A LYAPUNOV-TYPE STABILITY CRITERION USING L^α NORMS

MEIRONG ZHANG AND WEIGU LI

(Communicated by Carmen Chicone)

ABSTRACT. Let $q(t)$ be a T -periodic potential such that $\int_0^T q(t) dt < 0$. The classical Lyapunov criterion to stability of Hill's equation $-\ddot{x} + q(t)x = 0$ is $\|q_-\|_1 = \int_0^T |q_-(t)| dt \leq 4/T$, where q_- is the negative part of q . In this paper, we will use a relation between the (anti-)periodic and the Dirichlet eigenvalues to establish some lower bounds for the first anti-periodic eigenvalue. As a result, we will find the best Lyapunov-type stability criterion using L^α norms of q_- , $1 \leq \alpha \leq \infty$. The numerical simulation to Mathieu's equation shows that the new criterion approximates the first stability region very well.

1. INTRODUCTION AND MAIN RESULT

Let $q(t)$ be a periodic function of period $T > 0$ such that $q \in L^1(0, T)$. Recall that Hill's equation

$$(1) \quad -\ddot{x} + q(t)x = 0$$

is stable (in the sense of Lyapunov) if any solution $x(t)$ to (1) satisfies

$$\sup_{t \in \mathbb{R}} (|x(t)| + |\dot{x}(t)|) < \infty.$$

The stability of Hill's equation is a basic and an important problem in the theory of ordinary differential equations. Research on it goes back to the time of Lyapunov (see, e.g., [3]). Many theoretical results concerning this problem can be found in textbooks such as [5, 8]. Theoretically, the stability of (1) can be completely described using the periodic and the anti-periodic eigenvalues; see [8, Theorem 2.1].

A classical stability criterion given by Lyapunov, Krein and Borg (see [8, p. 46]) is as follows. Suppose that $q(t) \leq 0$ for a.e. $t \in \mathbb{R}$ and $q(t) < 0$ on a subset of positive measure. If

$$(2) \quad \|q\|_1 = \int_0^T |q(t)| dt \leq \frac{4}{T},$$

Received by the editors October 3, 2000 and, in revised form, June 15, 2001.

2000 *Mathematics Subject Classification*. Primary 34L15, 34D20, 34C25.

Key words and phrases. Hill's equation, Lyapunov stability, eigenvalue.

This project was supported by the National Natural Science Foundation of China, The National 973 Project of China, and The Excellent Personnel Supporting Plan of the Ministry of Education of China.

then (1) is stable. This can be shown using a Poincaré inequality. Condition (2) is the simplest criterion for the first stability interval. It is also best possible in the sense that for any $\varepsilon > 0$, there is some q such that

$$\|q\|_1 < \frac{4}{T} + \varepsilon,$$

while (1) is instable. Condition (2) has been generalized in several ways; see [8]. There are also many recent works on this problem and related ones such as determining the length of the stability intervals. See [4, 6, 7, 10].

In this paper, we will use certain Sobolev constants given by Talenti [9] and a relation between the (anti-)periodic and the Dirichlet eigenvalues to establish some lower bounds for the first anti-periodic values. Then we will give a stability criterion to (1) using the L^α ($1 \leq \alpha \leq \infty$) norms of the potential $q(t)$. The main result of this paper follows:

Theorem 1. *Let q be a T -periodic function such that $\int_0^T q(t)dt < 0$. Assume that $q \in L^\alpha(0, T)$ for some $1 \leq \alpha \leq \infty$. Then (1) is stable when*

$$(3) \quad \|q_-\|_\alpha < K(2\alpha^*), \quad \text{if } 1 < \alpha \leq \infty,$$

or

$$(4) \quad \|q_-\|_\alpha \leq K(\infty) = 4/T, \quad \text{if } \alpha = 1.$$

Here $\alpha^* = \alpha/(\alpha - 1)$ and $K(\cdot)$ are certain Sobolev constants which will be given explicitly in (8), and $q_-(t) = \max\{-q(t), 0\}$ is the negative part of $q(t)$, $\|\cdot\|_\alpha$ denotes the L^α norm on the interval $[0, T]$. Furthermore, the upper bounds $K(2\alpha^*)$ for $\|q_-\|_\alpha$ in (3) are best possible.

