EQUIVALENCE OF DOMAINS WITH ISOMORPHIC SEMIGROUPS OF ENDOMORPHISMS

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ABSTRACT. For two bounded domains Ω_1, Ω_2 in $\mathbb{C}$ whose semigroups of analytic endomorphisms $E(\Omega_1), E(\Omega_2)$ are isomorphic with an isomorphism $\varphi : E(\Omega_1) \rightarrow E(\Omega_2)$, Eremenko proved in 1993 that there exists a conformal or anticonformal map $\psi : \Omega_1 \rightarrow \Omega_2$ such that $\varphi f = \psi \circ f \circ \psi^{-1}$, for all $f \in E(\Omega_1)$.

In the present paper we prove an analogue of this result for the case of bounded domains in $\mathbb{C}^n$.

1. INTRODUCTION

A classical theorem of L. Bers says that every $\mathbb{C}$ -algebra isomorphism $H(A) \rightarrow H(B)$ of algebras of holomorphic functions in domains A and B in the complex plane has the form $f \mapsto f \circ \theta$, where $\theta : B \rightarrow A$ is a conformal isomorphism, or $f \mapsto \bar{f} \circ \theta$ with anticonformal θ . In particular, the algebras $H(A)$ and $H(B)$ are isomorphic if and only if the domains A and B are conformally equivalent. H. Iss'sa [9] obtained a similar theorem for fields of meromorphic functions on Stein spaces. A good reference for these results is [5].

Likewise, the question of recovering a topological space from the algebraic structure of its semigroup of continuous self-maps has been extensively studied [12].

In 1990, L. Rubel asked whether similar results hold for semigroups (under composition) $E(D)$ of holomorphic endomorphisms of a domain D . A. Hinkkanen constructed examples [6] which show that even non-homeomorphic domains in $\mathbb{C}$ can have isomorphic semigroups of endomorphisms. The reason is that the semigroup of endomorphisms of a domain can be too small to characterize this domain.

However, in 1993, A. Eremenko [4] proved that for two Riemann surfaces D_1, D_2 , which admit bounded nonconstant holomorphic functions, and such that the semigroups of analytic endomorphisms $E(D_1)$ and $E(D_2)$ are isomorphic with an isomorphism $\varphi : E(D_1) \rightarrow E(D_2)$, there exists a conformal or anticonformal map $\psi : D_1 \rightarrow D_2$ such that $\varphi f = \psi \circ f \circ \psi^{-1}$, for all $f \in E(D_1)$. In the present paper we investigate the analogue of this result for the case of bounded domains in $\mathbb{C}^n$. The theorems of Bers and Iss'sa, mentioned above, do not extend to arbitrary domains in $\mathbb{C}^n$.

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For a bounded domain Ω in $\mathbb{C}^n$ we denote by $E(\Omega)$ the semigroup of analytic endomorphisms of Ω under composition. In what follows, we say that a map is *(anti-) biholomorphic*, if it is biholomorphic or antibiholomorphic. We prove the following theorem.

Theorem 1. *Let Ω_1, Ω_2 be bounded domains in $\mathbb{C}^n, \mathbb{C}^m$ respectively, and suppose that there exists $\varphi : E(\Omega_1) \rightarrow E(\Omega_2)$, an isomorphism of semigroups. Then $n = m$ and there exists an (anti-) biholomorphic map $\psi : \Omega_1 \rightarrow \Omega_2$ such that*

$$(1) \quad \varphi f = \psi \circ f \circ \psi^{-1}, \quad \text{for all } f \in E(\Omega_1).$$

The existence of a homeomorphism ψ satisfying (1) follows from simple general considerations (Section 2). The hard part is proving that ψ is (anti-) biholomorphic. In dimension 1 this is done by linearization of holomorphic germs of $f \in E(\Omega)$ near an attracting fixed point. In several dimensions such linearization theory exists ([1], pp. 192–194), but it is too complicated (many germs with an attracting fixed point are non-linearizable, even formally). In Sections 3, 4 we show how to localize the problem. In Sections 5, 6 we describe, using only the semigroup structure, a large enough class of linearizable germs. Linearization of these germs permits us to reduce the problem to a matrix functional equation, which is solved in Section 7. In Section 8 we complete the proof that ψ is (anti-) biholomorphic.

Theorem 1 can be slightly generalized, namely one may assume that φ is an epimorphism. In Section 9 we prove the following theorem.

Theorem 2. *If $\varphi : E(\Omega_1) \rightarrow E(\Omega_2)$ is an epimorphism between semigroups, where Ω_1, Ω_2 are bounded domains in $\mathbb{C}^n, \mathbb{C}^m$ respectively, then φ is an isomorphism.*

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2. TOPOLOGY

For a bounded domain Ω in $\mathbb{C}^n$ we denote by $C(\Omega)$ the subsemigroup of $E(\Omega)$ consisting of constant maps. An endomorphism c_z is constant if it sends Ω to a point $z \in \Omega$. The subset $C(\Omega) \subset E(\Omega)$ can be described using only the semigroup structure as follows:

$$(2) \quad c \in C(\Omega) \text{ iff } \forall (f \in E(\Omega)), \quad (c \circ f = c).$$

It is clear that we have a bijection between constant endomorphisms of Ω and points of this domain as a set: to each z corresponds a unique $c_z \in C(\Omega)$ and vice versa, so we can identify the two. Under this identification, a subset of Ω corresponds to a subsemigroup of $C(\Omega)$.

