APPLICATION OF A RIESZ-TYPE FORMULA TO WEIGHTED BERGMAN SPACES

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ABSTRACT. Let $\mathbb{D}$ denote the unit disk in the complex plane. We consider a class of superbiharmonic weight functions $w: \mathbb{D} \rightarrow \mathbb{R}^+$ whose growth are subject to the condition $0 \leq w(z) \leq C(1 - |z|)$ for some constant C . We first establish a Reisz-type representation formula for w , and then use this formula to prove that the polynomials are dense in the weighted Bergman space with weight w .

1. INTRODUCTION

We denote by $\mathbb{D}$ the unit disk and by $\mathbb{T}$ the unit circle in the complex plane. A *weight function*, or simply a *weight*, in $\mathbb{D}$ is any continuous positive function $w: \mathbb{D} \rightarrow [0, +\infty[$.

The problem. A real-valued function w defined on the unit disk $\mathbb{D}$ is said to be *superbiharmonic* provided that w is locally integrable and the bi-laplacian $\Delta^2 w$ is a positive distribution on $\mathbb{D}$; here Δ stands for the Laplace operator defined by

$$\Delta = \Delta_z = \frac{\partial^2}{\partial z \partial \bar{z}} = \frac{1}{4} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right), \quad z = x + iy.$$

We consider a class of superbiharmonic weight functions w satisfying condition

$$(A) \quad 0 \leq w(z) \leq C(1 - |z|), \quad \text{for every } z \in \mathbb{D},$$

for some constant C . We shall prove that if w is a superbiharmonic function satisfying condition (A), then w enjoys a Riesz-type representation formula in terms of the biharmonic Green function for the operator Δ^2 , and the harmonic compensator. In the following we shall briefly introduce these notions.

The biharmonic Green function for the operator Δ^2 in the unit disk is the function $\Gamma(z, \zeta)$ which solves, for fixed ζ , the following boundary value problem:

$$\begin{cases} \Delta_z^2 \Gamma(z, \zeta) = \delta_\zeta(z) & z \in \mathbb{D}, \\ \Gamma(z, \zeta) = 0, & z \in \mathbb{T}, \\ \partial_{n(z)} \Gamma(z, \zeta) = 0, & z \in \mathbb{T}, \end{cases}$$

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where the symbol δ_ζ stands for the unit point mass at $\zeta \in \mathbb{D}$, and $\partial_{n(z)}$ denotes the inward normal derivative in the sense of distributions. It is well-known [4] that the biharmonic Green function has the form

$$\Gamma(z, \zeta) = |z - \zeta|^2 \log \left| \frac{z - \zeta}{1 - \bar{\zeta}z} \right|^2 + (1 - |z|^2)(1 - |\zeta|^2), \quad (z, \zeta) \in \mathbb{D} \times \mathbb{D}.$$

We define the *harmonic compensator* $H(\zeta, z)$ by

$$H(\zeta, z) = (1 - |z|^2) P(z, \zeta) = \frac{(1 - |z|^2)^2}{|1 - \bar{\zeta}z|^2}, \quad (\zeta, z) \in \mathbb{T} \times \mathbb{D},$$

where $P(z, \zeta)$ denotes the Poisson kernel for the unit disk. It turns out that superbiharmonic functions w satisfying condition (A) can be represented by a Riesz-type formula which takes into account the growth of w as z approaches the boundary. Indeed, we shall see that

$$(1-1) \quad w(z) = \int_{\mathbb{D}} \Gamma(z, \zeta) \Delta^2 w(\zeta) dA(\zeta) + \int_{\mathbb{T}} H(\zeta, z) \partial_{n(\zeta)} w(\zeta) d\sigma(\zeta), \quad z \in \mathbb{D},$$

where $dA = \pi^{-1} dx dy$ is the normalized area measure in $\mathbb{D}$ and $d\sigma = (2\pi)^{-1} d\theta$ denotes the normalized arc-length measure on $\mathbb{T}$. Moreover, the normal derivative in the second term should be understood in a generalized sense: $\partial_{n(\zeta)} w(\zeta) d\sigma(\zeta)$ is the weak-star limit of a family of positive continuous functions on the unit circle, so that it can be identified with a positive measure on the unit circle $\mathbb{T}$.

