ON IMAGES OF BOREL MEASURES UNDER BOREL MAPPINGS

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ABSTRACT. Let X and Y be metric spaces. We show that the tight images of a (fixed) tight Borel probability measure μ on X , under all Borel mappings $f: X \rightarrow Y$, form a closed set in the space of tight Borel probability measures on Y with the weak*-topology. In contrast, the set of images of μ under all continuous mappings from X to Y may not be closed. We also characterize completely the set of tight images of μ under Borel mappings. For example, if μ is non-atomic, then all tight Borel probability measures on Y can be obtained as images of μ , and as a matter of fact, one can always choose the corresponding Borel mapping to be of Baire class 2.

1. MAIN RESULT

Given a complete and separable metric space Z , we denote by $M_1^+(Z)$ the space of all Borel probability measures on Z , with the topology inherited from the weak*-topology on $C_b(Z)^*$; here $C_b(Z)$ is the Banach space of bounded continuous functions $h: Z \rightarrow \mathbb{R}$ and $C_b(Z)^*$ is its dual. The space $M_1^+(Z)$ is metrizable, Z being assumed separable; the Prohorov metric is one metric that metrizes $M_1^+(Z)$ (cf. [1], Theorem 5 in Appendix III).

The following is the main result of this note, answering in the affirmative a question raised by S. Slijepcevic.

Theorem 1. *Let X and Y be complete, separable metric spaces and μ a Borel probability measure on X . Then the set $\{\mu \circ f^{-1} \mid f: X \rightarrow Y \text{ Borel}\}$ is closed in $M_1^+(Y)$.*

The rest of this note is organized as follows. In this section we prove Theorem 1. In Section 2 we give an example showing that the result of Theorem 1 is in general not true for images under continuous mappings. In Section 3, and in particular in Theorem 2 there, we characterize completely the set of images $\{\mu \circ f^{-1} \mid f: X \rightarrow Y \text{ Borel}, \mu \circ f^{-1} \text{ tight}\}$ for a tight Borel probability measure μ , when X and Y are arbitrary metric spaces (so we drop the assumptions of completeness and separability on X and Y , but require that the measures considered

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be tight); this characterization underlies the proof of Theorem 1. We also show in Section 3 that if ν is a tight Borel image of μ , then one can always write ν as $\mu \circ f^{-1}$ with f of Baire class 2 (Theorem 2 again), and that this is optimal with respect to the class of f . Section 3 also contains a generalization of the main result: restricting attention to tight measures only, allows one to drop the assumptions of completeness and separability on X and Y in Theorem 1 (Corollary 1). Finally, in Section 4 we discuss how the set $\{\mu \circ f^{-1} \mid f: X \rightarrow Y \text{ Borel}, \mu \circ f^{-1} \text{ tight}\}$ sits in $C_b(Y)^*$ (rather than $M_1^+(Y)$).

We now proceed with the proof of Theorem 1. We first fix some notation, which is to be retained throughout this section. Assume that:

- (1) X and Y are complete, separable metric spaces, i.e., Polish spaces.
- (2) μ is a Borel probability measure on X .
- (3) $f_n: X \rightarrow Y$, $n \in \mathbb{N}$, is a sequence of Borel mappings, such that $\mu \circ f_n^{-1} \rightarrow \nu$, in the weak*-topology, for some Borel probability measure ν on Y .

We will then show the following, which proves Theorem 1.

Proposition 1. *There exists a Borel mapping $f: X \rightarrow Y$, such that $\nu = \mu \circ f^{-1}$.*

We will call $x \in X$ an atom of μ if $\mu(\{x\}) > 0$. The set of all atoms of μ will be denoted by A ; it is a countable set, finite or infinite and possibly empty. We will write μ_a for the atomic part of μ :

$$\mu_a(B) := \sum_{x \in B \cap A} \mu(\{x\});$$

μ_c will denote the non-atomic (or continuous) part of μ : $\mu_c(B) := \mu(B) - \mu_a(B)$. Note that $\mu = \mu_a + \mu_c$. We will call μ non-atomic if $\mu_a = 0$ and purely atomic if $\mu_c = 0$. Finally, for $x \in X$, δ_x will denote the measure

$$\delta_x(B) := \begin{cases} 1 & \text{if } x \in B, \\ 0 & \text{if } x \notin B. \end{cases}$$

Proposition 2 (Purely atomic case). *If μ is purely atomic, then $\nu = \mu \circ f^{-1}$, for some Borel $f: X \rightarrow Y$.*

Proof. Suppose that μ is purely atomic, say $\mu = \sum_i p_i \delta_{x_i}$, where $p_i > 0$ for all i . So $A = \{x_i\}$. We first claim that, for each atom x_i of μ , there exists a compact set $C_i \subseteq Y$, such that

$$(1.1) \quad f_n(x_i) \in C_i \quad \forall n \in \mathbb{N}.$$

Suppose not. Then for any compact $C \subseteq Y$, there exists an $n_C \geq 1$, such that

$$(1.2) \quad \mu \circ f_{n_C}^{-1}(C) \leq 1 - p_i.$$

Now note that since the sequence $\{\mu \circ f_n^{-1} : n \in \mathbb{N}\}$ converges weak* in $M_1^+(Y)$ it must be tight, by Prohorov's theorem ([1], Theorem 6.2); i.e., for each $\epsilon > 0$, there exists a compact set $C \subseteq Y$, such that

$$\mu \circ f_n^{-1}(C) \geq 1 - \epsilon \quad \forall n \in \mathbb{N}.$$

But for $0 < \epsilon < p_i$, this is violated by (1.2).

By (1.1) (and Cantor's diagonal method if A is infinite), we can find a subsequence $n_1 < n_2 < \dots$, such that, for each i , $f_{n_k}(x_i) \rightarrow y_i$, as $k \rightarrow \infty$, for some

$y_i \in Y$. Then

$$\mu \circ f_{n_k}^{-1} \longrightarrow \sum_i p_i \delta_{y_i} \quad (k \longrightarrow \infty)$$

and so, by our assumption that $\mu \circ f_n^{-1} \longrightarrow \nu$,

$$\nu = \sum_i p_i \delta_{y_i};$$

we note at the outset that some of the y_i in this representation of ν may be the same. As the set A is countable, the mapping $f(x_i) = y_i$ and $f(x) = y$ for $x \in A^c$, where y is any fixed point in Y , is Borel-measurable and $\nu = \mu \circ f^{-1}$. $\square$

We next turn to the non-atomic case. We begin with the case $X = Y = [0, 1]$.