Having defined points of a domain in terms of its semigroup structure of analytic endomorphisms, we can construct a map ψ between Ω_1 and Ω_2 as follows:

$$(3) \quad \psi(z) = w \quad \text{iff} \quad \varphi c_z = c_w.$$

So defined, ψ satisfies (1). Indeed, let $f \in E(\Omega_1)$, $f(z) = \zeta$. This is equivalent to

$$(4) \quad f \circ c_z = c_\zeta.$$

Applying φ to both sides of (4), we have

$$(5) \quad \varphi f \circ c_{\psi(z)} = c_{\psi(\zeta)}.$$

But (5) is equivalent to $\varphi f(\psi(z)) = \psi(\zeta) = \psi(f(z))$, which is (1).

We describe the topology of a domain Ω using its injective endomorphisms. A map $f \in E(\Omega)$ is injective if and only if

$$\forall (c' \in C(\Omega)) \forall (c'' \in C(\Omega)), ((f \circ c' = f \circ c'') \Rightarrow (c' = c'')).$$

We denote the class of injective endomorphisms of Ω by $E_i(\Omega)$. For every $f \in E_i(\Omega)$, $f_i(\Omega)$ is open [2]. The family $\{f(\Omega), f \in E_i(\Omega)\}$ of subsets of Ω forms a base of topology, because every $z \in \Omega$ has a neighborhood $f(\Omega)$, where $f(\zeta) = z + \lambda(\zeta - z)$, f belongs to $E_i(\Omega)$ for every λ such that $|\lambda|$ is small.

To summarize, we described subsets of Ω and the topology on it using only the semigroup structure of $E(\Omega)$. Since this is so, the semigroup structure also defines the notions of an open set, closed set, compact set, closure of a set.

Now we can easily prove continuity of the map ψ constructed above. Indeed, let $g(\Omega_2)$, $g \in E_i(\Omega_2)$, be a set from the base of topology of Ω_2 . We take $f = \varphi^{-1}g$. Then $f \in E_i(\Omega_1)$ and $\psi^{-1}(g(\Omega_2)) = f(\Omega_1)$, which proves that ψ is continuous. Since φ is an isomorphism, the same argument works to prove that ψ^{-1} is also continuous, and thus ψ is a homeomorphism.

Therefore the domains Ω_1 , Ω_2 are homeomorphic, and hence [8] they have the same dimension, i. e. $n = m$.

3. LOCALIZATION

We need the following lemma.

Lemma 1. *Suppose H is a semigroup with identity, and f an element of H with the following two properties:*

- (i) $hf = fh$, for every h in H ;
- (ii) $h_1f = h_2f$ implies $h_1 = h_2$, for every h_1 and h_2 in H .

Then there exist a semigroup S_f and a monomorphism $i : H \rightarrow S_f$ such that $i(f)$ is invertible in S_f and commutes with all elements of S_f . Moreover, the semigroup S_f satisfies the following universal property: for every semigroup S_1 with a monomorphism $i_1 : H \rightarrow S_1$ such that $i_1(f)$ is invertible in S_1 and commutes with all elements of S_1 , there exists a unique monomorphism $\hat{i}_1 : S_f \rightarrow S_1$ such that $i_1 = \hat{i}_1 \circ i$.

Remark 1. Uniqueness of $\hat{i}_1$ implies that the semigroup S_f with the universal property is unique up to an isomorphism.

Proof. We construct S_f as follows. First we consider formal expressions of the form hf^k , where $h \in H$ and k is an integer (it may be positive, negative or zero). Then we define a multiplication on this set: $h_1f^{k_1} * h_2f^{k_2} = h_1h_2f^{k_1+k_2}$. Next we consider a relation on the set of formal expressions: $h_1f^{k_1} \sim h_2f^{k_2}$ if $k_1 \leq k_2$ and $h_1 = h_2f^{k_2-k_1}$ in H , or $k_2 \leq k_1$ and $h_2 = h_1f^{k_1-k_2}$ in H . It is easy to verify that this is an equivalence relation and it is compatible with the operation $*$; that is, $x \sim y$, $u \sim v$ implies $x * u \sim y * v$.

Lastly, let S_f be the set of equivalence classes with the binary operation induced by $*$. For S_f to be a semigroup, we need to show that the binary operation $*$ is associative. Let $h_1f^{k_1} \sim h'_1f^{k'_1}$, $h_2f^{k_2} \sim h'_2f^{k'_2}$ and $h_3f^{k_3} \sim h'_3f^{k'_3}$. We need to show that $(h_1f^{k_1} * h_2f^{k_2}) * h_3f^{k_3} \sim h'_1f^{k'_1} * (h'_2f^{k'_2} * h'_3f^{k'_3})$. By the definition of the operation $*$, the last equivalence is the same as $h_1h_2h_3f^{k_1+k_2+k_3} \sim h'_1h'_2h'_3f^{k'_1+k'_2+k'_3}$. Assuming that $k_1 + k_2 + k_3 \leq k'_1 + k'_2 + k'_3$, we have essentially one possibility to

consider (the others are either similar or trivial): $k_1 \leq k'_1$, $k_2 \leq k'_2$, $k'_3 \leq k_3$. In this case $h_1 h_2 h_3 f^{k_3 - k'_3} = h'_1 h'_2 h'_3 f^{k'_1 - k_1 + k'_2 - k_2}$. Now we can use the cancellation property (ii) to get the desired equivalence.

