ON ANTOSIK'S LEMMA AND THE ANTOSIK-MIKUSINSKI BASIC MATRIX THEOREM

QU WENBO AND WU JUNDE

(Communicated by Jonathan M. Borwein)

ABSTRACT. That Antosik's Lemma is not a special case of the Antosik-Mikusinski Basic Matrix Theorem will be shown and, an equivalent form of the Antosik-Mikusinski Basic Matrix Theorem will also be presented in this paper.

In [1] and [2], Antosik proved two results which are called the Antosik-Mikusinski Basic Matrix Theorem and Antosik's Lemma, respectively. The theorem and lemma have been proven to be quite effective in treating various topics in Functional Analysis and Set Function Theory [1]–[6]. In [6, 2.2], Swartz thought that Antosik's Lemma is a special case of the Antosik-Mikusinski Basic Matrix Theorem. In [7], Li Ronglu presented the Uniform Convergent Principle. Now, we will show that Swartz's conclusion is incorrect and, the Uniform Convergent Principle is an equivalent form of the Antosik-Mikusinski Basic Matrix Theorem.

Lemma 1 (Antosik). *Let G be an abelian topological group and $x_{ij} \in G$ for $i, j \in \mathbf{N}$. Suppose that each strictly increasing sequence $\{m_i\}$ in $\mathbf{N}$ has a subsequence $\{n_i\}$ such that*

- (i) $\lim_i x_{n_i n_j} = 0$ for all $j \in \mathbf{N}$, and
- (ii) $\lim_i \sum_{j=1}^{\infty} x_{n_i n_j} = 0$.

Then $\lim x_{ii} = 0$.

Antosik observed that assumption (i) can be dropped if G is a locally convex space and posed the problem ([2]) of whether (i) can also be dropped in general. In [8], Weber solved this problem showing that assumption (i) is in fact superfluous.

A direct consequence of Lemma 1 is as follows:

Corollary 1. *Let G be an abelian topological group and $z_{ij} \in G$ for $i, j \in \mathbf{N}$. Suppose that*

- (I) $\lim_i z_{ij} = 0$ for each $j \in \mathbf{N}$, and
- (II) *for each strictly increasing sequence of positive integers $\{m_j\}$ in $\mathbf{N}$ there is a subsequence $\{n_j\}$ such that $\lim_i \sum_{j=1}^{\infty} z_{in_j} = 0$.*

Then $\lim_i z_{ii} = 0$.

Received by the editors June 20, 2000 and, in revised form, June 11, 2001.

2000 *Mathematics Subject Classification.* Primary 40C05.

Key words and phrases. Antosik lemma, basic matrix theorem, uniform convergence principle.

This work was supported by the NSF of China.

Now, we show that the following Antosik-Mikusinski Basic Matrix Theorem can be obtained from Corollary 1.

Theorem 1 (Antosik-Mikusinski). *Let G be an abelian topological group, $x_{ij} \in G$ for $i, j \in \mathbf{N}$. Suppose*

(III) $\lim_i x_{ij} = x_j$ exists for each j , and

(IV) *for each strictly increasing sequence of positive integers $\{m_j\}$ there is a subsequence $\{n_j\}$ such that $\{\sum_j x_{in_j}\}_{i=1}^\infty$ is a Cauchy sequence.*

Then $\lim_i x_{ij} = x_j$ uniformly for $j \in \mathbf{N}$. In particular, $\lim_i x_{ii} = 0$.

Proof. If the conclusion fails, then there exist a closed, symmetric neighbourhood V_0 of 0 in G and strictly increasing sequences of positive integers $\{p_k\}$ and $\{q_k\}$ such that

$$(1) \quad x_{p_k q_k} - x_{q_k} \notin V_0$$

for all k . Pick a closed, symmetric neighbourhood V_1 of 0 such that $V_1 + V_1 \subseteq V_0$. Note that $x_{p_i q_j} - x_{q_j} \rightarrow 0$ as $i \rightarrow \infty$ for $j \in \mathbf{N}$. Therefore, there exists a subsequence $\{m_i\}$ of $\{p_i\}$ such that

$$(2) \quad x_{m_i q_i} - x_{q_i} \in V_1$$

for $i \in \mathbf{N}$. We have

$$(3) \quad x_{p_i q_i} - x_{q_i} = (x_{p_i q_i} - x_{m_i q_i}) + (x_{m_i q_i} - x_{q_i}).$$

Consider the matrix $(x_{p_i q_j} - x_{m_i q_j})$ and note that the matrix satisfies conditions of Corollary 1. Consequently,

$$x_{p_i q_i} - x_{m_i q_i} \rightarrow 0$$

as $i \rightarrow \infty$, and

$$x_{p_i q_i} - x_{m_i q_i} \in V_1$$

for sufficiently large i . Hence, by (3) and (2)

$$x_{p_i q_i} - x_{q_i} \in V_1 + V_1 \subseteq V_0$$

for sufficiently large i . Which contradicts (1) and we established the result.

Thus, we have Lemma 1 $\Rightarrow$ Corollary 1 $\iff$ Theorem 1. Now, we show that Lemma 1 is not a special case of Theorem 1. $\square$

Example 1. Consider the matrix (x_{ij}) such that $x_{ij} = 0$ if $i \neq j+1$ and $x_{ii-1} = 1$. Then (x_{ij}) satisfies the assumptions of Lemma 1. If Lemma 1 was a special case of Theorem 1, then the columns should converge to 0 uniformly. But they do not converge to 0 uniformly.