Lemma 1. *If P and Q are Borel probability measures on $[0, 1]$, with P non-atomic, then $Q = P \circ f^{-1}$ for some Borel mapping $f: [0, 1] \rightarrow [0, 1]$.*

Proof. Let $F(x) := P([0, x])$, $x \in [0, 1]$, be the distribution function of P . Since P is non-atomic, F is continuous. It follows that if ξ is a random variable (on some probability space) with distribution P , then $F(\xi)$ has a uniform distribution on the unit interval; i.e., $P \circ F^{-1}$ is Lebesgue measure on the unit interval because F is continuous ([2], Exercise 14.3).

Now let $G: [0, 1] \rightarrow [0, 1]$ be the distribution function of Q and set $g(y) := \inf\{x \in [0, 1]: G(x) \geq y\}$ for $y \in [0, 1]$. Then, since $F(\xi)$ is uniform, $g(F(\xi))$ has distribution Q ; i.e., since $P \circ F^{-1}$ is Lebesgue measure, $(P \circ F^{-1}) \circ g^{-1} = Q$ (cf. [2], 2nd Proof of Theorem 14.1, p. 189). Thus $Q = P \circ (g \circ F)^{-1}$, with $g \circ F$ Borel. $\square$

The next result will allow us to pass from the unit interval $[0, 1]$ to general spaces.

Theorem K (Kuratowski). *Let Z be an uncountable, complete, separable metric space. Then there exists a bijection $\pi: Z \rightarrow [0, 1]$, such that both π and π^{-1} are Borel mappings.*

Proof. [7], Chapter 15, Theorem 10. $\square$

Proposition 3 (Non-atomic case). *If μ is non-atomic, then $\nu = \mu \circ f^{-1}$ for some Borel mapping $f: X \rightarrow Y$.*

Proof. We shall in fact show that

$$(1.3) \quad \{\mu \circ f^{-1} \mid f: X \rightarrow Y \text{ Borel}\} = M_1^+(Y)$$

when μ is non-atomic. Proposition 3 is then obviously a consequence of this fact.

First note that since μ is non-atomic, any set of positive μ -measure must be uncountable. Thus X is uncountable. Suppose now that Y is also uncountable. Let π_X and π_Y be the bijections of Kuratowski's theorem (Theorem K), corresponding to X and Y respectively. Set $\tilde{\mu} := \mu \circ \pi_X^{-1}$ and $\tilde{\nu} := \nu \circ \pi_Y^{-1}$; $\tilde{\mu}$ and $\tilde{\nu}$ are Borel probability measures on $[0, 1]$. Furthermore, $\tilde{\mu}$ is non-atomic, since π_X is injective and μ is non-atomic. It follows from Lemma 1 that $\tilde{\nu} = \tilde{\mu} \circ \tilde{f}^{-1}$ for some Borel $\tilde{f}: [0, 1] \rightarrow [0, 1]$. Now set $f := \pi_Y^{-1} \circ \tilde{f} \circ \pi_X$; then f is a Borel mapping of X into Y and $\nu = \mu \circ f^{-1}$.

To complete the proof we also need to consider the case where Y is countable, say $Y = \{y_j: j \in J\}$. Have disjoint Borel sets $X_j \subseteq X$, $j \in J$, such that $\mu(X_j) = \nu(\{y_j\})$ for each j . This is possible because μ is non-atomic. (For example, use the intermediate value theorem for the function $\tilde{F}(x) := \tilde{\mu}([0, x])$,

$x \in [0, 1]$, which is continuous and with $\tilde{F}(0) = 0$ and $\tilde{F}(1) = 1$, to obtain points $x_0 := 0, x_1, x_2, \dots$ in $[0, 1]$ such that $\tilde{\mu}(x_{j-1}, x_j) = \nu(\{y_j\})$ for all $j \in J$, and then set $X_j := \pi_X^{-1}(x_{j-1}, x_j)$. Here we identify J with $\{1, \dots, n\}$ for some $n \in \mathbb{N}$, or $\mathbb{N}$ itself, and the notation $\tilde{\mu}, \pi_X$ is that of the previous paragraph.) Define $f(x) = y_j$ for $x \in X_j, j \in J$, and $f(x) = y$ for $x \in X \setminus \bigcup_{j \in J} X_j$, where y is any fixed point in Y . Then $\nu = \mu \circ f^{-1}$ and f is Borel because the sets X_j are Borel and J is countable. $\square$

Proof of Proposition 1. If μ is either purely atomic or non-atomic, then we are done by the two preceding propositions. So assume that both $a := \mu_a(X) \neq 0$ and $c := \mu_c(X) \neq 0$. Then $a^{-1}\mu_a$ and $c^{-1}\mu_c$ are both probability measures.

Since $\{\mu \circ f_n^{-1}\}$ converges weak* in $M_1^+(Y)$, it is tight (Prohorov's theorem). It follows that each of $\{(a^{-1}\mu_a) \circ f_n^{-1}\}$ and $\{(c^{-1}\mu_c) \circ f_n^{-1}\}$ is tight. By Prohorov's theorem again (this time Theorem 6.1 of [1]), there exist probability measures ν^a and ν^c on Y , and a subsequence $n_1 < n_2 < \dots$, along which

$$(1.4) \quad (a^{-1}\mu_a) \circ f_{n_k}^{-1} \longrightarrow \nu^a \quad \text{and} \quad (c^{-1}\mu_c) \circ f_{n_k}^{-1} \longrightarrow \nu^c.$$

Propositions 2 and 3 now imply that $\nu^a = (a^{-1}\mu_a) \circ f_a^{-1}$ and $\nu^c = (c^{-1}\mu_c) \circ f_c^{-1}$, for some Borel mappings $f_a, f_c: X \rightarrow Y$. Since we are assuming that $\mu \circ f_n^{-1} \longrightarrow \nu$, (1.4) implies that

$$\nu(h) = \lim_{k \rightarrow \infty} \mu \circ f_{n_k}^{-1}(h) = \lim_{k \rightarrow \infty} \mu_a \circ f_{n_k}^{-1}(h) + \lim_{k \rightarrow \infty} \mu_c \circ f_{n_k}^{-1}(h) = a\nu^a(h) + c\nu^c(h),$$

for every bounded continuous function $h: Y \rightarrow \mathbb{R}$. It follows from this equality that $\nu = a\nu^a + c\nu^c$ (Theorem 1.3 of [1]), whence

$$\nu = \mu_a \circ f_a^{-1} + \mu_c \circ f_c^{-1}.$$

Recall that the set A of atoms of μ is countable, hence Borel. Thus the mapping $f(x) = f_a(x)$ if $x \in A$ and $f(x) = f_c(x)$ if $x \in A^c$ is Borel and $\nu = \mu \circ f^{-1}$. $\square$

The proof of Theorem 1 is now complete.