When the first stability region of the parametrized Mathieu equation

$$\ddot{x} + \lambda(1 + \varepsilon \cos t)x = 0 \quad (\lambda > 0, \varepsilon \in [-1, 1])$$

is considered, a suitable choice of α in (3) depending on ε shows that the stability condition (3) strongly approximates the stability region for all $\varepsilon \in [-1, 1]$.

2. PROOFS

Let $q(t)$ be a periodic function of period $T > 0$ such that $q \in L^1(0, T)$. Consider the eigenvalue problems of

$$(5) \quad Lx = -\ddot{x} + q(t)x = \lambda x$$

subject to the periodic boundary condition

$$(P) \quad x(0) - x(T) = \dot{x}(0) - \dot{x}(T) = 0,$$

or, to the anti-periodic boundary condition

$$(A) \quad x(0) + x(T) = \dot{x}(0) + \dot{x}(T) = 0.$$

The following is a well-known result concerning eigenvalues and stability of (5).

Theorem 2 ([8, Theorem 2.1]). *There exist*

$$\bar{\lambda}_0(q) < \underline{\lambda}_1(q) \leq \bar{\lambda}_1(q) < \underline{\lambda}_2(q) \leq \bar{\lambda}_2(q) < \cdots < \underline{\lambda}_k(q) \leq \bar{\lambda}_k(q) < \cdots$$

such that

(i) λ is an eigenvalue of (5)+(P) if and only if $\lambda = \underline{\lambda}_k(q)$ or $\bar{\lambda}_k(q)$ for $k = 0, 2, 4, \dots$;

(ii) λ is an eigenvalue of (5)+(A) if and only if $\lambda = \underline{\lambda}_k(q)$ or $\bar{\lambda}_k(q)$ for $k = 1, 3, 5, \dots$;

(iii) Equation (5) is stable if λ is in the intervals

$$(\bar{\lambda}_0(q), \underline{\lambda}_1(q)), (\bar{\lambda}_1(q), \underline{\lambda}_2(q)), \dots, (\bar{\lambda}_k(q), \underline{\lambda}_{k+1}(q)), \dots;$$

(iv) Equation (5) is unstable if λ is in the intervals

$$(-\infty, \bar{\lambda}_0(q)), (\underline{\lambda}_1(q), \bar{\lambda}_1(q)), \dots, (\underline{\lambda}_k(q), \bar{\lambda}_k(q)), \dots.$$

Our main theorem is proved using this theorem by showing that 0 is in the first stability interval $(\bar{\lambda}_0(q), \underline{\lambda}_1(q))$. To this end, we will establish a lower bound for the first anti-periodic eigenvalue $\underline{\lambda}_1(q)$ of (5)+(A).

Let us introduce some notation. For $1 \leq \alpha \leq \infty$, we use $\|\cdot\|_\alpha$ to denote the L^α norm in the Lebesgue space $L^\alpha(0, T)$.

Consider also the eigenvalues of (5) subject to the Dirichlet boundary condition

$$(D) \quad x(0) = x(T) = 0.$$

Then (5)+(D) has a sequence of eigenvalues

$$\lambda_1^D(q) < \lambda_2^D(q) < \dots < \lambda_k^D(q) < \dots.$$

It is well-known that the periodic and anti-periodic eigenvalues can be realized using the Dirichlet eigenvalues in the following way: For any $k \in \mathbb{N}$,

$$(6) \quad \underline{\lambda}_k(q) = \min\{\lambda_k^D(q_s) : s \in \mathbb{R}\}, \quad \bar{\lambda}_k(q) = \max\{\lambda_k^D(q_s) : s \in \mathbb{R}\},$$

where $q_s(\cdot)$ are translations of $q(\cdot)$: $q_s(t) \equiv q(t+s)$. Such a relation has also been generalized to the one-dimensional p -Laplacian; see [12, Theorem 4.3].