Application to weighted Bergman spaces. We say that a function f , analytic in $\mathbb{D}$, belongs to the *weighted Bergman space* $L_a^p(\mathbb{D}, w)$, $0 < p < +\infty$, provided that the following integral is finite:

$$\|f\|_{L_a^p(\mathbb{D}, w)}^p = \int_{\mathbb{D}} |f(z)|^p w(z) dA(z) < +\infty.$$

If $1 \leq p < +\infty$, it follows that $L_a^p(\mathbb{D}, w)$ is a Banach space of analytic functions with norm $\|\cdot\|_{L_a^p(\mathbb{D}, w)}$, and for $0 < p < 1$, it is a quasi-Banach space. For $p = 2$, the space $L_a^2(\mathbb{D}, w)$ is a Hilbert space with the inner product

$$\langle f, g \rangle_{L_a^2(\mathbb{D}, w)} = \int_{\mathbb{D}} f(z) \overline{g(z)} w(z) dA(z), \quad f, g \in L_a^2(\mathbb{D}, w).$$

In the case where the weight function w is identically 1, the corresponding space is called the *Bergman space* and is denoted by $L_a^p(\mathbb{D})$. Note that condition (A) entails

$$\int_{\mathbb{D}} w(z) dA(z) < +\infty,$$

which guarantees that the polynomials are contained in the weighted Bergman space $L_a^p(\mathbb{D}, w)$.

We devote the last section to an application of the representation formula (1-1) to the weighted Bergman space $L_a^p(\mathbb{D}, w)$ whose (non-radial) weight w is superbiharmonic and satisfies condition (A). It is shown that the polynomials are dense in such weighted Bergman spaces. We remark that if the weight function w is radial (w depends only on $|z|$), the result is well-known [9]. The question of weighted polynomial approximation for such weights was raised by H. Hedenmalm in [5], p. 114. We answer this question in the positive.

If we weaken condition (A) above, it is possible to find a representation formula for w with one more term. This is done in [1], where the author joint with Hedenmalm succeeded to establish a Riesz representation formula for w .

2. A RIESZ-TYPE REPRESENTATION FORMULA

In this section we will find a representation formula for a superbiharmonic function w satisfying the condition $0 \leq w(z) \leq C(1 - |z|)$ for some constant C . The following lemma asserts that $\Delta^2 w(z)$ is integrable against the area measure $(1 - |z|^2)^2 dA(z)$ in the unit disk. The (rather long) proof uses the assumption (A) on w , twice the application of Green's formula together with some sharp estimates of the laplacians and the normal derivatives involved. We omit the details and refer the reader to [1], Lemma 3.1.

Lemma 2.1. *Let w be a superbiharmonic function satisfying $0 \leq w(z) \leq C(1 - |z|)$. Then*

$$\int_{\mathbb{D}} (1 - |z|^2)^2 \Delta^2 w(z) dA(z) < +\infty.$$

Proof. The assertion is a consequence of Lemma 3.1 in [1]. $\square$

In the following lemma we collect some facts on the biharmonic Green function for the unit disk. The proof of parts (c) and (d) can be found in [2], Lemma 2.3. Part (a) can be verified by a direct argument using the defining formula for $\Gamma(z, \zeta)$, and finally, (b) is an immediate consequence of (a).

Lemma 2.2. *Let $\Gamma(z, \zeta)$ denote the biharmonic Green function for the unit disk. Then*

(a) $\Gamma(z, \zeta)$ has the power series representation

$$\Gamma(z, \zeta) = \sum_{n=0}^{\infty} \frac{[(1 - |z|^2)(1 - |\zeta|^2)]^{n+2}}{(n+1)(n+2)|1 - \bar{z}\zeta|^{2(n+1)}}, \quad (z, \zeta) \in \mathbb{D} \times \mathbb{D},$$

(b) $\Gamma(z, \zeta) > 0$, for every $(z, \zeta) \in \mathbb{D} \times \mathbb{D}$,

(c) for every $(z, \zeta) \in \mathbb{D} \times \mathbb{D}$ we have

$$\frac{1}{2} \frac{(1 - |z|^2)^2 (1 - |\zeta|^2)^2}{|1 - \bar{\zeta}z|^2} \leq \Gamma(z, \zeta) \leq \frac{(1 - |z|^2)^2 (1 - |\zeta|^2)^2}{|1 - \bar{\zeta}z|^2},$$

(d) for every $(z, \zeta) \in \mathbb{D} \times \mathbb{D}$ we have

$$(1 - |z|)^2 (1 - |\zeta|^2)^2 \leq 2\Gamma(z, \zeta) \leq (1 + |z|)^2 (1 - |\zeta|^2)^2.$$

We now combine the preceding lemmas and obtain the following proposition.