2. IMAGES UNDER CONTINUOUS MAPPINGS

It is natural to ask whether Theorem 1 remains true when one replaces the set $\{\mu \circ f^{-1} \mid f \text{ Borel}\}$ by $\{\mu \circ f^{-1} \mid f \text{ continuous}\}$. The answer is easily seen to be negative. In particular, it is easy to construct a sequence of continuous functions $f_n: [0, 1] \rightarrow [0, 1]$, and a Borel probability measure μ on $[0, 1]$, such that $\mu \circ f_n^{-1} \longrightarrow \nu$ for some probability measure ν , which is not of the form $\mu \circ g^{-1}$ with g continuous. The following is an elementary example.

Example 1. Take μ to be Lebesgue measure on $[0, 1]$, let

$$f_n(x) = \begin{cases} 2x/3 & \text{for } 0 \leq x < x_n, \\ 2x/3 + 1/3 & \text{for } 1/2 \leq x \leq 1, \end{cases}$$

and define f_n to be linear in $[x_n, 1/2)$ and so that f_n is continuous on $[0, 1]$; here $\{x_n\}$ is any sequence with $0 < x_1 < x_2 < \dots \uparrow 1/2$. Then $f_n \longrightarrow f$, where

$$f(x) = \begin{cases} 2x/3 & \text{for } 0 \leq x < 1/2, \\ 2x/3 + 1/3 & \text{for } 1/2 \leq x \leq 1. \end{cases}$$

Hence $\mu \circ f_n^{-1} \longrightarrow \nu$, where $\nu = \mu \circ f^{-1}$, and ν cannot be written as $\mu \circ g^{-1}$ for some continuous g . For $\nu(a, b) = 0$ for any interval with $1/3 < a < b < 2/3$, and so if $\nu = \mu \circ g^{-1}$, then $g^{-1}(a, b)$ must be of Lebesgue measure zero. If g

is continuous however, $g^{-1}(a, b)$ is open and must therefore either have positive Lebesgue measure or be empty. But $g^{-1}(a, b)$ can also not be empty when g is continuous and $\nu = \mu \circ g^{-1}$, because we must have $g([0, 1]) = [0, 1]$ for such a g .

3. FURTHER RESULTS AND GENERALIZATIONS

Section 1 contains actually more information than is stated in Theorem 1. For example (1.3) shows that *any* Borel probability measure on Y can be obtained as an image of *any* non-atomic μ on X . Furthermore, the assumptions of completeness and separability on X and Y are superfluous throughout Section 1, provided one restricts attention to tight probability measures (see below for the definition). In the present section we drop these assumptions on X and Y altogether, give a complete characterization of the tight measures ν on Y that can be obtained as Borel images of a given tight measure μ on X (Theorem 2) and extend Theorem 1 to arbitrary metric spaces (Corollary 1).

Given metric spaces X and Y , write

$$B(\mu) := \{\mu \circ f^{-1} \mid f: X \rightarrow Y \text{ Borel}\}$$

for the set of images of a Borel measure μ on X under Borel mappings from X to Y . We also retain the notation of Section 1 pertaining to measures: for a measure κ on a metric space Z we write κ_a and κ_c for its atomic and continuous parts respectively and denote by A_κ its possibly empty and always countable set of atoms. Furthermore, having dropped the assumption that Z is Polish, we now denote by $M_1^+(Z)$ the space of *tight* Borel probability measures on Z . Recall that a Borel probability measure κ on a metric space Z is tight if, given any $\epsilon > 0$, there exists a compact set K such that $\kappa(K^c) < \epsilon$.

Note. A few remarks regarding terminology are perhaps in order. Recall that every finite Borel measure κ on a metric space Z is regular in the weak sense: for each Borel set B , and any $\epsilon > 0$, there exist an open set O and a closed set C , such that $C \subseteq B \subseteq O$ and $\kappa(O \setminus C) < \epsilon$ ([7], Chapter 15, Proposition 11). A Borel probability measure κ is tight iff it is regular in the strong sense, i.e., for each $\epsilon > 0$, the closed set C above can be chosen to be compact. Recall that when Z is Polish, every Borel probability measure is strongly regular, hence tight (cf. [7], Chapter 15, Proposition 18, or [1], Theorem 1.4). Thus the notation $M_1^+(Z)$ introduced here agrees with the one already introduced for Polish spaces in Section 1 (i.e., for Polish spaces, $M_1^+(Z)$ contains *all* Borel probability measures on Z).

3.1. Characterization of Borel images of a tight measure. Part (i) of the following theorem is in the heart of the proof of Theorem 1 (and Corollary 1).

Theorem 2. *Let X and Y be metric spaces, μ a tight Borel probability measure on X and ν a tight Borel measure on Y . Write*

$$\mu_a = \sum_{i \in I} p_i \delta_{x_i} \quad (A_\mu = \{x_i : i \in I\}; p_i > 0 \ \forall i \in I)$$

and

$$\nu_a = \sum_{j \in J} q_j \delta_{y_j} \quad (A_\nu = \{y_j : j \in J\}; q_j > 0 \ \forall j \in J)$$

for the atomic parts of μ and ν respectively, with I and J possibly empty. Then:

- (i) $\nu \in B(\mu)$ iff ν is a probability measure, and there exists a partition $I = \bigcup_{j \in J} I_j$ of the atoms of μ , with some I_j possibly empty, such that

$$q_j \geq \sum_{i \in I_j} p_i \quad \forall j \in J.$$

- (ii) If $\nu \in B(\mu)$, then one can always write $\nu = \mu \circ f^{-1}$ with f of Baire class 2.