In this way, we have corrected the incorrect statement in ([6, 2.2]).

Now, we use Theorem 1 to prove Theorem 2 below.

Theorem 2 (Uniform Convergent Principle). *Let G be an abelian topological group and let Ω be a sequentially compact topological space. Let $\{f_i\}$ be a sequence of sequentially continuous G -valued functions defined on Ω . If each strictly increasing sequence $\{m_j\}$ in $\mathbf{N}$ has a subsequence $\{n_j\}$ such that for each $\omega \in \Omega$, the series $\sum_j f_{n_j}(\omega)$ converges and $\sum_j f_{n_j} : \Omega \rightarrow G$ is sequentially continuous, then $\lim_j f_j(\omega) = 0$ uniformly with respect to $\omega \in \Omega$.*

Proof. We will show that $f_j(\omega) \rightarrow 0$ uniformly for ω in Ω or, equivalently, for each sequence $\{\omega_i\}$ in Ω ,

$$(4) \quad f_i(\omega_i) \rightarrow 0$$

as $i \rightarrow \infty$. Let $\{\omega_i\}$ be a sequence of Ω . Since Ω is sequentially compact, there exists a subsequence $\{\omega_{n_i}\}$ of $\{\omega_i\}$ and ω_0 such that $\omega_{n_i} \rightarrow \omega_0$. Consider the matrix $(f_{n_j}(\omega_{n_i}) - f_{n_j}(\omega_0))$. Note that the matrix satisfies conditions of Theorem 1. Therefore,

$$f_{n_i}(\omega_{n_i}) - f_{n_i}(\omega_0) \rightarrow 0.$$

Since $f_{n_i}(\omega_0) \rightarrow 0$, we get $f_{n_i}(\omega_{n_i}) \rightarrow 0$. Hence, by assumption conditions, (4) holds. Thus, Theorem 1 $\implies$ Theorem 2. $\square$

Theorem 3. *Theorem 2 is an equivalent form of Theorem 1.*

Proof. Since Corollary 1 $\iff$ Theorem 1 $\implies$ Theorem 2, we only need to show that Theorem 2 $\implies$ Corollary 1.

Indeed, assume the conditions of Corollary 1 are satisfied. Let $\Omega = \{\frac{1}{i}, 0\}_{i=1}^\infty$, for $x, y \in \Omega$, put $d(x, y) = |x - y|$. Then (Ω, d) is a sequentially compact topological space. Let $f_j : \Omega \rightarrow G$ satisfy that if $\omega = \frac{1}{i}$, $f_j(\omega) = z_{ij}$; if $\omega = 0$, $f_j(\omega) = 0$. It is easily shown that each f_j is continuous, and for each strictly increasing sequence $\{m_j\}$ in $\mathbf{N}$ has subsequence $\{n_j\}$ such that for $i \in \mathbf{N}$, the series $\sum_j z_{in_j}$ is convergent and $\lim_i \sum_j z_{in_j} = 0$. Thus, for each $\omega \in \Omega$, the series $\sum_j f_{n_j}(\omega)$ is convergent and $\sum_j f_{n_j} : \Omega \rightarrow G$ is continuous. It follows from Theorem 2 that $\lim_j f_j(\omega) = 0$ uniformly with respect to $\omega \in \Omega$. In particular, $\lim_j f_j(\frac{1}{j}) = \lim_j z_{jj} = 0$. So Corollary 1 is proved. $\square$

ACKNOWLEDGEMENT

The authors would like to thank Professor Antosik and Professor Li Ronglu for their useful remarks and suggestions.

REFERENCES

1. P. Antosik, C. Swartz, Matrix Methods in Analysis, Lecture Notes in Mathematics, 1113, Springer-Verlag, Berlin, 1985. MR **87b**:46079
2. P. Antosik, A lemma on matrices and its applications, Contemp. Math. 52(1986), 89-95. MR **87f**:46022
3. Wu Junde, Li Ronglu, An Orlicz-Pettis theorem with applications to $\mathcal{A}$ -spaces, Studia Sci. Math. Hungary. 35(1999), 353-358. MR **2001g**:46004
4. Wu Junde, Li Ronglu, Unconditional convergent series on locally convex spaces, Taiwanese J. Math. (4)2000, 253-259. MR **2001b**:46009
5. Wu Junde, Li Ronglu, Hypocontinuity and uniform boundedness for bilinear maps, Studia Sci. Math. Hungary. (35)1999, 133-138. MR **2000a**:46003
6. C. Swartz, Infinite Matrices and the Gliding Hump, World Sci. Publ., Singapore, 1996. MR **98b**:46002
7. Li Ronglu, Cho Minhyung, A uniform convergent principle, J. Harbin Institute of Technology 24(1992), 107-108. MR **94a**:46016
8. H. Weber, A diagonal theorem. Answer to a question of Antosik, Bull. Polish Acad. Sci. Math., 41(1993), 95-102. MR **98c**:22001

DEPARTMENT OF MATHEMATICS, HARBIN INSTITUTE OF TECHNOLOGY, HARBIN 150006, PEOPLE'S REPUBLIC OF CHINA

DEPARTMENT OF MATHEMATICS, ZHEJIANG UNIVERSITY, HANGZHOU 310027, PEOPLE'S REPUBLIC OF CHINA

E-mail address: WJD@math.zju.edu.cn