Proposition 2.3. *Let w be a superbiharmonic weight function satisfying the condition $0 \leq w(z) \leq C(1 - |z|)$. Let $\Gamma(z, \zeta)$ denote the biharmonic Green function for the unit disk. Then*

$$\int_{\mathbb{D}} \Gamma(z, \zeta) \Delta^2 w(\zeta) dA(\zeta) < +\infty, \quad z \in \mathbb{D}.$$

Proof. According to Lemma 2.2(d), for every $(z, \zeta) \in \mathbb{D} \times \mathbb{D}$,

$$(1 - |z|)^2 (1 - |\zeta|^2)^2 \leq 2\Gamma(z, \zeta) \leq (1 + |z|)^2 (1 - |\zeta|^2)^2.$$

Hence

$$\Gamma(z, \zeta) \leq 2(1 - |\zeta|^2)^2, \quad (z, \zeta) \in \mathbb{D} \times \mathbb{D},$$

from which follows

$$\int_{\mathbb{D}} \Gamma(z, \zeta) \Delta^2 w(\zeta) dA(\zeta) \leq 2 \int_{\mathbb{D}} (1 - |\zeta|^2)^2 \Delta^2 w(\zeta) dA(\zeta) < +\infty,$$

in accordance with Lemma 2.1. $\square$

Proposition 2.4. *Let w be a superbiharmonic weight function satisfying the condition $0 \leq w(z) \leq C(1 - |z|)$. Define the function v by the formula:*

$$v(z) = \int_{\mathbb{D}} \Gamma(z, \zeta) \Delta^2 w(\zeta) dA(\zeta), \quad z \in \mathbb{D}.$$

Then for every integer k we have

$$\lim_{r \rightarrow 1^-} \int_{\mathbb{T}} \frac{\bar{z}^k v(rz)}{1 - r^2} d\sigma(z) = 0.$$

In fact,

$$\sup_k \left| \int_{\mathbb{T}} \frac{\bar{z}^k v(rz)}{1 - r^2} d\sigma(z) \right| \leq \int_{\mathbb{T}} \frac{v(rz)}{1 - r^2} d\sigma(z) \rightarrow 0, \quad \text{as } r \rightarrow 1^-.$$

Proof. For an integer k and $0 < r < 1$ we define

$$(2-1) \quad C(k, r) = \int_{\mathbb{T}} \frac{\bar{z}^k v(rz)}{1 - r^2} d\sigma(z).$$

We first make the following simple, but important, observation. Since the function $v(rz)/(1 - r^2)$ is nonnegative, it follows that for every integer k we have

$$0 \leq |C(k, r)| \leq \int_{\mathbb{T}} \left| \frac{\bar{z}^k v(rz)}{1 - r^2} \right| d\sigma(z) = C(0, r).$$

Hence it suffices to verify the validity of the statement of the proposition for $k = 0$; that is to show that $C(0, r) \rightarrow 0$ as $r \rightarrow 1^-$. To this end, we note that

$$0 \leq C(0, r) = \int_{\mathbb{T}} \int_{\mathbb{D}} \frac{\Gamma(rz, \zeta)}{1 - r^2} \Delta^2 w(\zeta) dA(\zeta) d\sigma(z).$$

It now follows from Fubini's theorem and Lemma 2.2(c) that

$$(2-2) \quad \begin{aligned} 0 \leq C(0, r) &\leq \int_{\mathbb{D}} \int_{\mathbb{T}} \frac{(1 - r^2|z|^2)^2 (1 - |\zeta|^2)^2}{(1 - r^2)|1 - rz\bar{\zeta}|^2} \Delta^2 w(\zeta) d\sigma(z) dA(\zeta) \\ &= \left(\int_{\mathbb{D}} (1 - |\zeta|^2)^2 \Delta^2 w(\zeta) dA(\zeta) \right) \int_{\mathbb{T}} \frac{1 - r^2}{|1 - rz\bar{\zeta}|^2} d\sigma(z). \end{aligned}$$