Remark. Part (i) of this theorem

- (A) simply reflects the obvious fact that the atoms of a measure cannot decompose when we take an image of it (but several atoms, or non-atomic mass, can aggregate to form a single atom), and
 (B) also states that there are no further restrictions for a tight measure ν on Y to be a Borel image of μ (except of course that μ and ν must also have the same total mass).

Thus in particular, if μ is non-atomic, then $B(\mu) \cap M_1^+(Y) = M_1^+(Y)$, as already shown in (1.3) when X and Y are Polish, while when μ is purely atomic, then $B(\mu)$ consists precisely of all purely atomic measures of the form $\nu = \sum_{j \in J} q_j \delta_{y_j}$ with

$$q_j = \sum_{i \in I_j} p_i \quad \forall j \in J,$$

where $\{I_j : j \in J\}$ ranges over all possible partitions $I = \bigcup_{j \in J} I_j$ of the atoms of μ .

Before giving the proof of Theorem 2 we briefly recall the terminology on Baire classes and collect a couple of elementary facts that will be used in the proof, for the convenience of the reader.

A Borel mapping $f: X \rightarrow Y$ is said to be of Baire class 0 if it is continuous; f is of class 1 if $f^{-1}(O)$ is an F_σ -set for every open set O in Y ; it is of class 2 if, for every open O in Y , $f^{-1}(O)$ is a $G_{\delta\sigma}$ -set, etc.; see Kuratowski [6] for these definitions. The composition of two mappings of classes a and b , respectively, is of class $a + b$. And if $X = X_1 \cup X_2$ with each X_i an $F_{\sigma\delta}$ and $f|_{X_i}$ is of class 2 for each $i = 1, 2$, then f is of class 2. These elementary facts will be used below and can be found in [6] (Theorem 2 in §31, III and Theorem 2 in §31, IV).

Proof of Theorem 2. One direction in (i) is entirely trivial. If $\nu = \mu \circ f^{-1}$ for some Borel $f: X \rightarrow Y$, then ν is obviously a probability measure, and $I_j := \{i \in I : x_i \in f^{-1}(\{y_j\})\}$ gives the desired partition of the atoms of μ .

For the other direction, consider a tight Borel probability measure ν on Y , for which a partition $I = \bigcup_{j \in J} I_j$ such that $q_j \geq \sum_{i \in I_j} p_i$ exists. We seek to find a mapping $f: X \rightarrow Y$, of Baire class 2, such that $\nu = \mu \circ f^{-1}$; this will then also prove (ii).

First observe that we may, without loss of generality, assume that μ is non-atomic. For if the theorem is true for non-atomic measures, have a Baire class 2 map $f_c: X \rightarrow Y$ such that $\nu' = \mu_c \circ f_c^{-1}$, where ν' is the measure on Y with $\nu'_c = \nu_c$ and $\nu'(\{y_j\}) = q_j - \sum_{i \in I_j} p_i$ for each $j \in J$, and then define $f: X \rightarrow Y$ by setting $f = f_c$ on $X \setminus A_\mu$ and $f(x_i) := y_j$ for $i \in I_j$, $j \in J$. As $f|_{X \setminus A_\mu}$ is of class 2 and $f|_{A_\mu}$ of class 1 (since A_μ is countable), and as each of the sets $X \setminus A_\mu$ and A_μ is an $F_{\sigma\delta}$ (being G_δ and F_σ , respectively), the resulting mapping f is of class 2 on X , and clearly $\nu = \mu \circ f^{-1}$.

We may further assume that X and Y are σ -compact. For, since μ and ν are assumed to be tight, one can find σ -compact subsets $X' \subseteq X$ and $Y' \subseteq Y$ of

full μ - and ν -measure, respectively. If $f: X' \rightarrow Y'$ is a Baire 2 mapping with $\nu|_{Y'} = (\mu|_{X'}) \circ f^{-1}$, extend f to all of X by letting it be constant on $X \setminus X'$; the resulting mapping will be of class 2 on all of X , for the same reasons as in the previous paragraph, and again one will have $\nu = \mu \circ f^{-1}$, by the very definition of X' and Y' .

So assume from now on that X and Y are σ -compact and that μ is non-atomic. The idea of course is to embed X and Y into $[0, 1]$ and then use Lemma 1 to construct an f that carries μ to ν , as in the proof of Proposition 3. A little care is needed in doing this however, because here we also want to control the class of the resulting mapping f . We remark that a shorter and less involved argument for this part of the theorem can be given if one does not pay attention to the class of f , i.e., if one only cares to obtain part (i) of Theorem 2; see the Remark following this proof.

Let $e_X: X \rightarrow \mathbb{R}^\infty$ be the mapping given by $e_X(x) := (\rho_X(x, d_1), \rho_X(x, d_2), \dots)$, where ρ_X is a metric on X and $\{d_n: n \in \mathbb{N}\}$ is a countable dense subset of X . As is well known, e_X is a homeomorphism of X onto $e_X(X)$ (cf. [6], §22, II, Remark 3, or [1], Appendix I, p. 219). Define $e_Y: Y \rightarrow \mathbb{R}^\infty$ similarly.

Let G be a zero-dimensional, uncountable, G_δ -subset of $\mathbb{R}^\infty$ which has both full $\mu \circ e_X^{-1}$ - and full $\nu \circ e_Y^{-1}$ -measure. Such a set G may be constructed as follows. Let κ be the Borel measure $\kappa := \mu \circ e_X^{-1} + \nu \circ e_Y^{-1}$ on $\mathbb{R}^\infty$. Since, for each $n \geq 1$, $x \mapsto \kappa(\{(x_m)_{m \in \mathbb{N}} \in \mathbb{R}^\infty: x_n \leq x\})$ is a non-decreasing function of $x \in \mathbb{R}$, there exists a countable, dense in $\mathbb{R}$ set $\{q_{n,k}: k \in \mathbb{N}\}$ such that $\kappa(\{(x_m)_{m \in \mathbb{N}} \in \mathbb{R}^\infty: x_n = q_{n,k}\}) = 0$ for each $k \in \mathbb{N}$. The set

$$G := \bigcap_n \bigcap_k \{(x_m)_{m \in \mathbb{N}} \in \mathbb{R}^\infty: x_n \neq q_{n,k}\}$$

then has the desired properties. It follows that G is homeomorphic to a closed subset of $\mathbb{I} := [0, 1] \setminus \mathbb{Q}$ (cf. [6], §36, II, Theorem 2). Let $h: G \rightarrow h(G) \subseteq \mathbb{I}$ be a homeomorphism.