The semigroup H is embedded into S_f via $i : h \mapsto [hf^0]$. The element $i(f) = [\text{id}f]$, where id is the identity in H , is invertible in S_f with the inverse $[\text{id}f^{-1}]$. Clearly, $[\text{id}f]$ commutes with all elements of S_f .

Now, suppose that $S_1, i_1 : H \rightarrow S_1$ is a semigroup and a monomorphism, such that $i_1(f)$ is invertible in S_1 and commutes with all elements of S_1 . Then we define

$$\hat{i}_1([hf^k]) = i_1(h)(i_1(f))^k.$$

This definition does not depend on a representative of $[hf^k]$. Indeed, suppose $h_1 f^{k_1} \sim h_2 f^{k_2}$ and assume $k_1 \leq k_2$. Then $h_1 = h_2 f^{k_2 - k_1}$, and thus $i_1(h_1) = i_1(h_2) i_1(f)^{k_2 - k_1}$. Hence $i_1(h_1) i_1(f)^{k_1} = i_1(h_2) i_1(f)^{k_2}$.

So defined, $\hat{i}_1$ is a homomorphism:

$$\begin{aligned} \hat{i}_1([h_1 f^{k_1}][h_2 f^{k_2}]) &= \hat{i}_1([h_1 h_2 f^{k_1 + k_2}]) \\ &= i_1(h_1 h_2) i_1(f)^{k_1 + k_2} = i_1(h_1) i_1(h_2) i_1(f)^{k_1} i_1(f)^{k_2} \\ &= i_1(h_1) i_1(f)^{k_1} i_1(h_2) i_1(f)^{k_2} = \hat{i}_1([h_1 f^{k_1}]) \hat{i}_1([h_2 f^{k_2}]). \end{aligned}$$

The relation $\hat{i}_1 \circ i = i_1$ holds, since $\hat{i}_1([hf^0]) = i_1(h)$ for all $h \in H$.

Uniqueness of $\hat{i}_1$ is clear. Lemma 1 is proved. $\square$

4. EXTENSION OF φ

Following [4], we say that for a bounded domain Ω an element $f \in E(\Omega)$ is *good* at $z \in \Omega$, denoted by $f \in G_z(\Omega)$, if

1. z is a unique fixed point of f ;
2. $f(\Omega)$ has compact closure in Ω ;
3. f is injective in Ω .

Property 3 of a good element was already stated in terms of the semigroup structure of Ω . Since the topology on Ω was described using only the semigroup structure, Property 2 can also be stated in these terms. Property 1 can be expressed in terms of the semigroup structure as

$$(f \circ c_z = c_z) \wedge ((f \circ c_\zeta = c_\zeta) \Rightarrow (c_\zeta = c_z)).$$

Since f is an endomorphism of a domain, all eigenvalues λ of its linear part at z satisfy $|\lambda| \leq 1$ [10]. Moreover, $|\lambda| < 1$ because the closure of $f(\Omega)$ is a compact set in Ω . The injectivity of f implies [2] that it is biholomorphic onto $f(\Omega)$ and the Jacobian determinant of f does not vanish at any point of Ω .

It is clear that for every $z \in \Omega$ a good element f at z exists. For example, we can take $f(\zeta) = z + \lambda(\zeta - z)$ with sufficiently small $|\lambda|$.

Consider a good element $f \in G_z(\Omega)$ and its commutant $H_f(\Omega)$ in $E(\Omega)$:

$$H_f(\Omega) = \{h \in E(\Omega) : hf = fh\}.$$

Clearly $H_f(\Omega)$ is a subsemigroup of $E(\Omega)$. The element f , being good (hence injective), satisfies the cancellation property (ii) of Lemma 1 in $H_f(\Omega)$. Thus, by Lemma 1, we have the extension S_f of $H_f(\Omega)$ in which f is invertible and commutes with all elements of S_f . In the case of analytic endomorphisms we can embed $H_f(\Omega)$ into the subsemigroup of A_z , the semigroup of germs of analytic mappings at z under composition, consisting of elements that commute with the germ of f

and containing the germ of f^{-1} . We use the universal property of Lemma 1 to conclude that S_f is isomorphic to a subsemigroup of A_z . We identify S_f with this semigroup, i. e. we consider elements of S_f as germs of analytic mappings at z .

In proving that ψ is (anti-) biholomorphic we need to show that it is so in a neighborhood of every point of Ω_1 . Since an (anti-) biholomorphic type of a domain is preserved by translations in $\mathbb{C}^n$, it is enough to show that ψ is (anti-) biholomorphic in a neighborhood of $0 \in \mathbb{C}^n$, assuming that Ω_1 and Ω_2 contain 0 and $\psi(0) = 0$.

