$$

$$(2) \quad E(r, 0; x, y) = \left[\frac{1}{r} \cdot \frac{y^r - x^r}{\ln y - \ln x} \right]^{1/r}, \quad r(x-y) \neq 0,$$

$$(3) \quad E(r, r; x, y) = e^{-1/r} \left(\frac{x^{x^r}}{y^{y^r}} \right)^{1/(x^r - y^r)}, \quad r(x-y) \neq 0,$$

$$(4) \quad \begin{aligned} E(0, 0; x, y) &= \sqrt{xy}, & x &\neq y, \\ E(r, s; x, x) &= x, & x &= y. \end{aligned}$$

Define a function g by

$$(5) \quad g(t) = g(t; x, y) = \begin{cases} \frac{(y^t - x^t)}{t}, & t \neq 0, \\ \ln y - \ln x, & t = 0. \end{cases}$$

It is easy to see that g can be expressed in integral form as

$$(6) \quad g(t; x, y) = \int_x^y u^{t-1} du, \quad t \in \mathbb{R},$$

and

$$(7) \quad g^{(n)}(t) = \int_x^y (\ln u)^n u^{t-1} du, \quad t \in \mathbb{R}.$$

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Therefore, the extended mean values can be represented [8, 19] in terms of g by

$$(8) \quad E(r, s; x, y) = \begin{cases} \left(\frac{g(s; x, y)}{g(r; x, y)} \right)^{1/(s-r)}, & (r-s)(x-y) \neq 0, \\ \exp \left(\frac{\partial g(r; x, y)/\partial r}{g(r; x, y)} \right), & r = s, x - y \neq 0, \end{cases}$$

and

$$(9) \quad \ln E(r, s; x, y) = \begin{cases} \frac{1}{s-r} \int_r^s \frac{\partial g(t; x, y)/\partial t}{g(t; x, y)} dt, & (r-s)(x-y) \neq 0, \\ \frac{\partial g(r; x, y)/\partial r}{g(r; x, y)}, & r = s, x - y \neq 0. \end{cases}$$

Leach and Sholander [3] showed that $E(r, s; x, y)$ are increasing with both r and s , or with both x and y . The monotonicities of E have also been researched by the author and others in [1] and [13]–[19] using different ideas and simpler methods.

Leach and Sholander [4] and Páles [7] solved the problem of comparison of E ; that is, they found necessary and sufficient conditions for the parameters r, s, u, v in order that

$$(10) \quad E(r, s; x, y) \leq E(u, v; x, y)$$

be satisfied for all positive x and y .

Most of two variable means are special cases of E , for example [2],

$$(11) \quad E(1, 2; x, y) = A(x, y), \quad E(1, 1; x, y) = I(x, y), \quad E(0, 1; x, y) = L(x, y).$$

They are called the arithmetic mean, the identric mean, and the logarithmic mean, respectively.

Recently, the concepts of mean values have been generalized by the author in [9]–[12].

The main purpose of this paper is to verify the logarithmic convexity of the extended mean values $E(r, s; x, y)$. As applications, an inequality among the arithmetic mean, the identric mean and the logarithmic mean is established; two open problems are proposed. As by-products, an inequality for the exponential function is obtained.

2. LOGARITHMIC CONVEXITY OF $E(r, s; x, y)$

In order to prove our main result, the following lemma is necessary.

Lemma 1 ([19]). *If $f(t)$ is an increasing integrable function on I , then the arithmetic mean of function $f(t)$,*

$$(12) \quad \phi(r, s) = \begin{cases} \frac{1}{s-r} \int_r^s f(t) dt, & r \neq s, \\ f(r), & r = s, \end{cases}$$

is also increasing with both r and s on I .

If f is a twice-differentiable convex function, then the function $\phi(r, s)$ is also convex with both r and s on I .

Proof. Direct calculation yields

$$(13) \quad \frac{\partial \phi(r, s)}{\partial s} = \frac{1}{(s-r)^2} \left[(s-r)f(s) - \int_r^s f(t) dt \right],$$

$$(14) \quad \frac{\partial^2 \phi(r, s)}{\partial s^2} = \frac{(s-r)^2 f'(s) - 2(s-r)f(s) + 2 \int_r^s f(t) dt}{(s-r)^3} \equiv \frac{\varphi(r, s)}{(s-r)^3},$$

$$(15) \quad \frac{\partial \varphi(r, s)}{\partial s} = (s-r)^2 f''(s).$$

In the case of $f(t)$ being increasing, we have $\partial \phi(r, s)/\partial s \geq 0$, thus $\phi(r, s)$ increases in both r and s , since $\phi(r, s) = \phi(s, r)$.