We need also certain Sobolev constants. For any $1 \leq \alpha \leq \infty$, let $K(\alpha)$ be the best Sobolev constant in the inequality

$$C\|u\|_\alpha^2 \leq \|\dot{u}\|_2^2 \quad \text{for all } u \in \mathcal{H} := H_0^1(0, T),$$

i.e.,

$$(7) \quad K(\alpha) = \inf_{u \in \mathcal{H} \setminus \{0\}} \frac{\|\dot{u}\|_2^2}{\|u\|_\alpha^2}.$$

Proposition 3. (i) The constants $K(\alpha)$ are given by

$$(8) \quad K(\alpha) = \begin{cases} \frac{2\pi}{\alpha T^{1+2/\alpha}} \left(\frac{2}{2+\alpha}\right)^{1-2/\alpha} \left(\frac{\Gamma(\frac{1}{\alpha})}{\Gamma(\frac{1}{2}+\frac{1}{\alpha})}\right)^2, & \text{if } 1 \leq \alpha < \infty, \\ \frac{4}{T}, & \text{if } \alpha = \infty. \end{cases}$$

(ii) Let $1 \leq \alpha < \infty$. Then the infimum in (7) can only be attained by functions $u = c u_\alpha(t)$, where $c \neq 0$ and $u_\alpha(t)$ is

$$(9) \quad u_\alpha(t) = \begin{cases} F_\alpha^{-1}(2F_\alpha(1)t/T), & \text{if } t \in [0, T/2], \\ F_\alpha^{-1}(2F_\alpha(1)(1-t/T)), & \text{if } t \in [T/2, T], \end{cases}$$

where $F_\alpha : [0, 1] \rightarrow \mathbb{R}$ is given by

$$F_\alpha(u) = \int_0^u \frac{du}{(1-u^\alpha)^{1/2}}.$$

Proof. These results are given in [9]. See also [11] for some generalizations. $\square$

Now we establish the following lower bound for the first Dirichlet eigenvalue $\lambda_1^D(q)$ of (5).

Theorem 4. Let q be as before. Suppose that $q \in L^\alpha(0, T)$ for some $1 \leq \alpha \leq \infty$ and $\theta = 1 - \|q_-\|_\alpha / K(2\alpha^*) \geq 0$. Then

$$(10) \quad \lambda_1^D(q) \geq \theta(\pi/T)^2.$$

Proof. Let us introduce the quadratic form in $\mathcal{H}$:

$$Q(u) = \int_0^T (\dot{u}^2 + q(t)u^2)dt, \quad u \in \mathcal{H}.$$

The following is a standard result.

$$(11) \quad \lambda_1^D(q) = \inf_{u \in \mathcal{H} \setminus \{0\}} \frac{Q(u)}{\|u\|_2^2}.$$

Moreover, equality is attained if and only if u is an eigenfunction for $\lambda_1^D(q)$.

Now let $u \in \mathcal{H}$ and $u \neq 0$. Then

$$\begin{aligned} (12) \quad Q(u) &= \int_0^T (\dot{u}^2 + q(t)u^2)dt \\ &\geq \int_0^T \dot{u}^2 dt - \int_0^T q_-(t)u^2 dt \\ &\geq \|\dot{u}\|_2^2 - \|q_-\|_\alpha \|u^2\|_{\alpha^*} \\ &= \|\dot{u}\|_2^2 - \|q_-\|_\alpha \|u\|_{2\alpha^*}^2 \\ &\geq \|\dot{u}\|_2^2 - \frac{\|q_-\|_\alpha}{K(2\alpha^*)} \|\dot{u}\|_2^2 = \theta \|\dot{u}\|_2^2, \end{aligned}$$

where the Hölder inequality and (7) are used in the proof. From these estimates, one has

$$\frac{Q(u)}{\|u\|_2^2} \geq \theta \frac{\|\dot{u}\|_2^2}{\|u\|_2^2} \geq \theta(\pi/T)^2.$$