Let $\varphi : E(\Omega_1) \rightarrow E(\Omega_2)$ be an isomorphism of the semigroups, f a good element, $f \in G_0(\Omega_1)$, and $H_f(\Omega_1)$ the commutant of f . Then clearly $H_g(\Omega_2) = \varphi(H_f(\Omega_1))$ is the commutant of $g = \varphi f$. By Lemma 1, we have the extensions S_f , S_g of $H_f(\Omega_1)$ and $H_g(\Omega_2)$ respectively, and by the universal property of this lemma the isomorphism φ extends to an isomorphism

$$\Phi : S_f \rightarrow S_g.$$

5. SYSTEM OF PROJECTIONS AND LINEARIZATION

Let Ω be a bounded domain in $\mathbb{C}^n$. We say that a good element $f \in G_0(\Omega)$ is *very good* at 0, and write $f \in VG_0(\Omega)$, if the corresponding semigroup $S_f \subset A_0$ constructed in Section 4 contains a system of elements, which we call a system of projections, $\{p_i\}_{i=1}^n$ with the following properties:

- (a) $\forall (i = 1, \dots, n), (p_i \neq 0)$;
- (b) $\forall (i = 1, \dots, n), (p_i^2 = p_i)$;
- (c) $\forall (i, j = 1, \dots, n, i \neq j), (p_i p_j = 0)$.

There does exist a very good element, since we can take f to be a homothetic transformation at 0 with sufficiently small coefficient, p_i a projection on the i 'th coordinate of the standard coordinate system. Clearly, $p_i f = f p_i$ and there exists k such that $p_i f^k \in E(\Omega)$, and hence $p_i \in S_f$. From now on, we fix a very good element $f \in VG_0(\Omega)$, associated semigroups $H_f(\Omega)$, S_f and a system of projections $\{p_i\}$.

We introduce another subsemigroup of $E(\Omega)$:

$$P_f(\Omega) = \{h \in G_0(\Omega) \cap H_f(\Omega), \quad h p_i = p_i h, \quad i = 1, \dots, n\},$$

where the commutativity relations are in $S_f \subset A_0$. Notice that $P_f(\Omega) \neq \emptyset$ since f belongs to it.

Lemma 2. *For every $h \in P_f(\Omega)$ there exists a biholomorphic germ θ_h at $0 \in \mathbb{C}^n$ such that $\theta_h h = \Lambda \theta_h$, where $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_n)$ is an invertible diagonal matrix which is similar to $dh(0)$ in $GL(n, \mathbb{C})$.*

Proof. The relations $p_i \neq 0$, $p_i^2 = p_i$, $p_i p_j = 0$, $i \neq j$, imply that for $P_i = dp_i(0)$, the linear part of p_i at 0, we have $P_i \neq 0$, $P_i^2 = P_i$, $P_i P_j = 0$, $i \neq j$. Since the matrices P_i commute, there exists [7] a matrix $A \in GL(n, \mathbb{C})$ such that $P'_i = A P_i A^{-1} = \Delta_i = \text{diag}(0, \dots, 1, \dots, 0)$, where the only non-zero entry appears in the i 'th place.

Since $p_i^2 = p_i$, $i = 1, \dots, n$, we can use the argument given in [10] to linearize p_i , i.e. there exists a biholomorphic germ ξ_i at 0 such that $\xi_i p_i = P_i \xi_i$, $d\xi_i(0) = \text{id}$, $i = 1, \dots, n$. The map ξ_i is constructed in [10] as follows:

$$\xi_i = \text{id} + (2P_i - \text{id})(p_i - P_i), \quad i = 1, \dots, n.$$

If we take $\xi'_i = A\xi_i$, we have $\xi'_i p_i = P'_i \xi'_i$. For simplicity of notations, we assume that ξ_i itself conjugates p_i to a diagonal matrix, that is, $P_i = P'_i$ (in this case P_i is not necessarily $dp_i(0)$, but rather $Adp_i(0)A^{-1}$; $d\xi_i(0) = A$). For every $i = 1, \dots, n$ we have $h_i P_i = P_i h_i$, where $h_i = \xi_i h \xi_i^{-1}$. Let $H_i = dh_i(0)$. Then $H_i P_i = P_i H_i$, and hence in the i 'th row and the i 'th column the matrix H_i has only one non-zero entry, λ_i , which is located at their intersection. Thus λ_i has to be an eigenvalue of H_i , and hence of the linear part of h . In particular, $0 < |\lambda_i| < 1$.

Let $I_i : \mathbb{C} \rightarrow \mathbb{C}^n$ be the embedding $z \mapsto (0, \dots, z, \dots, 0)$, where the only non-zero entry is z , which is in the i 'th place; and $\pi_i : \mathbb{C}^n \rightarrow \mathbb{C}$, a projection $(z_1, \dots, z_n) \mapsto z_i$, corresponding to the i 'th axis. For every $i = 1, \dots, n$, the map $\pi_i h_i I_i$ sends a neighborhood of 0 in $\mathbb{C}$ into $\mathbb{C}$, and its derivative at 0, λ_i , is an eigenvalue of h . Hence ([3], p. 31) $\pi_i h_i I_i$ is linearized by the unique solution $\eta_{h,i}$ of the Schröder equation

$$(6) \quad \eta(\pi_i h_i I_i) = \lambda_i \eta, \quad \eta(0) = 0, \quad \eta'(0) = 1.$$