But

$$\int_{\mathbb{T}} \frac{1}{|1 - rz\bar{\zeta}|^2} d\sigma(z) = \frac{1}{1 - |r\zeta|^2} \int_{\mathbb{T}} \frac{1 - |r\zeta|^2}{|z - r\zeta|^2} d\sigma(z) = \frac{1}{1 - |r\zeta|^2}.$$

This together with (2-2) yields

$$(2-3) \quad 0 \leq C(0, r) \leq \int_{\mathbb{D}} \frac{1 - r^2}{1 - r^2|\zeta|^2} (1 - |\zeta|^2)^2 \Delta^2 w(\zeta) dA(\zeta).$$

To deal with the area integral appearing in (2-3) we write

$$d\lambda(\zeta) = (1 - |\zeta|^2)^2 \Delta^2 w(\zeta) dA(\zeta), \quad \zeta \in \mathbb{D}.$$

We note that $d\lambda(\zeta)$ is a positive measure on the unit disk, and according to Lemma 2.1 the integral of the constant function 1 against the measure $d\lambda$ is finite. Hence (2-3) can be written as

$$(2-4) \quad 0 \leq C(0, r) \leq \int_{\mathbb{D}} \frac{1 - r^2}{1 - r^2 |\zeta|^2} d\lambda(\zeta).$$

Since the integrand in (2-4) is nonnegative and bounded from above by 1, we can apply the dominated convergence theorem to the right-hand side of (2-4) to obtain

$$0 \leq \lim_{r \rightarrow 1^-} C(0, r) \leq \lim_{r \rightarrow 1^-} \int_{\mathbb{D}} \frac{1 - r^2}{1 - r^2 |\zeta|^2} d\lambda(\zeta) = 0,$$

from which the proposition follows. $\square$

Corollary 2.5. *With the same conditions as in the preceding proposition*

$$\sup_k \left| \int_{\mathbb{T}} \bar{z}^k v(rz) d\sigma(z) \right| = o(1 - r) \quad \text{as } r \rightarrow 1^-.$$

We have prepared the ground for the main result of this section.

Theorem 2.6. *Let w be a superbiharmonic weight function satisfying the condition $0 \leq w(z) \leq C(1 - |z|)$. Then*

$$w(z) = \int_{\mathbb{D}} \Gamma(z, \zeta) \Delta^2 w(\zeta) dA(\zeta) + \int_{\mathbb{T}} H(\zeta, z) \partial_{n(\zeta)} w(\zeta) d\sigma(\zeta), \quad z \in \mathbb{D},$$

where $H(\zeta, z)$ is the harmonic compensator and $\partial_n w$ is the generalized normal derivative of w ; in the sense that $\partial_{n(\zeta)} w(\zeta) d\sigma(\zeta)$ is a positive measure on the unit circle.

Proof. We define the real-valued function v by

$$v(z) = \int_{\mathbb{D}} \Gamma(z, \zeta) \Delta^2 w(\zeta) dA(\zeta), \quad z \in \mathbb{D}.$$

According to Proposition 2.3 the function v is well-defined. Moreover, v is nonnegative, because $\Gamma(z, \zeta)$ is positive in $\mathbb{D} \times \mathbb{D}$, and $\Delta^2 w \geq 0$ by our assumption.

We shall first verify that the weight function w can be written as

$$w(z) = v(z) + (1 - |z|^2)h(z), \quad z \in \mathbb{D},$$

where h is a harmonic function in the unit disk. It is clear that $w - v$ is a biharmonic function, hence according to the Almansi representation formula for biharmonic functions (see [6], Lemma 3.1) there exist two real-valued harmonic functions h and u such that

$$w(z) - v(z) = u(z) + (1 - |z|^2)h(z), \quad z \in \mathbb{D}.$$

We fix a real number r , $0 < r < 1$, and an integer n . Then we have

$$w(rz) - v(rz) = u(rz) + (1 - r^2)h(rz), \quad z \in \mathbb{T},$$

from which it follows that

$$\begin{aligned} \int_{\mathbb{T}} \bar{z}^n (w(rz) - v(rz)) d\sigma(z) \\ = \int_{\mathbb{T}} \bar{z}^n u(rz) d\sigma(z) + (1 - r^2) \int_{\mathbb{T}} \bar{z}^n h(rz) d\sigma(z) \\ = \hat{u}_r(n) + (1 - r^2) \hat{h}_r(n). \end{aligned}$$