In the case of $f''(t) \geq 0$, $\varphi(r, s)$ increases with s . Since $\varphi(r, r) = 0$, we have $\partial^2 \phi(r, s)/\partial s^2 \geq 0$. Therefore $\phi(r, s)$ is convex with respect to either r or s , since $\phi(r, s) = \phi(s, r)$. This completes the proof. $\square$

By formula (9) and the above Lemma, we can see that, in order to prove the logarithmic convexity of the extended mean values $E(r, s; x, y)$, it suffices to verify the convexity of function

$$g'(t)/g(t) \triangleq g'_t(t; x, y)/g(t; x, y) \triangleq (\partial g(t; x, y)/\partial t)/g(t; x, y)$$

with respect to t , where $g(t) = g(t; x, y)$ is defined by (5) or (6).

Straightforward computation results in

$$(16) \quad \left(\frac{g'(t)}{g(t)} \right)' = \frac{g''(t)g(t) - [g'(t)]^2}{g^2(t)},$$

$$(17) \quad \left(\frac{g'(t)}{g(t)} \right)'' = \frac{g^2(t)g'''(t) - 3g(t)g'(t)g''(t) + 2[g'(t)]^3}{g^3(t)}.$$

For $y > x = 1$, expanding $g(t; 1, y)$ into series at $t_0 = 0$ with respect to t directly gives us

$$(18) \quad \begin{aligned} g(t; 1, y) &= \sum_{i=0}^{\infty} \frac{(\ln y)^{i+1}}{(i+1)!} t^i, \\ g'_t(t; 1, y) &= \sum_{i=0}^{\infty} \frac{(i+1)(\ln y)^{i+2}}{(i+2)!} t^i, \\ g''_t(t; 1, y) &= \sum_{i=0}^{\infty} \frac{(i+1)(i+2)(\ln y)^{i+3}}{(i+3)!} t^i, \\ g'''_t(t; 1, y) &= \sum_{i=0}^{\infty} \frac{(i+1)(i+2)(i+3)(\ln y)^{i+4}}{(i+4)!} t^i. \end{aligned}$$

From the four fundamental operations of arithmetic and suitable properties of series, we have

$$(19) \quad g^2(t; 1, y) = 2 \sum_{k=0}^{\infty} \frac{(2^{k+1} - 1)(\ln y)^{k+2}}{(k+2)!} t^k,$$

$$(20) \quad g(t; 1, y)g'_t(t; 1, y) = \sum_{k=0}^{\infty} \frac{(k+1)(2^{k+2} - 1)(\ln y)^{k+3}}{(k+3)!} t^k,$$

$$(21) \quad g(t; 1, y)g_t''(t; 1, y) + [g_t'(t; 1, y)]^2 = \sum_{k=0}^{\infty} \frac{(k+1)(k+2)(2^{k+3}-1)(\ln y)^{k+4}}{(k+4)!} t^k.$$

By standard arguments for series, we can get two combinatorial identities:

$$(22) \quad \sum_{i=0}^k (i+1) \binom{k+3}{i+2} = (k+1)(2^{k+2}-1),$$

$$(23) \quad \sum_{i=0}^k (i+1)(k-i+1) \binom{k+4}{i+2} = 2(k+3)(1+k \cdot 2^{k+1}).$$