Since $P_i I_i = I_i$, $\pi_i P_i I_i = \text{id}_{\mathbb{C}}$, we can rewrite (6) as

$$\eta_{h,i} \pi_i h_i P_i I_i = \lambda_i \eta_{h,i} \pi_i P_i I_i, \quad \text{or} \quad \eta_{h,i} \pi_i h_i P_i = \lambda_i \eta_{h,i} \pi_i P_i.$$

But $h_i P_i = P_i h_i$, and so

$$(7) \quad \eta_{h,i} \pi_i P_i h_i = \lambda_i \eta_{h,i} \pi_i P_i.$$

The equation (7), in its turn, is equivalent to

$$(8) \quad \eta_{h,i} \pi_i \xi_i p_i h = \lambda_i \eta_{h,i} \pi_i \xi_i p_i.$$

We denote

$$(9) \quad \theta_{h,i} = \eta_{h,i} \pi_i \xi_i p_i,$$

a map from a neighborhood of 0 in $\mathbb{C}^n$ into $\mathbb{C}$. Then (8) becomes $\theta_{h,i} h = \lambda_i \theta_{h,i}$. Now we define

$$\theta_h = (\theta_{h,1}, \dots, \theta_{h,n}),$$

which is a germ of an analytic map at 0. This germ linearizes h :

$$\theta_h h = (\theta_{h,1} h, \dots, \theta_{h,n} h) = (\lambda_1 \theta_{h,1}, \dots, \lambda_n \theta_{h,n}) = \Lambda \theta_h,$$

where $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_n)$ is an invertible diagonal matrix, which has eigenvalues of $dh(0)$ on its diagonal.

The germ θ_h is biholomorphic. Indeed,

$$\theta_{h,i} = \eta_{h,i} \pi_i \xi_i p_i = \eta_{h,i} \pi_i P_i \xi_i, \quad i = 1, \dots, n.$$

Using the chain rule, we see that $d\theta_h(0) = A$, where A is an invertible diagonal matrix that diagonalizes P_i . We conclude that θ_h is biholomorphic. Lemma 2 is proved. $\square$

6. SIMULTANEOUS LINEARIZATION

Using Lemma 2, we can linearize elements of $P_f(\Omega)$. Namely, for every $h \in P_f(\Omega)$ there exists θ_h (constructed in Section 5) such that $\theta_h h = \Lambda_h \theta_h$, where Λ_h is an invertible diagonal matrix. In particular, we can linearize f :

$$\theta_f f = \Lambda_f \theta_f,$$

where the germ θ_f is biholomorphic at 0, and Λ_f is an invertible diagonal matrix.

Lemma 3. *For every $h \in P_f(\Omega)$ we have $\theta_h = \theta_f$.*

Proof. Let us consider the germ

$$(10) \quad \theta = \Lambda_f^{-1} \theta_h f,$$

which is clearly biholomorphic. We have

$$\theta h = \Lambda_f^{-1} \theta_h f h = \Lambda_f^{-1} \theta_h h f = \Lambda_f^{-1} \Lambda_h \theta_h f = \Lambda_h \Lambda_f^{-1} \theta_h f = \Lambda_h \theta.$$

Using (10), we write the equation $\theta h = \Lambda_h \theta$ in the coordinate form:

$$(1/\lambda_{f,i}) \theta_{h,i} f h = (\lambda_{h,i}/\lambda_{f,i}) \theta_{h,i} f, \quad i = 1, \dots, n.$$

By (9) and the definition of ξ_i ,

$$(1/\lambda_{f,i}) \eta_{h,i} \pi_i P_i f h_i = (\lambda_{h,i}/\lambda_{f,i}) \eta_{h,i} \pi_i P_i f_i, \quad i = 1, \dots, n,$$

where $f_i = \xi_i f \xi_i^{-1}$. Using the commutativity relations $f_i P_i = P_i f_i$, $h_i P_i = P_i h_i$, which hold since $\{p_i\} \subset S_f$, $h \in P_f(\Omega)$, we get

$$\begin{aligned} (1/\lambda_{f,i}) \eta_{h,i} \pi_i f_i h_i P_i &= (\lambda_{h,i}/\lambda_{f,i}) \eta_{h,i} \pi_i f_i P_i, \quad \text{or} \\ (1/\lambda_{f,i}) \eta_{h,i} \pi_i f_i h_i I_i &= (\lambda_{h,i}/\lambda_{f,i}) \eta_{h,i} \pi_i f_i I_i, \quad i = 1, \dots, n. \end{aligned}$$

This is the same as

$$((1/\lambda_{f,i}) \eta_{h,i} \pi_i f_i I_i) (\pi_i h_i I_i) = \lambda_{h,i} ((1/\lambda_{f,i}) \eta_{h,i} \pi_i f_i I_i), \quad i = 1, \dots, n,$$

since h_i locally preserves the i 'th coordinate axis ($h_i P_i = P_i h_i$). It is easily seen that

$$\begin{aligned} ((1/\lambda_{f,i}) \eta_{h,i} \pi_i f_i I_i)(0) &= 0, \\ ((1/\lambda_{f,i}) \eta_{h,i} \pi_i f_i I_i)'(0) &= 1. \end{aligned}$$