Here $\hat{u}(n)$ stands for the n -th Fourier coefficient of u ; moreover, $u_r(z) = u(rz)$. Since u and h are harmonic, it follows that

$$(2-5) \quad \int_{\mathbb{T}} \bar{z}^n w(rz) d\sigma(z) - \int_{\mathbb{T}} \bar{z}^n v(rz) d\sigma(z) = r^{|n|} \hat{u}(n) + (1 - r^2) r^{|n|} \hat{h}(n).$$

Letting now $r \rightarrow 1^-$, we see that the right-hand side of (2-5) tends to $\hat{u}(n)$. As for the left-hand side of (2-5) we see that the first integral tends to zero, as $r \rightarrow 1^-$. Indeed, we know from our assumption on w that

$$0 \leq w(rz) \leq C(1 - r|z|) = C(1 - r), \quad \text{for } z \in \mathbb{T},$$

and hence

$$0 \leq \left| \int_{\mathbb{T}} \bar{z}^n w(rz) d\sigma(z) \right| \leq \int_{\mathbb{T}} w(rz) d\sigma(z) \leq C(1 - r) \rightarrow 0, \quad \text{as } r \rightarrow 1^-.$$

The fact that the second integral on the left-hand side of (2-5) approaches zero as $r \rightarrow 1^-$ follows from Corollary 2.5. Hence for any integer n , we have $\hat{u}(n) = 0$, from which it follows that u is identically zero. This implies that

$$(2-6) \quad w(z) - v(z) = (1 - |z|^2)h(z), \quad z \in \mathbb{D},$$

where h is some harmonic function in the unit disk $\mathbb{D}$.

We fix once again $0 < r < 1$ and use (2-6) to write

$$(2-7) \quad \frac{w(rz)}{1 - r^2} - \frac{v(rz)}{1 - r^2} = h(rz), \quad z \in \mathbb{T}.$$

It then follows from Proposition 2.4 that for every integer k we have

$$(2-8) \quad \lim_{r \rightarrow 1^-} \int_{\mathbb{T}} \bar{z}^k \frac{w(rz)}{1 - r^2} d\sigma(z) = \lim_{r \rightarrow 1^-} \int_{\mathbb{T}} \bar{z}^k h(rz) d\sigma(z) = \hat{h}(k).$$

Note that the limit on the right-hand side of (2-8) exists, since h is a harmonic function. In fact the equality (2-8) holds for every trigonometric polynomial p on the unit circle, that is,

$$(2-9) \quad \lim_{r \rightarrow 1^-} \int_{\mathbb{T}} p(z) \frac{w(rz)}{1 - r^2} d\sigma(z) = \lim_{r \rightarrow 1^-} \int_{\mathbb{T}} p(z) h(rz) d\sigma(z).$$

We shall see that

$$(2-10) \quad \sup_{0 < r < 1} \|h_r\|_{L^1(\mathbb{T})} < +\infty.$$

To see this, we first use (2-7) to write

$$(2-11) \quad \int_{\mathbb{T}} |h(rz)| d\sigma(z) \leq \int_{\mathbb{T}} \frac{w(rz)}{1 - r^2} d\sigma(z) + \int_{\mathbb{T}} \frac{v(rz)}{1 - r^2} d\sigma(z).$$

The first integral in (2-11) is bounded by the constant 1, as was observed earlier. The uniform boundedness of the second integral in (2-11) follows from Proposition 2.4 for $k = 0$. Hence (2-10) holds. This L^1 -boundedness of the functions h_r implies

that there exists a unique real-valued Borel measure μ on the unit circle such that h is the Poisson integral of this measure (see [10], Theorem 11.30); or

$$(2-12) \quad h(z) = \int_{\mathbb{T}} \frac{1 - |z|^2}{|1 - \bar{\zeta}z|^2} d\mu(\zeta), \quad z \in \mathbb{D}.$$