By further computation, the following expansions are obtained:

$$(24) \quad \begin{aligned} g^2(t; 1, y)g_t'''(t; 1, y) &\equiv \sum_{k=0}^{\infty} \alpha_k (\ln y)^{k+6} t^k \\ &= \sum_{k=0}^{\infty} \frac{2}{(k+6)!} \sum_{i=0}^k (i+1)(i+2)(i+3)(2^{k-i+1}-1) \binom{k+6}{i+4} (\ln y)^{k+6} t^k, \end{aligned}$$

$$(25) \quad \begin{aligned} g(t; 1, y)g_t'(t; 1, y)g_t''(t; 1, y) &\equiv \sum_{k=0}^{\infty} \beta_k (\ln y)^{k+6} t^k \\ &= \sum_{k=0}^{\infty} \frac{1}{(k+6)!} \sum_{i=0}^k (i+1)(i+2)(k-i+1)(2^{k-i+2}-1) \binom{k+6}{i+3} (\ln y)^{k+6} t^k, \end{aligned}$$

$$(26) \quad \begin{aligned} g_t'(t; 1, y)[g_t'(t; 1, y)]^2 + g(t; 1, y)g_t''(t; 1, y) &\equiv \sum_{k=0}^{\infty} \gamma_k (\ln y)^{k+6} t^k \\ &= \sum_{k=0}^{\infty} \frac{1}{(k+6)!} \sum_{i=0}^k (i+1)(i+2)(k-i+1)(2^{i+3}-1) \binom{k+6}{i+4} (\ln y)^{k+6} t^k. \end{aligned}$$

Proposition 1. For an arbitrary nonnegative integer k , we have

$$(27) \quad \begin{aligned} 2 \sum_{i=0}^k (i+1)(i+2) \{ (i+3)2^{k-i+1} + (k-i+1)2^{i+3} - (k+4) \} \binom{k+6}{i+4} \\ \leq 5 \sum_{i=0}^k (i+1)(i+2)(k-i+1)(2^{k-i+2}-1) \binom{k+6}{i+3}. \end{aligned}$$

Proof. The inequality (27) is equivalent to

$$(28) \quad \begin{aligned} \sum_{j=0}^{k+6} (j-3)(j-2) [(k-j+5)2^j + (j-1)2^{k-j+6} - 2(k+4)] \binom{k+6}{j} \\ + (k^2 + 12k + 38)(2^{k+6} + k + 3) \\ \leq 5 \sum_{j=0}^{k+6} (j-2)(j-1)(k-j+4)(2^{k-j+5}-1) \binom{k+6}{j}. \end{aligned}$$

Using expansions-into-series of $(1 + 2x)^n$ and its derivatives, we get

$$\begin{aligned}
 (29) \quad & \sum_{i=0}^n 2^i \binom{n}{i} = 3^n, \\
 & \sum_{i=0}^n i 2^i \binom{n}{i} = 2n 3^{n-1}, \\
 & \sum_{i=0}^n i(i-1) 2^i \binom{n}{i} = 4n(n-1) 3^{n-2}, \\
 & \sum_{i=0}^n i(i-1)(i-2) 2^i \binom{n}{i} = 8n(n-1)(n-2) 3^{n-3}.
 \end{aligned}$$

Similarly, using expansions-into-series of $(1 + x/2)^n$ and its derivatives, we have

$$\begin{aligned}
 (30) \quad & \sum_{j=0}^n \frac{1}{2^j} \binom{n}{j} = \left(\frac{3}{2}\right)^n, \\
 & \sum_{j=0}^n \frac{j}{2^j} \binom{n}{j} = \frac{n}{2} \left(\frac{3}{2}\right)^{n-1}, \\
 & \sum_{j=0}^n \frac{j(j-1)}{2^j} \binom{n}{j} = \frac{n(n-1)}{4} \left(\frac{3}{2}\right)^{n-2}, \\
 & \sum_{j=0}^n \frac{j(j-1)(j-2)}{2^j} \binom{n}{j} = \frac{n(n-1)(n-2)}{8} \left(\frac{3}{2}\right)^{n-3}.
 \end{aligned}$$

The following formulae are also well-known:

$$\begin{aligned}
 (31) \quad & \sum_{j=0}^n \binom{n}{j} = 2^n, \\
 & \sum_{j=0}^n j \binom{n}{j} = n 2^{n-1}, \\
 & \sum_{j=0}^n j(j-1) \binom{n}{j} = n(n-1) 2^{n-2}, \\
 & \sum_{j=0}^n j(j-1)(j-2) \binom{n}{j} = n(n-1)(n-2) 2^{n-3}.
 \end{aligned}$$