A normalized solution to a Schröder equation is unique, though; thus we have

$$\eta_{h,i}(\pi_i f_i I_i) = \lambda_{f,i} \eta_{h,i}, \quad \eta_{h,i}(0) = 0, \quad \eta'_{h,i}(0) = 1.$$

Using the uniqueness argument again, we obtain $\eta_{h,i} = \eta_{f,i}$, and hence $\theta_h = \theta_f$. The lemma is proved. $\square$

According to Lemma 3, the single biholomorphic germ θ_f conjugates the subsemigroup $P_f(\Omega)$ to some subsemigroup D_f of invertible diagonal matrices in D_n , the set of all $n \times n$ diagonal matrices with entries in $\mathbb{C}$. We show that D_f contains all invertible diagonal matrices with sufficiently small entries. To do this, first we extend θ_f to an analytic map on the whole domain Ω using the formula

$$\theta_f = \Lambda_f^{-l} \theta_f f^l,$$

where l is chosen so large that $\text{Cl}\{f^l(\Omega)\}$ is contained in a neighborhood of 0 where θ_f is originally defined and biholomorphic; the symbol Cl denotes closure. From the procedure of extending θ_f to Ω we see that it is one-to-one and bounded in the whole domain.

Now, let $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_n)$ be a matrix such that $\text{Cl}\{\Lambda \theta_f(\Omega)\} \subset W$, where W is a neighborhood of $0 \in \mathbb{C}^n$ for which $\text{Cl}\{\theta_f^{-1} W\} \subset \Omega$. Such a matrix Λ exists since θ_f is bounded in Ω . Consider $h = \theta_f^{-1} \Lambda \theta_f$, which belongs to $G_0(\Omega)$. The map h commutes with f and all p_i 's. Indeed, using the formula $\theta_f f \theta_f^{-1} = \Lambda_f$, we conclude that $h f = f h$ is equivalent to $\Lambda \Lambda_f = \Lambda_f \Lambda$, which is a true relation since both matrices Λ and Λ_f are diagonal. The relations $h p_i = p_i h$, $i = 1, \dots, n$, are

verified similarly, using the formula $\theta_f p_i \theta_f^{-1} = P_i$, which follows from the definition of θ_f .

7. SOLVING A MATRIX EQUATION

We proved that for an element $f \in VG_0(\Omega)$ there exists a biholomorphic germ θ_f conjugating the semigroup $P_f(\Omega)$ to a subsemigroup $D_f \subset D_n$, which contains all invertible diagonal matrices with sufficiently small entries.

Let $f \in VG_0(\Omega_1)$ and $g = \varphi f$. Then $g \in VG_0(\Omega_2)$, and there is an isomorphism

$$\Phi: S_f \rightarrow S_g.$$

For the mappings f and g we have

$$\theta_f f = \Lambda_f \theta_f, \quad \theta_g g = M_g \theta_g,$$

where Λ_f, M_g are invertible diagonal matrices.

Let us consider the germ $L = \theta_g \psi \theta_f^{-1}$. This germ conjugates the semigroups D_f, D_g :

$$\begin{aligned} L \Lambda L^{-1} &= \theta_g \psi \theta_f^{-1} \Lambda \theta_f \psi^{-1} \theta_g^{-1} \\ &= \theta_g \psi h \psi^{-1} \theta_g^{-1} = \theta_g j \theta_g^{-1} = M, \end{aligned}$$

where $h \in P_f$, $\theta_f h = \Lambda \theta_f$; $j = \varphi h$, $\theta_g j = M \theta_g$.

Define $R(\Lambda) = L \Lambda L^{-1}$. Then $R: D_f \rightarrow D_g$,

$$R(\Lambda_1 \Lambda_2) = R(\Lambda_1) R(\Lambda_2), \quad \Lambda_1, \Lambda_2 \in D_f.$$

In what follows, we will identify D_n with the multiplicative semigroup $\mathbb{C}^n$ ($D_n \cong \mathbb{C}^n$) in the obvious way and consider a topology on D_n induced by the standard topology on $\mathbb{C}^n$.

We are going to extend R to an isomorphism of D_n . First, we denote by $\overline{D}_f, \overline{D}_g$ the closures of D_f, D_g in D_n , and for $\Lambda \in \overline{D}_f$ we set

$$R(\Lambda) = \lim R(\Lambda_k), \quad \Lambda_k \rightarrow \Lambda, \quad \Lambda_k \in D_f.$$

This limit exists and does not depend on the sequence $\{\Lambda_k\}$, which follows from the fact that $\psi^{\pm 1}, \theta_f^{\pm 1}, \theta_g^{\pm 1}$ are continuous. The map R is an isomorphism of topological semigroups $\overline{D}_f$ and $\overline{D}_g$ (the inverse of R has a similar representation).

Next, we extend the map R to D_n as

$$R(\Gamma) = R(\Gamma \Lambda) R(\Lambda)^{-1}, \quad \Gamma \in D_n,$$

where $\Lambda \in D_f$ is chosen so that $\Gamma \Lambda \in \overline{D}_f$. This definition does not depend on the choice of Λ . Indeed, since all matrices in question are diagonal (hence commute), the relation $R(\Gamma \Lambda_1) R(\Lambda_1)^{-1} = R(\Gamma \Lambda_2) R(\Lambda_2)^{-1}$ is equivalent to $R(\Gamma \Lambda_1) R(\Lambda_2) = R(\Gamma \Lambda_2) R(\Lambda_1)$, which holds.