Thus (10) follows the characterization (11) on $\lambda_1^D(q)$. $\square$

Remark. By the relation (6), the first anti-periodic eigenvalue $\underline{\lambda}_1(q)$ can be realized by $\lambda_D^1(q_{s_0})$ for some s_0 . Note that $\|(q_{s_0})_-\|_\alpha = \|q_-\|_\alpha$. Thus, under the assumption of Theorem 4, one has

$$(13) \quad \underline{\lambda}_1(q) \geq \theta(\pi/T)^2 = \left(\frac{\pi}{T}\right)^2 \left(1 - \frac{\|q_-\|_\alpha}{K(2\alpha^*)}\right).$$

Proof of Theorem 1. First, it is well-known that the zeroth periodic eigenvalue $\bar{\lambda}_0(q) \leq T^{-1} \int_0^T q(t)dt$; see [8, Theorem 4.4]. By the assumption of Theorem 1, $\bar{\lambda}_0(q) < 0$ in this case. On the other hand, if (3) holds, then $\theta > 0$. By (13) the first anti-periodic eigenvalue $\underline{\lambda}_1(q) \geq \theta(\pi/T)^2 > 0$. Thus 0 is inside the interval $(\bar{\lambda}_0(q), \underline{\lambda}_1(q))$. Now Theorem 2 shows that equation (1), which corresponds to (5) with $\lambda = 0$, is stable. If (4) holds, we know from (13) that $\underline{\lambda}_1(q) \geq \theta(\pi/T)^2 \geq 0$. We assert that $\underline{\lambda}_1(q)$ is always positive even when $\theta = 0$. Let us simply prove that $\lambda_1^D(q) > 0$ in this case. Suppose that u_0 is an eigenfunction of (5)+(D) associated with $\lambda_1^D(q)$. Proceeding as in the proof of (12), we have

$$(14) \quad \lambda_1^D(q) \|u_0\|_2^2 \geq \|\dot{u}_0\|_2^2 - \|q_-\|_1 \|u_0\|_\infty^2 > (4/T - \|q_-\|_1) \|u_0\|_\infty^2.$$

Note that the last inequality must be strict since $u_0 \in \mathcal{H}$ cannot be the extremal of the inequality

$$(4/T)\|y\|_\infty^2 \leq \|\dot{y}\|_2^2$$

which (up to multiplication by constants) is the linear spline

$$y = u_\infty(t) = \begin{cases} (2/T)t, & t \in [0, T/2], \\ -(2/T)t + 2, & t \in [T/2, T], \end{cases}$$

It then follows from (4) and from (14) that $\lambda_1^D(q) > 0$. Similarly, $\underline{\lambda}_1(q) > 0$ in this case.

Finally, we prove the last statement in Theorem 1. Suppose that $1 < \alpha \leq \infty$. Let

$$u(t) = u_{2\alpha^*}(t) \in \mathcal{H}$$

and

$$q(t) = q^\eta(t) := -\eta u_{2\alpha^*}^{2\alpha^*/\alpha}(t),$$

where $u_\beta(t)$ is given by (9) and η is a positive parameter. By Proposition 3, one has

$$\|u\|_{2\alpha^*}^2 = \frac{1}{K(2\alpha^*)} \|\dot{u}\|_2^2.$$

It is easy to check that

$$\|q_-\|_\alpha = \eta \|u\|_{2\alpha^*}^{2\alpha^*/\alpha}$$

and

$$\int_0^T q(t)u^2 dt = -\eta \|u\|_{2\alpha^*}^{2\alpha^*} = -\|q_-\|_\alpha \|u\|_{2\alpha^*}^2.$$

Thus

$$(15) \quad Q(u) = \int_0^T (\dot{u}^2 + q(t)u^2) dt = (1 - \|q_-\|_\alpha / K(2\alpha^*)) \|\dot{u}\|_2^2.$$