Moreover, the measure μ is the weak-star limit of the measures $d\mu_r = h_r d\sigma$ (see [8], p. 33). This means that for every trigonometric polynomial p on the unit circle we have

$$\begin{aligned} \int_{\mathbb{T}} p(z) d\mu(z) &= \lim_{r \rightarrow 1^-} \int_{\mathbb{T}} p(z) h_r(z) d\sigma(z) \\ &= \lim_{r \rightarrow 1^-} \int_{\mathbb{T}} p(z) \frac{w(rz)}{1 - r^2} d\sigma(z) - \lim_{r \rightarrow 1^-} \int_{\mathbb{T}} p(z) \frac{v(rz)}{1 - r^2} d\sigma(z). \end{aligned}$$

It is implicit in (2-9) that the last limit in the above displayed formula is zero from which it follows that

$$(2-13) \quad \int_{\mathbb{T}} p(z) d\mu(z) = \lim_{r \rightarrow 1^-} \int_{\mathbb{T}} p(z) \frac{w(rz)}{1 - r^2} d\sigma(z).$$

Now, (2-13) implies that for every nonnegative trigonometric polynomial p on the unit circle

$$\int_{\mathbb{T}} p(z) d\mu(z) \geq 0,$$

proving that μ is a positive measure on $\mathbb{T}$. This together with (2-12) implies that h is a positive harmonic function in the unit disk. Therefore we obtain from (2-6)

$$0 \leq w(z) - v(z) = (1 - |z|^2)h(z) \leq w(z) \leq C(1 - |z|), \quad z \in \mathbb{D},$$

so that $0 \leq h(z) \leq C$, that is, h is a nonnegative bounded harmonic function in the unit disk. It follows from (2-6) and (2-12) that for every $z \in \mathbb{D}$,

$$\begin{aligned} w(z) - v(z) &= (1 - |z|^2)h(z) = \int_{\mathbb{T}} \frac{(1 - |z|^2)^2}{|1 - \bar{\zeta}z|^2} d\mu(\zeta) \\ (2-14) \quad &= \int_{\mathbb{T}} H(\zeta, z) d\mu(\zeta), \end{aligned}$$

where $H(\zeta, z)$ is the harmonic compensator defined by

$$H(\zeta, z) = (1 - |z|^2)P(z, \zeta) = \frac{(1 - |z|^2)^2}{|1 - \bar{\zeta}z|^2}, \quad (\zeta, z) \in \mathbb{T} \times \mathbb{D},$$

in which $P(z, \zeta)$ denotes the Poisson kernel for the unit disk.

The measure μ in the representation formula (2-14) was obtained as a weak-star limit of the functions $w_r(z)/(1 - r^2)$, for $z \in \mathbb{T}$. This weak-star limit can be regarded as the generalized normal derivative of the function w on the boundary. The proof of the theorem is now complete. $\square$

3. APPLICATION TO WEIGHTED BERGMAN SPACES

We start with a lemma which says that the harmonic compensator has some kind of monotonicity property with respect to its second argument. The proof uses the method of [2], p. 285 (see also [7], p. 64, where the result is attributed to the current author).

Lemma 3.1. *Let $z = e^{i\theta} \in \mathbb{T}$ be fixed, and consider*

$$H(e^{i\theta}, \zeta) = \frac{(1 - |\zeta|^2)^2}{|1 - \bar{\zeta}e^{i\theta}|^2}, \quad \zeta \in \mathbb{D}.$$

Then for $0 < r < 1$ and $|\zeta| < r$, we have

$$\frac{d}{dr} \left(rH\left(e^{i\theta}, \frac{\zeta}{r}\right) \right) > 0.$$

Proof. For $0 < r < 1$ and $|\zeta| < r$ we have

$$H\left(e^{i\theta}, \frac{\zeta}{r}\right) = \frac{(r^2 - |\zeta|^2)^2}{r^2 |r - \bar{\zeta}e^{i\theta}|^2}.$$

A computation reveals that

$$\frac{d}{dr} H\left(e^{i\theta}, \frac{\zeta}{r}\right) = 4 \frac{r^2 - |\zeta|^2}{r |r - \bar{\zeta}e^{i\theta}|^2} - \frac{(r^2 - |\zeta|^2)^2}{|r - \bar{\zeta}e^{i\theta}|^2} \left(\frac{2}{r^3} + \frac{1}{r^2(r - \bar{\zeta}e^{i\theta})} + \frac{1}{r^2(r - \zeta e^{-i\theta})} \right),$$

and therefore

$$r \frac{d}{dr} H\left(e^{i\theta}, \frac{\zeta}{r}\right) = \frac{4(r^2 - |\zeta|^2)}{|r - \bar{\zeta}e^{i\theta}|^2} - \frac{(r^2 - |\zeta|^2)^2}{|r - \bar{\zeta}e^{i\theta}|^2} \left(\frac{2}{r^2} + \frac{1}{r(r - \bar{\zeta}e^{i\theta})} + \frac{1}{r(r - \zeta e^{-i\theta})} \right).$$