Next let $\pi_X: e_X^{-1}(G) \rightarrow [0, 1]$ and $\pi_Y: e_Y^{-1}(G) \rightarrow [0, 1]$ be the mappings defined by $\pi_X := h \circ e_X$ and $\pi_Y := h \circ e_Y$, respectively. π_X and π_Y are homeomorphisms of $e_X^{-1}(G)$ and $e_Y^{-1}(G)$ onto subsets $\tilde{X}$ and $\tilde{Y}$ of $\mathbb{I}$, and $\tilde{X} = h(e_X(X) \cap G)$ and $\tilde{Y} = h(e_Y(Y) \cap G)$ are F_σ in $\mathbb{I}$ because X and Y are assumed to be σ -compact and $h(G)$ is closed in $\mathbb{I}$.

Since μ is non-atomic, and since π_X is an injection, the measure $\tilde{\mu} := \mu \circ \pi_X^{-1}$ defines a non-atomic Borel probability measure on $[0, 1]$. Consequently, by Lemma 1, there exists a mapping $\tilde{f}: [0, 1] \rightarrow [0, 1]$ such that $\tilde{\nu} = \tilde{\mu} \circ \tilde{f}^{-1}$, where $\tilde{\nu} := \nu \circ \pi_Y^{-1}$. Define $f := \pi_Y^{-1} \circ \tilde{f} \circ \pi_X$ on $\pi_X^{-1}(\tilde{X} \cap \tilde{f}^{-1}(\tilde{Y}))$ and then extend it to be constant on the complement of this set. Obviously, $\nu = \mu \circ f^{-1}$.

We now consider the class of f . One sees upon inspecting the proof of Lemma 1 that $\tilde{f}: [0, 1] \rightarrow [0, 1]$ is always a non-decreasing function. This implies that if I is an interval, then $\tilde{f}^{-1}(I)$ is also an interval, which may or may not contain its endpoints. In particular, if I is an interval which is open as a subset of $[0, 1]$ (i.e., $I = (a, b)$, $I = [0, b)$, $I = (a, 1]$, or $I = [0, 1]$), then $\tilde{f}^{-1}(I)$ is an open interval union with a finite set, the latter consisting of one, both, or none of the endpoints of this interval.¹

¹ As a matter of fact $\tilde{f}$ is also left-continuous, so the left endpoint is never included in the inverse image of an open interval, unless it equals 0.

It follows that:

- (a) $\tilde{f}^{-1}(O)$ is an F_σ in $[0, 1]$ whenever O is an open subset of $[0, 1]$;
- (b) for any set C closed in $[0, 1]$, $\tilde{f}^{-1}(C)$ is a closed $\setminus$ countable set.

By (a), $\tilde{f}$ is of class 1 on $[0, 1]$. By (b), $\tilde{f}^{-1}(\tilde{Y})$ is of the form $\tilde{f}^{-1}(\tilde{Y}) = F_1 \setminus F_2$ with F_1 and F_2 both F_σ in $[0, 1]$: $\tilde{Y} = F \setminus \mathbb{Q}$, where F is an F_σ in $[0, 1]$ (recall that $\tilde{Y}$ is F_σ in $\mathbb{I} = [0, 1] \setminus \mathbb{Q}$), whence by (b)

$$\tilde{f}^{-1}(\tilde{Y}) = \tilde{f}^{-1}(F) \setminus \tilde{f}^{-1}(\mathbb{Q}) = (F_1 \setminus C) \setminus \tilde{f}^{-1}(\mathbb{Q})$$

with F_1 an F_σ in $[0, 1]$ and C countable. Since $\tilde{f}^{-1}(\mathbb{Q})$ is also an F_σ , because by the monotonicity of $\tilde{f}$ again inverse images of singletons are either empty, single points, or intervals, $\tilde{f}^{-1}(\tilde{Y}) = F_1 \setminus F_2$ with each F_i an F_σ in $[0, 1]$, as asserted.

Since π_X is continuous, $\pi_X^{-1}(\tilde{X} \cap \tilde{f}^{-1}(\tilde{Y}))$ is of the form

$$\pi_X^{-1}(\tilde{X} \cap \tilde{f}^{-1}(\tilde{Y})) = F'_1 \setminus F'_2$$

with each F'_i an F_σ in $e_X^{-1}(G)$; since $e_X^{-1}(G)$ is itself a G_δ in X , it follows that $\pi_X^{-1}(\tilde{X} \cap \tilde{f}^{-1}(\tilde{Y}))$ is of the form

$$\pi_X^{-1}(\tilde{X} \cap \tilde{f}^{-1}(\tilde{Y})) = F' \cap G'$$

with F' an F_σ in X and G' a G_δ in X . Thus:

- both $\pi_X^{-1}(\tilde{X} \cap \tilde{f}^{-1}(\tilde{Y}))$ and its complement are $F_{\sigma\delta}$ -subsets of X ;
- f is of class 1 when restricted to $\pi_X^{-1}(\tilde{X} \cap \tilde{f}^{-1}(\tilde{Y}))$, since $f = \pi_Y^{-1} \circ \tilde{f} \circ \pi_X$ on this set, and $\tilde{f}$ is of class 1 while both π_X and π_Y^{-1} are continuous;
- f is of class 0 on the complement of $\pi_X^{-1}(\tilde{X} \cap \tilde{f}^{-1}(\tilde{Y}))$, being constant on that set.