Let $\eta = \eta_0$ be such that $\theta = 1 - \|q_-\|_\alpha / K(2\alpha^*) = 0$, i.e.,

$$\eta_0 = K(2\alpha^*) / \|u_{2\alpha^*}\|_{2\alpha^*}^{2\alpha^*/\alpha}.$$

For this $q = q^{\eta_0}$, (13) implies that $\underline{\lambda}_1(q) \geq 0$. On the other hand, it follows from (15) that

$$\underline{\lambda}_1(q) \leq \lambda_1^D(q) = \inf_{y \in \mathcal{H} \setminus \{0\}} \frac{Q(y)}{\|y\|_2^2} \leq 0.$$

Thus, $\underline{\lambda}_1(q) = 0$ in this case. Consequently, if one takes $\eta > 0$ a little bit bigger than η_0 , then $q = q^\eta$ does not satisfy (3), i.e., $\|q_-\|_\alpha > K(2\alpha^*)$. It follows again from (15) that $\underline{\lambda}_1(q) < 0$. This means that 0 is out of the stable interval $(\bar{\lambda}_0(q), \underline{\lambda}_1(q))$ and (1) will be unstable. $\square$

Assume that $q(t) = -w(t)$ in (1), where $w(t)$ is T -periodic and satisfies $w(t) \geq 0$ for a.e. t , $w(t) > 0$ on a subset of positive measure. (We write this as $w \succ 0$.) Instead of making use of eigenvalues of (5), one may consider the weighted (anti-) periodic eigenvalues of

$$(16) \quad -\ddot{x} = \lambda w(t)x$$

subject to (P) or (A). Let us use $\{\bar{\mu}_n(w) : n \in \mathbb{Z}^+\}$ and $\{\underline{\mu}_n(w) : n \in \mathbb{N}\}$ to denote the complete sequence of all weighted periodic and anti-periodic eigenvalues of (16). Then $\bar{\mu}_0(w) = 0$. Like the estimate (13), we have the following lower bound on the first weighted anti-periodic eigenvalue.

Theorem 5. *Assume that $w \succ 0$. If $w \in L^\alpha(0, T)$ for some $1 \leq \alpha \leq \infty$, then*

$$(17) \quad \underline{\mu}_1(w) \geq \frac{K(2\alpha^*)}{\|w\|_\alpha} \quad \text{if } 1 < \alpha \leq \infty,$$

$$(18) \quad \underline{\mu}_1(w) > \frac{K(\infty)}{\|w\|_1} \quad \text{if } \alpha = 1.$$

Since equation (1) is stable if $1 \in (0, \underline{\mu}_1(w))$, one can use (17) and (18) to obtain the same stability conditions (3) and (4) when $q = -w$, $w \succ 0$.

3. AN EXAMPLE

In this section we give an example to illustrate our stability criterion.

Consider the first stability region of Mathieu's equation

$$(19) \quad \ddot{x} + \lambda(1 + \varepsilon \cos t)x = 0,$$

where $\varepsilon \in [-1, 1]$ and $\lambda > 0$. Let $w^\varepsilon(t) = 1 + \varepsilon \cos t$, $T = 2\pi$. The first stability region of (19) is

$$S_1 := \left\{ (\lambda, \varepsilon) : 0 < \lambda < \underline{\mu}_1(w^\varepsilon) \right\}.$$

Let us approximate S_1 by several stability criteria. The classical stability result (4) yields the stability of (19) only when

$$(20) \quad 0 < \lambda \leq H_0(\varepsilon) := \frac{4}{2\pi\|w^\varepsilon\|_1} = \frac{1}{\pi^2} \approx 0.10132$$

for all $\varepsilon \in [-1, 1]$. Such a result is not satisfactory because when $\varepsilon = 0$, for example, the first stability interval is $0 < \lambda < 1/4$.

Now we use the criterion (3) by choosing α depending upon ε . Then equation (19) is stable when λ satisfies

$$0 < \lambda < \frac{K(2\alpha^*)}{\|w^\varepsilon\|_\alpha} =: H_1(\varepsilon, \alpha)$$

for some $\alpha \in [1, +\infty]$. Thus (19) is stable when

$$(21) \quad 0 < \lambda < H_1(\varepsilon) := \sup_{1 \leq \alpha \leq \infty} H_1(\varepsilon, \alpha).$$

A numerical evaluation shows that (21) is a very good approximation to S_1 for all $\varepsilon \in [-1, 1]$. See Figure 1.