The extended map R is clearly an isomorphism of D_n onto itself. Thus we have

$$(11) \quad R(\Lambda' \Lambda'') = R(\Lambda') R(\Lambda''), \quad \Lambda', \Lambda'' \in D_n.$$

Injectivity of R and (11) imply that $R(\Delta_i) = \Delta_j$ for all i , where $j = j(i)$ depends on i ; $j(i)$ is a permutation on $\{1, \dots, n\}$ (we recall that $\Delta_i = \text{diag}(0, \dots, 1, \dots, 0)$). This is because $\{\Delta_i\}_{i=1}^n$ is the only system in D_n with the following relations: $\Delta_i \neq 0$, $\Delta_i^2 = \Delta_i$, $\Delta_i \Delta_j = 0$, $i \neq j$.

Since all matrices Λ and their images $R(\Lambda)$ are diagonal, we can consider the matrix equation (11) as n scalar equations:

$$(12) \quad r_j(\lambda'_1 \lambda''_1, \dots, \lambda'_n \lambda''_n) = r_j(\lambda'_1, \dots, \lambda'_n) r_j(\lambda''_1, \dots, \lambda''_n), \quad j = 1, \dots, n,$$

where r_j are components of R . If we rewrite the equation $R(\Delta_i \Lambda) = \Delta_j R(\Lambda)$ in the coordinate form, we see that $r_j(\lambda_1, \dots, \lambda_n) = r_j(0, \dots, \lambda_i, \dots, 0) = q_j(\lambda_i)$; that is, each r_j depends on only one of the λ_i 's. For each j the corresponding equation in (12) in terms of the q_j 's becomes

$$q_j(\lambda'_i \lambda''_i) = q_j(\lambda'_i) q_j(\lambda''_i).$$

This equation has ([4], p. 130) either the constant solution $q_j(\lambda_i) = 1$, or

$$q_j(\lambda_i) = \lambda_i^{\alpha_{ij}} \bar{\lambda}_i^{\beta_{ij}}, \quad \alpha_{ij}, \beta_{ij} \in \mathbb{C}, \quad \alpha_{ij} - \beta_{ij} = \pm 1.$$

Going back to the function L , we have

$$(13) \quad L \operatorname{diag}(\lambda_1, \dots, \lambda_n) = \operatorname{diag}(\lambda_{i(1)}^{\alpha_1} \bar{\lambda}_{i(1)}^{\beta_1}, \dots, \lambda_{i(n)}^{\alpha_n} \bar{\lambda}_{i(n)}^{\beta_n}) L, \\ \alpha_i - \beta_i = \pm 1, \quad i = 1, \dots, n,$$

where $i(j)$ is the inverse permutation to $j(i)$.

Let us choose and fix $(\mu_1, \dots, \mu_n)$ such that $(1/\mu_1, \dots, 1/\mu_n)$ belongs to a neighborhood W_0 of $0 \in \mathbb{C}^n$ where L is defined, and let W_1 be a neighborhood of $0 \in \mathbb{C}^n$ such that $(\mu_1 z_1, \dots, \mu_n z_n) \in W_0$, whenever $(z_1, \dots, z_n) \in W_1$. Then from (13) we have

$$L(z_1, \dots, z_n) = L \operatorname{diag}(\mu_1 z_1, \dots, \mu_n z_n) (1/\mu_1, \dots, 1/\mu_n) \\ = \operatorname{diag}((\mu_{i(1)} z_{i(1)})^{\alpha_1} (\overline{\mu_{i(1)} z_{i(1)}})^{\beta_1}, \dots, (\mu_{i(n)} z_{i(n)})^{\alpha_n} (\overline{\mu_{i(n)} z_{i(n)}})^{\beta_n}) \\ \times L(1/\mu_1, \dots, 1/\mu_n) = B(z_1^{\alpha_1} \bar{z}_1^{\beta_1}, \dots, z_n^{\alpha_n} \bar{z}_n^{\beta_n}),$$

where B is a constant matrix. The last formula is the explicit expression for L .

8. PROVING THAT ψ IS (ANTI-) BIHOLOMORPHIC

To prove that ψ is (anti-) biholomorphic is the same as to prove that L is (anti-) biholomorphic, because the relation $L = \theta_g \circ \psi \circ \theta_f^{-1}$ holds. We showed that

$$(14) \quad L(z_1, \dots, z_n) = B(z_1^{\alpha_1} \bar{z}_1^{\beta_1}, \dots, z_n^{\alpha_n} \bar{z}_n^{\beta_n}), \quad \alpha_i - \beta_i = \pm 1, \quad i = 1, \dots, n,$$