It follows that f is of class 2 on X . □

Remark. A more direct argument for part (i) of Theorem 2 can be given when one does not care about the class of the f carrying a given μ to a given ν . First reduce, as in the above proof, to the case where μ is non-atomic and X and Y are σ -compact. Define e_X and e_Y as before and then use Theorem K to obtain a Borel isomorphism $\pi: \mathbb{R}^\infty \rightarrow [0, 1]$. Then, $\pi_X := \pi \circ e_X$ and $\pi_Y := \pi \circ e_Y$ are Borel isomorphisms from X and Y onto Borel subsets $\tilde{X}$ and $\tilde{Y}$ of $[0, 1]$, respectively. Now use Lemma 1 as in the above proof to obtain a Borel $\tilde{f}: \tilde{X} \rightarrow \tilde{Y}$ with $\nu = \mu \circ \tilde{f}^{-1}$. The essential ingredients of this argument are of course (1) tightness, which allows one to restrict attention to σ -compact subsets of X and Y of full measure, and (2) the fact that any σ -compact metric space is Borel isomorphic to a Borel subset of $[0, 1]$ (as the above argument shows), so that $\tilde{X}$ and $\tilde{Y}$ are Borel in $[0, 1]$. Completeness and separability, which is what we assumed of X and Y in Proposition 3, are thus unnecessary, provided one restricts oneself to tight measures.

3.2. Optimality in Theorem 2. Theorem 2 is optimal with respect to the Baire class of f in a representation $\nu = \mu \circ f^{-1}$ of a $\nu \in B(\mu) \cap M_1^+(Y)$. Example 1 in Section 2 shows that it may be impossible to write a $\nu \in B(\mu) \cap M_1^+(Y)$ as $\nu = \mu \circ f^{-1}$ with f of class 0. The following example shows that it may also be impossible to choose f to be of class 1.

Example 2. Let κ be an arbitrary probability measure concentrated on the rationals in $[0, 1]$ and such that $\kappa(\{q\}) > 0$ for each $q \in \mathbb{Q} \cap [0, 1]$. Let $X = Y = [0, 1]$

and λ be Lebesgue measure on $[0, 1]$. Define

$$\mu = \frac{1}{2}\lambda + \frac{1}{2}\kappa,$$

$$f(x) = \begin{cases} 0 & \text{for } x \in \mathbb{Q} \cap [0, 1], \\ x/2 + 1/2 & \text{for } x \in [0, 1] \setminus \mathbb{Q}, \end{cases}$$

and let $\nu = \mu \circ f^{-1}$. Suppose now that $\nu = \mu \circ g^{-1}$ with g of class 1. Then $g^{-1}(\{0\})$ is a G_δ -set, G say, containing $\mathbb{Q} \cap [0, 1]$. Hence G is also dense. Furthermore, $F := g^{-1}([0, 1/2])$ is F_σ , contains G , and also has $\lambda(F) = 0$. But this is impossible by virtue of the Baire category theorem: If $F = \bigcup_n F_n$ with the F_n closed, each F_n is nowhere dense, because F^c is a set of full measure contained in F_n^c . If $G^c = \bigcup_n C_n$ with the C_n closed, then each C_n is also nowhere dense, because C_n^c contains G , which is dense. But then $X = \bigcup_n (F_n \cup C_n)$, a countable union of nowhere dense sets.

The following example shows that it may be impossible to choose f of class 1, even when μ is purely atomic. This example is due to S. Papadopoulos.

Example 3. Let p_n , $n \in \mathbb{N}$, be (strictly) positive numbers, which sum to 1, and such that no p_n can be written as a sum of other p_j 's. This can be done, for example, by choosing $p_1 > 1/2$ and then $p_n > (1 - p_1 - \dots - p_{n-1})/2$ for each $n > 1$. Let μ be the probability measure which puts mass p_n at q_n , where $\{q_n : n \in \mathbb{N}\}$ is an enumeration of the rationals in $[0, 1]$. Let $Q_1 \cup Q_2$ be a partition of $\mathbb{Q} \cap [0, 1]$ with each Q_i dense, and let $f : [0, 1] \rightarrow [0, 1]$ be any function which is one-to-one on the rationals and sends Q_1 into $[0, 1/3]$ and Q_2 into $[2/3, 1]$. The measure $\nu = \mu \circ f^{-1}$, on $[0, 1]$, cannot be written as $\nu = \mu \circ g^{-1}$ with g of class 1. For we must have $f = g$ on $\mathbb{Q} \cap [0, 1]$, by our choice of the p_n , and then $g^{-1}([0, 1/3])$ and $g^{-1}([2/3, 1])$ are both dense G_δ -sets, which are also disjoint. This is again impossible by Baire's theorem.

3.3. Extension of Theorem 1 to arbitrary metric spaces. Recall that, for an arbitrary metric space Z , we denote by $M_1^+(Z)$ the set of *tight* Borel probability measures on Z . Endowed with the weak*-topology inherited from $C_b(Z)^*$, this space is still metrizable by the Prohorov metric (cf. [1], Theorem 5 in Appendix III).

Corollary 1. *Let X and Y be metric spaces and $\mu \in M_1^+(X)$. Then $B(\mu) \cap M_1^+(Y)$ is closed in $M_1^+(Y)$.*

Proof. Recall the proof of Theorem 1. The assumptions of completeness and separability on X and Y were used

- (1) when applying Prohorov's theorem in the form of Theorem 6.2 of [1] (i.e., that weak*-convergence implies tightness) in the proofs of Propositions 1 and 2;
- (2) when applying Theorem K in the proof of Proposition 3, to get (1.3) for general X and Y from Lemma 1.

Prohorov's theorem remains true for arbitrary metric spaces when one restricts attention to tight Borel probability measures. In particular, one can replace Theorem 6.2 of [1] by Theorem 8 in Appendix III of the same reference. One can also replace (1.3) by

$$(3.1) \quad B(\mu) \cap M_1^+(Y) = M_1^+(Y)$$

for arbitrary X and Y and μ non-atomic, which is part of the conclusion of Theorem 2 (see Remark after Theorem 2; see also the Remark following the proof of Theorem 2 in this connection). Thus the proof of Theorem 1 goes through essentially verbatim to prove Corollary 1. $\square$

4. CONCLUDING REMARKS

4.1. Let Y be an arbitrary metric space. Since Y is a completely regular topological space, $C_b(Y)^*$ may be identified, via an isometric isomorphism, with the space $M(\check{Y})$ of regular, signed, Borel measures on the Stone-Čech compactification $\check{Y}$ of Y (cf. [3], Chapter V, Corollary 6.4). The set $M_b(Y)$ of finite, tight, signed, Borel measures on Y is obviously contained in $M(\check{Y})$ (a signed measure being tight iff both its positive and negative parts are).