Substitution of (29), (30) and (31) into (28) and simplification give us

$$(32) \quad (k^3 + 15k^2 + 74k + 114) + (k^3 + 15k^2 + 74k + 168)2^{k+3} - 2 \times 3^{k+6} \leq 0.$$

From Taylor's expansion

$$(33) \quad a^x = \sum_{i=0}^{\infty} \frac{(\ln a)^i}{i!} x^i,$$

the inequality (32) reduces to

$$\begin{aligned}
 & 1458 + (666 + 1344 \ln 2)k + 8[84(\ln 2)^2 + 74 \ln 2 + 15]k^2 + k^3 \\
 (34) \quad & + 8 \sum_{i=3}^{\infty} \left[\frac{168(\ln 2)^3}{i(i-1)(i-2)} + \frac{74(\ln 2)^2}{(i-1)(i-2)} + \frac{15 \ln 2}{i-2} + 1 \right] \frac{(\ln 2)^{i-3}}{(i-3)!} \cdot k^i \\
 & \leq 1458 \sum_{i=0}^{\infty} \frac{(\ln 3)^i}{i!} \cdot k^i.
 \end{aligned}$$

To prove inequality (34), by equating the coefficients of k^i in (34), it is sufficient to verify that

$$(35) \quad 4[168(\ln 2)^3 + 74(\ln 2)^2 i + 15i(i-1) \ln 2 + i(i-1)(i-2)] \leq 729(\ln 2)^3 (\log_2 3)^i$$

holds for $i \geq 4$.

Let

$$(36) \quad \begin{aligned} \psi(x) = & 4\{x^3 + (15 \ln 2 - 3)x^2 + [2 - 15 \ln 2 + 74(\ln 2)^2]x + 168(\ln 2)^3\} \\ & - 729(\ln 2)^3 (\log_2 3)^x \end{aligned}$$

for $x \geq 4$. By standard argument, we have

$$\begin{aligned}
 \psi'(x) &= 4\{3x^2 + 2(15 \ln 2 - 3)x + 2 - 15 \ln 2 + 74(\ln 2)^2\} \\
 &\quad - 729(\ln 2)^3 \ln(\log_2 3)(\log_2 3)^x, \\
 \psi''(x) &= 4\{6x + 2(15 \ln 2 - 3)\} - 729(\ln 2)^3 [\ln(\log_2 3)]^2 (\log_2 3)^x, \\
 \psi'''(x) &= 24 - 729(\ln 2)^3 [\ln(\log_2 3)]^3 (\log_2 3)^x, \\
 \psi^{(4)}(x) &= -729(\ln 2)^3 (\log_2 3)^x (\ln(\log_2 3))^4 < 0.
 \end{aligned}$$

Direct computation yields

$$\begin{aligned}
 \psi'''(4) &= 24 - 729(\ln 2)^3 [\ln(\log_2 3)]^3 (\log_2 3)^4 < -125, \\
 \psi''(4) &= 4(18 + 30 \ln 2) - 729(\ln 2)^3 [\ln(\log_2 3)]^2 (\log_2 3)^4 < -169, \\
 \psi'(4) &= 4\{26 + 105 \ln 2 + 74(\ln 2)^2\} - 729(\ln 2)^3 \ln(\log_2 3)(\log_2 3)^4 < -168, \\
 \psi(4) &= 4\{24 + 180 \ln 2 + 296(\ln 2)^2 + 168(\ln 2)^3\} - 729(\ln 2)^3 (\log_2 3)^4 < -144.
 \end{aligned}$$

Since the function $\psi'''(x)$ is decreasing, then $\psi'''(x) < 0$ for $x \geq 4$, and $\psi''(x) < 0$, $\psi'(x) < 0$, and then $\psi(x) < 0$ for $x \geq 4$. Inequality (35) follows. This completes the proof. $\square$

Remark 1. In fact, inequality $\psi(x) < 0$ holds for $x \geq 0$.