Let us recall a lower bound result in [11] on the first weighted Dirichlet eigenvalue $\mu_1^D(w)$ of (16)+(D). This is based essentially on the L^2 norm of the primitive $W(t) = \int^t w(t)dt$ and is proved using Opial's inequality in [1, 2].

Theorem 6 ([11, Theorem 4.4]). *Let $w \succ 0$ and $W(t)$ be a primitive of $w(t)$. Define*

$$\kappa_1(\nu) = \left(2 \int_0^{T/2} t(W(t) - \nu)^2 dt \right)^{1/2},$$

$$\kappa_2(\nu) = \left(2 \int_{T/2}^T (T-t)(W(t) - \nu)^2 dt \right)^{1/2}.$$

Then the first weighted Dirichlet eigenvalue has the following lower bound:

$$\mu_1^D(w) \geq \frac{1}{\min_{\nu \in \mathbb{R}} \max\{\kappa_1(\nu), \kappa_2(\nu)\}}.$$

Now we apply Theorem 6 to derive another stability result of (19). Note that $w_s^\varepsilon(t) = 1 + \varepsilon \cos(t+s)$. Take a primitive of $w_s^\varepsilon(t)$ as $W_s^\varepsilon(t) = t + \varepsilon \sin(t+s)$. Then

$$\begin{aligned} \kappa_1^2(\nu) &= \left[\frac{\pi^4 + \pi^2 \varepsilon^2}{2} - 4\varepsilon(4 - \pi^2) \cos s - 8\pi\varepsilon \sin s - \frac{\pi\varepsilon^2 \sin 2s}{2} \right] \\ &\quad + \left[-\frac{4\pi^3}{3} - 4\pi\varepsilon \cos s + 8\varepsilon \sin s \right] \nu + \pi^2 \nu^2, \\ \kappa_2^2(\nu) &= \left[\frac{11\pi^4 + 3\pi^2 \varepsilon^2}{6} - 4\varepsilon(4 + \pi^2) \cos s - 8\pi\varepsilon \sin s + \frac{\pi\varepsilon^2 \sin 2s}{2} \right] \\ &\quad + \left[-\frac{8\pi^3}{3} + 4\pi\varepsilon \cos s + 8\varepsilon \sin s \right] \nu + \pi^2 \nu^2. \end{aligned}$$

Let

$$\nu_0 = \frac{4\pi^3 - 24\pi\varepsilon \cos s + 3\varepsilon^2 \sin 2s}{4\pi^2 - 24\varepsilon \cos s}.$$

By Theorem 6, we have the following lower bound on $\mu_1^D(w_s^\varepsilon)$:

$$\begin{aligned} \mu_1^D(w_s^\varepsilon) &\geq \frac{1}{\min_{\nu \in \mathbb{R}} \max\{\kappa_1(\nu), \kappa_2(\nu)\}} = \frac{1}{\kappa_1(\nu_0)} \\ &= 4\sqrt{3}(\pi^2 - 6\varepsilon \cos s) / \left[(8\pi^8 + 24\pi^6 \varepsilon^2) - 96\pi^4 \varepsilon(8 + \pi^2 + 3\varepsilon^2) \cos s \right. \\ &\quad + 288\pi^2 \varepsilon^2(32 + \pi^2 + 3\varepsilon^2) \cos^2 s - 27648\varepsilon^3 \cos^3 s \\ &\quad \left. + 288\pi^2 \varepsilon^3 \sin s \sin 2s - 27\varepsilon^4(32 - \pi^2) \sin^2 2s \right]^{1/2} =: H_2(\varepsilon, s). \end{aligned}$$