When Y is locally compact and separable, sequences in $M_1^+(Y)$ can only converge to elements in $M_1^+(Y)$. (Unfortunately, we could not locate a reference for this fact in the literature, so we give an argument below.) Thus in this case Corollary 1 also shows that the set of tight Borel images $B(\mu) \cap M_1^+(Y)$, of a tight Borel probability measure μ on X , is *sequentially* closed in $C_b(Y)^*$. When Y is actually compact, $M_1^+(Y)$ is a compact subset of $C_b(Y)^* = C(Y)^*$, and hence $B(\mu)$ is closed in $C_b(Y)^*$. (Thus $B(\mu)$ is compact when Y is compact.) When Y is not compact however, the set $B(\mu) \cap M_1^+(Y)$ may not be closed in $C_b(Y)^*$, as the following argument shows.

First observe that when Y is not compact, $M_1^+(Y)$ is not closed in $M(\check{Y})$: for if $y \in \check{Y} \setminus Y$, then $y_\alpha \rightarrow y$ for some net $\{y_\alpha\} \subseteq Y$, as Y is dense in $\check{Y}$; consequently $\delta_{y_\alpha} \rightarrow \delta_y$ in the weak*-topology, showing that $M_1^+(Y)$ is not closed in $M(\check{Y})$. But then, if μ is any non-atomic element of $M_1^+(X)$, $B(\mu) \cap M_1^+(Y) = M_1^+(Y)$, by Theorem 2, and so $B(\mu) \cap M_1^+(Y)$ is not closed in $C_b(Y)^* \simeq M(\check{Y})$.

One is naturally led to wonder whether the fact that $M_1^+(Z)$ is sequentially closed in $C_b(Z)^*$ remains true for arbitrary metric spaces Z . As above, this would of course automatically imply that $B(\mu) \cap M_1^+(Y)$ is sequentially closed in $C_b(Y)^*$ for arbitrary metric spaces X and Y and $\mu \in M_1^+(X)$, but this question is also of independent interest.

Note added in proof. It turns out that $M_1^+(Z)$ is sequentially closed in $C_b(Z)^*$ whenever the former contains *all* Borel probability measures on Z (cf. [4]); thus in particular, besides the case where Y is locally compact and separable, $B(\mu) \cap M_1^+(Y)$ is also sequentially closed in $C_b(Y)^*$ whenever Y is Polish. In general however, $M_1^+(Z)$ need *not* be sequentially closed in $C_b(Z)^*$ (cf. [1], comment following Theorem 8 in Appendix III), and neither does $B(\mu) \cap M_1^+(Y)$ need to be sequentially closed in $C_b(Y)^*$. For an example of the latter (taken from [1]), take Y to be a subset of the unit interval with inner Lebesgue measure 0 and outer Lebesgue measure 1, with the topology it inherits from $[0, 1]$. If ν is outer Lebesgue measure on the Borel subsets of Y , every compact set in Y is Borel in $[0, 1]$ and therefore has ν -measure 0, as it has inner measure 0; thus ν is non-tight. On the other hand, one can find measures ν_n of *finite* support on Y such that $\nu_n \rightarrow \nu$ (cf. [1], Theorem 4 in Appendix III). Then however, if μ is Lebesgue measure on $X := [0, 1]$, one can easily find Borel functions $f_n : X \rightarrow Y$ such that $\nu_n = \mu \circ f_n^{-1}$, whence $\mu \circ f_n^{-1} \in M_1^+(Y)$, $\mu \circ f_n^{-1} \rightarrow \nu$, yet $\nu \notin M_1^+(Y)$.

4.2. We now present an argument for the following fact, which was used above. *If Z is a separable, locally compact metric space and $\{\mu_n\}$ a sequence of measures in $M_1^+(Z)$ which converges weak* to some $\mu \in C_b(Z)^* \simeq M(\check{Z})$, then $\mu \in M_1^+(Z)$.*

Note. Since a locally compact and separable metric space is σ -compact, all Borel probability measures on such a space Z are tight; thus, as in the case of Polish spaces, $M_1^+(Z)$ again contains *all* Borel probability measures on Z .

In view of Prohorov's theorem and the preceding Note, it suffices to show that convergence of a sequence of Borel probability measures $\{\mu_n\}$ on Z in $C_b(Z)^*$ implies tightness of the sequence.

Assume $\{\mu_n\} \subseteq M_1^+(Z)$ is not tight. Then for some (and hence all sufficiently small) $\varepsilon > 0$ we have that

$$(4.1) \quad \limsup_{n \rightarrow \infty} \mu_n(K^c) \geq \varepsilon \quad \text{for all compact } K \subseteq Z.$$

Fix ϵ to be the supremum of all those ε for which (4.1) holds and let $\epsilon' < \epsilon < \epsilon''$, with $\epsilon'' - \epsilon' \leq \epsilon'/2$.

Let O_n , $n \in \mathbb{N}$, be a countable collection of open sets with compact closures, whose union is all of Z ; the existence of such a sequence is guaranteed by local compactness and separability. First have a compact set K_0 with $\mu_n(K_0^c) \leq \epsilon''$ for all sufficiently large n , say $n \geq N$. Next, use (4.1) to choose $n_1 \geq N$ with $\mu_{n_1}(K_0^c) > \epsilon'$, and then use tightness to choose a compact $K_1 \subseteq K_0^c$ with $\mu_{n_1}(K_1) \geq \epsilon'$. Having chosen $K_0, \dots, K_{i-1}$, $i > 1$, use (4.1) to choose $n_i > n_{i-1}$, such that the μ_{n_i} -measure of the complement of the compact set $(K_0 \cup K_1 \cup \dots \cup K_{i-1}) \cup (\overline{O}_1 \cup \dots \cup \overline{O}_{i-1})$ is $> \epsilon'$, and then have a compact set K_i , with

$$(4.2) \quad K_i \subseteq [(K_0 \cup K_1 \cup \dots \cup K_{i-1}) \cup (\overline{O}_1 \cup \dots \cup \overline{O}_{i-1})]^c,$$

such that $\mu_{n_i}(K_i) \geq \epsilon'$.