Corollary 1. For any nonnegative number $x \geq 0$, we have

$$(37) \quad \frac{1 + 3^{x+3}}{1 + 2^{x+3}} \geq \frac{x^3 + 15x^2 + 74x + 168}{54}.$$

Proposition 2. If $y > x = 1$, then, for $t \geq 0$,

$$(38) \quad g^2(t; 1, y)g_t'''(t; 1, y) - 3g(t; 1, y)g_t'(t; 1, y)g_t''(t; 1, y) + 2[g_t'(t; 1, y)]^3 \leq 0.$$

Proof. It is clear that

$$\begin{aligned}
 & g^2(t)g'''(t) - 3g(t)g'(t)g''(t) + 2[g'(t)]^3 \\
 (39) \quad & = g^2(t)g'''(t) - 5(g(t)g'(t))g''(t) + 2g'(t)[g(t)g''(t) + (g'(t))^2] \\
 & = \sum_{k=0}^{\infty} (\alpha_k - 5\beta_k + 2\gamma_k)(\ln y)^{k+6} t^k.
 \end{aligned}$$

Furthermore, Proposition 1 implies

$$(40) \quad \alpha_k - 5\beta_k + 2\gamma_k \leq 0, \quad k \geq 0.$$

The proof of Proposition 2 is completed. $\square$

Theorem 1. For all fixed $x, y > 0$ and $s \in [0, +\infty)$ (or $r \in [0, +\infty)$, respectively), the extended mean values $E(r, s; x, y)$ are logarithmically concave in r (or in s , respectively) on $[0, +\infty)$. For all fixed $x, y > 0$ and $s \in (-\infty, 0]$ (or $r \in (-\infty, 0]$, respectively), the extended mean values $E(r, s; x, y)$ are logarithmically convex in r (or in s , respectively) on $(-\infty, 0]$.

Proof. The combination of Proposition 2 with equality (17) proves that $\frac{g'_t(t; 1, y)}{g(t; 1, y)}$ is concave on $[0, +\infty)$ with t for $y > x = 1$. Therefore, from Lemma 1, it follows that the extended mean values $E(r, s; 1, y)$ are logarithmically concave on $[0, +\infty)$ with respect to either r or s for $y > x = 1$.

By standard arguments, we obtain

$$(41) \quad E(r, s; x, y) = xE(r, s; 1, \frac{y}{x}),$$

$$(42) \quad E(-r, -s; x, y) = \frac{xy}{E(r, s; x, y)}.$$

Hence, $E(r, s; x, y)$ are logarithmically concave on $[0, +\infty)$ with either r or s , respectively, and logarithmically convex on $(-\infty, 0]$ in either r or s , respectively. The proof of Theorem 1 is completed. $\square$

Corollary 2. The logarithmic means $L(r; x, y) = (L(x^r, y^r))^{1/r}$ and the extended logarithmic means $S_r(x, y) = E(r, 1; x, y)$ are logarithmically concave on $[0, +\infty)$ with respect to r .

Corollary 3. For any $y > x > 0$, if $t \geq 0$, then

$$(43) \quad g^2(t; x, y)g_t'''(t; x, y) - 3g(t; x, y)g_t'(t; x, y)g_t''(t; x, y) + 2[g_t'(t; x, y)]^3 \leq 0.$$

If $t \leq 0$, the inequality (43) reverses.

As concrete applications of the logarithmic convexity of $E(r, s; x, y)$, an inequality of mean values is given as follows.

Theorem 2. Let $A(x, y)$, $L(x, y)$ and $I(x, y)$ denote the arithmetic mean, the logarithmic mean, and the identric mean of two variables x and y . Then, for $x \neq y$, we have

$$(44) \quad I(x, y) > \frac{L(x, y) + A(x, y)}{2}.$$

Proof. P. Montel [5, p. 19] verified that a positive function $f(x)$ is logarithmically convex if and only if $x \mapsto e^{ax}f(x)$ is a convex function for all real values of a . Thus, applied this conclusion to the function $e^{ar}E(r, s; x, y)$, for any given $a \in \mathbb{R}$ and $x, y > 0$, $x \neq y$, we get

$$(45) \quad I(x, y) \geq \frac{e^{-a}L(x, y) + e^aA(x, y)}{2} \geq \sqrt{A(x, y)L(x, y)}.$$

The proof is complete. $\square$

3. MISCELLANEA

A function $f(t)$ is said to be absolutely monotonic on (a, b) if it has derivatives of all orders and $f^{(k)}(t) \geq 0$, $t \in (a, b)$, $k \in \mathbb{N}$.