Hence the first weighted anti-periodic has the following lower bound:

$$\begin{aligned} \underline{\mu}_1(w^\varepsilon) &= \min_s \mu_1^D(w_s^\varepsilon) \geq \min_s H_2(\varepsilon, s) \\ &= \left(\frac{6}{\pi^4 + 96|\varepsilon| + 3\pi^2 \varepsilon^2} \right)^{1/2} =: H_2(\varepsilon). \end{aligned}$$

By Theorem 5, equation (19) is stable if

$$(22) \quad 0 < \lambda < H_2(\varepsilon), \quad \varepsilon \in [-1, 1].$$

This estimate also strongly approximates S_1 . For example, if $\varepsilon = 0$, then $H_2(0) \doteq 0.2481 \approx 1/4$.

Using the Sobolev constants in (8), one can numerically evaluate the function $H_1(\varepsilon)$. In Figure 1, we have plotted, from the left to right, the curves $\lambda = H_0(\varepsilon) \equiv$

$1/\pi^2$, $\lambda = H_2(\varepsilon)$, $\lambda = H_1(\varepsilon)$, and $\lambda = \underline{\mu}_1(w^\varepsilon)$. It can be seen that the estimates given in (21) and (22) are almost the same as the first stability region S_1 .

FIGURE 1. The first stability region of (19).

ACKNOWLEDGMENT

The authors are most grateful to the referee for his consideration and his outline of a stronger result (Theorem 4) which makes the main theorem applicable to a wider class of potentials.

REFERENCES

- [1] P. R. Beesack and K. M. Das, Extensions of Opial's inequality, *Pacific J. Math.* **26** (1968), 215–232. MR **39**:385
- [2] R. C. Brown and D. B. Hinton, Opial's inequality and oscillation of 2nd order equations, *Proc. Amer. Math. Soc.* **125** (1997), 1123–1129. MR **97f**:34018
- [3] S. A. Dovbysh, Some historic remarks on the Hill equation, in "Selected Questions of Algebra, Geometry and Discrete Mathematics" (Eds., O. B. Lupanov, et al.), pp. 22–26, Moscow State Univ., Moscow, 1992. (in Russian)
- [4] A. Grigis, Estimations asymptotiques des intervalles d'instabilité pour l'équation de Hill, *Ann. Sci. Ec. Norm. Super. IV Ser.* **20** (1987), 641–672. MR **89e**:34056
- [5] J. K. Hale, "Ordinary Differential Equations", Second Edition, Krieger, New York, 1980. MR **82e**:34001
- [6] S. M. Khrisanov, The mean-square generalized Hill's equations, *Ukr. Math. J.* **34** (1983), 480–486.
- [7] E. Korotyaev, Estimates for the Hill operator, I, *J. Differential Equations* **162** (2000), 1–26. MR **2000m**:34194

- [8] W. Magnus and S. Winkler, “Hill’s Equations”, corrected reprint of 1966 edition, Dover, New York, 1979. MR **80k**:34001
- [9] G. Talenti, Best constant in Sobolev inequality, *Ann. Mat. Pura Appl.* (4)**110** (1976), 353–372. MR **57**:3846
- [10] M. I. Weinstein and J. B. Joseph, Asymptotic behavior of stability regions for Hill’s equation, *SIAM J. Appl. Math.* **47** (1987), 941–958. MR **88j**:34056
- [11] M. Zhang, Nonuniform nonresonance of semilinear differential equations, *J. Differential Equations* **166** (2000), 33–50. MR **2001k**:34043
- [12] M. Zhang, The rotation number approach to eigenvalues of the one-dimensional p -Laplacian with periodic potentials, *J. London Math. Soc.* **64** (2001), 125–143.

DEPARTMENT OF MATHEMATICAL SCIENCES, TSINGHUA UNIVERSITY, BEIJING 100084, PEOPLE’S REPUBLIC OF CHINA

E-mail address: `mzhang@math.tsinghua.edu.cn`

SCHOOL OF MATHEMATICAL SCIENCES, PEKING UNIVERSITY, BEIJING 100871, PEOPLE’S REPUBLIC OF CHINA

E-mail address: `weigu@math.pku.edu.cn`