Let A be the union of the K_i over even $i \geq 0$ and B the union over odd i . The sets A and B are disjoint, by (4.2), and, using (4.2) again, it is easy to see that $A \cup B = \bigcup_{i \geq 0} K_i$ is closed (the O_n are open and cover Z). Define a function $f \in C_b(Z)$ by setting $f = 0$ on A , $f = 1/\epsilon'$ on B , and extending f continuously onto all of Z and so that $0 \leq f \leq 1/\epsilon'$. Then $\mu_{n_i}(f) \geq 1$ for all odd i , and $\mu_{n_i}(f) \leq (\epsilon'' - \epsilon') \cdot (1/\epsilon') \leq 1/2$ for i even and positive, so that $\{\mu_{n_i}\}$ cannot converge in $C_b(Z)^*$.

Note. As pointed out by the referee, the above argument is really a variant of the argument proving Shur's theorem, namely that strong and weak convergence in ℓ^1 are equivalent ([5], Theorem 99), and one can, as a matter of fact, derive the above statement—of sequential closedness of $M_1^+(Z)$ for locally compact separable Z —as a consequence of Shur's theorem.

4.3. We conclude with one more observation. Let $M_b^+(Z)$ denote the space of finite, non-negative, tight, Borel measures on an arbitrary metric space Z , with the weak*-topology. Then, part of the argument in the Proof of Proposition 1 provides a proof of the following general fact (which, as we learned, seems to be part of the folklore); this fact was essentially used (and proved) implicitly in the proof of Proposition 1. *If Λ and K are closed sets in $M_b^+(Z)$, then their algebraic sum $K + \Lambda$ is also closed.*

The complete argument is as follows. First note that $M_b^+(Z)$ is metrizable: If ρ_1 denotes the Prohorov metric on $M_1^+(Z)$, the metric defined by

$$\rho_b(\kappa, 0) := \kappa(Z)$$

for all $\kappa \in M_b^+(Z)$, where 0 is the measure identically equal to 0, and by

$$\rho_b(\kappa, \lambda) := |\kappa(Z) - \lambda(Z)| + \rho_1(\kappa', \lambda') \cdot \min\{\kappa(Z), \lambda(Z)\}$$

for $\kappa, \lambda \in M_b^+(Z) \setminus \{0\}$, where κ' and λ' are the measures κ and λ respectively normalized to be probability measures, provides a metric on $M_b^+(Z)$ compatible with the weak*-topology. Thus sequences are enough to characterize closed sets in $M_b^+(Z)$.

Let $\{\kappa_n + \lambda_n\}$ be a sequence in $K + A$ that converges to some element μ in $M_b^+(Z)$. We consider two cases:

Case 1: One (or both) of the numerical sequences $\{\kappa_n(Z)\}$ and $\{\lambda_n(Z)\}$, say $\{\kappa_n(Z)\}$, has a subsequence $n_1 < n_2 < \dots$ that converges to 0. Then $\kappa_{n_j} \rightarrow 0$ in $M_b^+(Z)$, whence also $\lambda_{n_j} \rightarrow \mu$. Thus $\mu = 0 + \mu$ with $0 \in K$ and $\mu \in A$, because we have assumed K and A to be closed.

Case 2: Both $\inf_n \kappa_n(Z) > 0$ and $\inf_n \lambda_n(Z) > 0$. Since $\kappa_n + \lambda_n \rightarrow \mu$ in $M_b^+(Z)$, the normalized measures $\mu_n := (\kappa_n + \lambda_n) / [\kappa_n(Z) + \lambda_n(Z)]$ converge in $M_1^+(Z)$, and therefore form a tight sequence, by Prohorov's theorem. It follows that each of the normalized sequences $\{\kappa'_n\}$ and $\{\lambda'_n\}$ is tight and therefore has a convergent subsequence, again by Prohorov's theorem. Thus $\kappa'_{n_j} \rightarrow \kappa'$ and $\lambda'_{n_j} \rightarrow \lambda'$, for some subsequence $\{n_j\}$ and probability measures κ' and λ' on Z . Now choose a further subsequence $\{n_{j(i)}\}$, such that the limits $\lim_i \kappa_{n_{j(i)}}(Z)$ and $\lim_i \lambda_{n_{j(i)}}(Z)$ exist; this is possible because the convergence of $\{\kappa_n + \lambda_n\}$ to an element $\mu \in M_b^+(Z)$ implies that $\{\kappa_n(Z)\}$ and $\{\lambda_n(Z)\}$ are bounded sequences of real numbers. Then $\kappa_{n_{j(i)}} \rightarrow \kappa$ and $\lambda_{n_{j(i)}} \rightarrow \lambda$, where $\kappa := \kappa' \cdot \lim_i \kappa_{n_{j(i)}}(Z)$ and λ is defined similarly. It follows that $\mu = \kappa + \lambda$ with $\kappa \in K$ and $\lambda \in A$, because we have assumed K and A to be closed.

Note. Since Z is an arbitrary metric space in the above argument, one is using one direction of Prohorov's theorem in the form of Theorem 8 in Appendix III of [1] rather than Theorem 6.2 of the same reference, just as in the proof of Corollary 1.

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REFERENCES

1. P. Billingsley (1968) *Convergence of Probability Measures*, Wiley, New York. MR **38**:1718
2. P. Billingsley (1995) *Probability and Measure*, 3rd ed., Wiley, New York. MR **95k**:60001
3. J. B. Conway (1990) *A Course in Functional Analysis*, 2nd ed., Springer, New York. MR **91e**:46001
4. D. Gatzouras (2001) *On Weak Convergence of Probability Measures in Metric Spaces*, preprint.
5. P. Habala, P. Hájek, V. Zizler (1996) *Introduction to Banach Spaces* [I], Matfyzpress, vydavatelství Matematicko-fyzikální fakulty Univerzity Karlovy.

6. K. Kuratowski (1966) *Topology*, Vol. 1, Academic Press, New York. MR **36**:840
7. H. L. Royden (1988) *Real Analysis*, 3rd ed., Macmillan Publishing Company, New York. MR **90g**:00004

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