A function $f(t)$ is said to be completely monotonic on (a, b) if it has derivatives of all orders and $(-1)^k f^{(k)}(t) \geq 0$, $t \in (a, b)$, $k \in \mathbb{N}$.

The famous Bernstein-Widder theorem [22] states that a function $f(x)$, $x \in (0, +\infty)$, is absolutely monotone if and only if there exists a bounded and nondecreasing function $\sigma(t)$ such that the integral

$$(46) \quad f(x) = \int_0^{+\infty} e^{xt} d\sigma(t)$$

converges for $0 \leq x < +\infty$, and a function $f(x)$, $x \in (0, +\infty)$, is completely monotone if and only if there exists a bounded and nondecreasing function $\eta(t)$ such that the integral

$$(47) \quad f(x) = \int_0^{+\infty} e^{-xt} d\eta(t)$$

converges for $0 \leq x < +\infty$.

Proposition 3. Suppose $F(t) = \int_a^b p(u)f^t(u) du$, $t \in \mathbb{R}$, $p(u) \not\equiv 0$, is a nonnegative and continuous function, and $f(u)$ a positive and continuous function on a given interval $[a, b]$. Then

$$(48) \quad F^{(n)}(t) = \int_a^b p(u)f^t(u) [\ln f(u)]^n du.$$

If $f(u) \geq 1$, $F(t)$ is absolutely monotone on $(-\infty, +\infty)$; if $0 < f(u) < 1$, then $F(t)$ is completely monotone on $(-\infty, +\infty)$. Moreover, $F(t)$ is absolutely convex on $(-\infty, +\infty)$.

Proof. This is obvious. $\square$

Corollary 4 ([17, 18]). The function $g(t; x, y)$ is absolutely and regularly monotonic on $(-\infty, +\infty)$ for $y > x > 1$, or on $(0, +\infty)$ for $y > x^{-1} > 1$, completely and regularly monotonic on $(-\infty, +\infty)$ for $0 < x < y < 1$, or on $(-\infty, 0)$ for $1 < y < x^{-1}$. Furthermore, $g(x)$ is absolutely convex on $(-\infty, +\infty)$.

The generalized weighted mean values $M_{p,f}(r, s; x, y)$, with two parameters r and s , are defined in [9, 10] by

$$(49) \quad M_{p,f}(r, s; x, y) = \left(\frac{\int_x^y p(u) f^s(u) \, du}{\int_x^y p(u) f^r(u) \, du} \right)^{1/(s-r)}, \quad (r-s)(x-y) \neq 0,$$

$$(50) \quad M_{p,f}(r, r; x, y) = \exp \left(\frac{\int_x^y p(u) f^r(u) \ln f(u) \, du}{\int_x^y p(u) f^r(u) \, du} \right), \quad x - y \neq 0,$$

$$M_{p,f}(r, s; x, x) = f(x),$$

where $x, y, r, s \in \mathbb{R}$, $p(u) \not\equiv 0$ is a nonnegative and integrable function and $f(u)$ a positive and integrable function on the interval between x and y .

It is clear that $E(r, s; x, y)$ is a special case of $M_{p,f}(r-1, s-1; x, y)$ applied to $p(u) \equiv 1$, $f(u) = u$, and $M^{[r]}(f; p; x, y) = M_{p,f}(r, 0; x, y)$.

The basic properties and monotonicities of $M_{p,f}(r, s; x, y)$ were studied in [9, 15, 16, 20].

At last, we propose the following two open problems.

Open Problem 1. If $f(x)$ is an absolutely or completely monotonic function on the interval $(-\infty, +\infty)$, then the following inequality holds for $0 \leq x < +\infty$ or reverses for $-\infty < x \leq 0$:

$$(51) \quad f^2(x)f'''(x) - 3f(x)f'(x)f''(x) + 2[f'(x)]^3 \leq 0.$$

Open Problem 2. Suppose $p(u)$ is nonnegative and continuous, and $f(u)$ positive and continuous on a given interval $[a, b]$. If $f(u) \geq 1$ or $0 < f(u) < 1$, then the generalized weighted mean values $M_{p,f}(r, s; x, y)$ are logarithmically concave on $(0, +\infty)$ with respect to either r or s , respectively, or logarithmically convex on $(-\infty, 0)$ with respect to either r or s , respectively.

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