in a neighborhood W_1 of 0 . From the representation (14) we see that L is $\mathbb{R}$ -differentiable and non-degenerate in $W_1 \setminus \bigcup_{k=1}^n \{(z_1, \dots, z_n) : z_k = 0\}$. Since this is true for every point in the domain Ω_1 , the map ψ is $\mathbb{R}$ -differentiable and non-degenerate everywhere, with the possible exception of an analytic set. Let us remove this set from Ω_1 , as well as its image under ψ from Ω_2 . We call the domains obtained in this way Ω' , Ω'' . Now the map $\psi : \Omega' \rightarrow \Omega''$ is $\mathbb{R}$ -differentiable and non-degenerate everywhere. It is clear that if we prove that ψ is (anti-) biholomorphic between Ω' , Ω'' , then it is (anti-) biholomorphic between Ω_1 , Ω_2 due to a standard continuation argument [11]. So we can think that ψ is $\mathbb{R}$ -differentiable and non-degenerate in Ω_1 itself. The map L thus has to be $\mathbb{R}$ -differentiable and non-degenerate at 0 . However, this is the case if and only if $\alpha_i + \beta_i = 1$, $i = 1, \dots, n$. Together with the equation $\alpha_i - \beta_i = \pm 1$ it gives us that either $\alpha_i = 1$, $\beta_i = 0$, or $\alpha_i = 0$, $\beta_i = 1$.

It remains to show that either $\alpha_i = 1$ and $\beta_i = 0$, or $\alpha_i = 0$ and $\beta_i = 1$, simultaneously for all i . Suppose, by way of contradiction, that we have $L(z_1, \dots, z_n) = B(\dots, z_i, \dots, \bar{z}_j, \dots)$. Then

$$L^{-1}(w_1, \dots, w_n) = (\dots, l_i(w_1, \dots, w_n), \dots, l_j(\bar{w}_1, \dots, \bar{w}_n), \dots),$$

where l_i, l_j are linear analytic functions. Let us look at an endomorphism f_0 of Ω_1 in the form

$$f_0 = \theta_f^{-1} \lambda(\dots, \theta_{f,i} \theta_{f,j}, \dots, \theta_{f,j}, \dots) \theta_f,$$

where $\theta_{f,i} \theta_{f,j}$ is in the i 'th place and $\theta_{f,j}$ in the j 'th; $|\lambda|$ is sufficiently small. Using (1) and the definition of L , we have

$$\theta_g \varphi f_0 \theta_g^{-1} = \theta_g \psi f_0 \psi^{-1} \theta_g^{-1} = L \theta_f f_0 \theta_f^{-1} L^{-1}.$$

So,

$$\begin{aligned} \theta_g \varphi f_0 \theta_g^{-1}(w_1, \dots, w_n) \\ = B'(\dots, l_i(w_1, \dots, w_n) l_j(\bar{w}_1, \dots, \bar{w}_n), \dots, \bar{l}_j(w_1, \dots, w_n), \dots) \end{aligned}$$

for some constant matrix B' . This map, and hence φf_0 , is not analytic, though, in a neighborhood of 0, which is a contradiction. Thus L , and hence ψ , is either analytic or antianalytic in a neighborhood of 0.

Theorem 1 is proved completely.

9. PROOF OF THEOREM 2

Since φ is an epimorphism, it takes constant endomorphisms of Ω_1 to constant endomorphisms of Ω_2 , which follows from (2). Thus we can define a map $\psi: \Omega_1 \rightarrow \Omega_2$ as in (3). Following the same steps as in verifying (1), we obtain

$$(15) \quad \varphi f \circ \psi = \psi \circ f, \quad \text{for all } f \in E(\Omega_1).$$

We will show that (15) implies bijectivity of ψ . The map ψ is surjective. Indeed, let $w \in \Omega_2$, and let c_w be the corresponding constant endomorphism. Since φ is an epimorphism, there exists $f \in E(\Omega_1)$ such that $\varphi f = c_w$. If we plug this f into (15), we get

$$\psi f(z) = w$$

for all $z \in \Omega_1$. Thus ψ is surjective.

To prove that ψ is injective, we show that for every $w \in \Omega_2$, the full preimage of w under ψ , $\psi^{-1}(w)$, consists of one point.

Assume for contradiction that $S_w = \psi^{-1}(w)$ consists of more than one point for some $w \in \Omega_2$. The set S_w cannot be all of Ω_1 , since ψ is surjective. For $z_0 \in \partial S_w \cap \Omega_1$ we can find $z_1 \in S_w$ and $\zeta \notin S_w$ which are arbitrarily close to z_0 . Let z_2 be a fixed point of S_w different from z_1 . Consider a homothetic transformation h such that $h(z_1) = z_1$, $h(z_2) = \zeta$. Since the domain Ω_1 is bounded, we can choose points z_1 and ζ sufficiently close to each other so that h belongs to $E(\Omega_1)$. Applying (15) to h , we obtain

$$\begin{aligned} \varphi h(w) &= \varphi h \circ \psi(z_1) = \psi \circ h(z_1) = \psi(z_1) = w; \\ \varphi h(w) &= \varphi h \circ \psi(z_2) = \psi \circ h(z_2) = \psi(\zeta) \neq w. \end{aligned}$$

The contradiction shows injectivity of ψ . Thus we have proved that ψ is bijective.

According to (15) we have

$$\varphi f = \psi \circ f \circ \psi^{-1}, \text{ for all } f \in E(\Omega_1),$$

which implies that φ is an isomorphism.

Theorem 2 is proved.

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