It then follows that

$$\begin{aligned} \frac{d}{dr} \left(rH\left(e^{i\theta}, \frac{\zeta}{r}\right) \right) &= H\left(e^{i\theta}, \frac{\zeta}{r}\right) + r \frac{d}{dr} H\left(e^{i\theta}, \frac{\zeta}{r}\right) \\ &= \frac{(r^2 - |\zeta|^2)^2}{r |r - \bar{\zeta}e^{i\theta}|^2} \left(\frac{r - |\zeta|}{r(r + |\zeta|)} + \frac{2}{r - |\zeta|} - \frac{1}{r - \bar{\zeta}e^{i\theta}} - \frac{1}{r - \zeta e^{-i\theta}} \right) \\ &\geq \frac{(r^2 - |\zeta|^2)^2 (r - |\zeta|)}{r^2 |r - \bar{\zeta}e^{i\theta}|^2 (r + |\zeta|)} > 0, \end{aligned}$$

because for $|\zeta| < r$ we have

$$\frac{1}{r - \zeta e^{-i\theta}} + \frac{1}{r - \bar{\zeta}e^{i\theta}} = 2 \operatorname{Re} \left(\frac{1}{r - \zeta e^{-i\theta}} \right) \leq \frac{2}{|r - \zeta e^{-i\theta}|} \leq \frac{2}{r - |\zeta|}.$$

The proof is now complete. $\square$

In the next lemma we prove that $r\Gamma(z, \frac{\zeta}{r})$ has the same monotonicity property as well.

Lemma 3.2. *For $0 < r < 1$, $z \in \mathbb{D}$, and $|\zeta| < r$, the function $r\Gamma(z, \frac{\zeta}{r})$ is an increasing function of r .*

Proof. We shall use the following formula, due to Hadamard, which represents the biharmonic Green function $\Gamma(z, \zeta)$ in terms of the harmonic compensator $H(z, \zeta)$ (see [6], p. 74):

$$\Gamma(z, \zeta) = \frac{1}{\pi} \int_{\max\{|z|, |\zeta|\}}^1 \int_{-\pi}^{\pi} H\left(e^{i\theta}, \frac{z}{s}\right) H\left(e^{i\theta}, \frac{\zeta}{s}\right) s d\theta ds, \quad (z, \zeta) \in \mathbb{D} \times \mathbb{D}.$$

Let us denote by

$$K(z, \zeta) = \chi_{[\max\{|z|, |\zeta|\}, 1]}, \quad (z, \zeta) \in \mathbb{D} \times \mathbb{D},$$

where χ_I stands for the characteristic function of the interval I . Assuming that $|\zeta| < r$, we obtain

$$r \Gamma\left(z, \frac{\zeta}{r}\right) = \frac{1}{\pi} \int_0^1 \int_{-\pi}^{\pi} K\left(z, \frac{\zeta}{r}\right) H\left(e^{i\theta}, \frac{z}{s}\right) rs H\left(e^{i\theta}, \frac{\zeta}{rs}\right) d\theta ds.$$

It is easy to see that the function $K(z, \frac{\zeta}{r})$ increases with r . On the other hand, since $\frac{|\zeta|}{r} < s$, we have $|\zeta| < rs$, and hence the function

$$rs H\left(e^{i\theta}, \frac{|\zeta|}{rs}\right)$$

increases with r , according to Lemma 3.1. This completes the proof. $\square$

Proposition 3.3. *Let w be a superbiharmonic weight function satisfying the condition $0 \leq w(z) \leq C(1 - |z|)$. Then the function $rw(\frac{z}{r})$, for $0 < r < 1$ and $|z| < r$, is increasing in r .*

Proof. This is an immediate consequence of Theorem 2.6, Lemma 3.1, and Lemma 3.2. $\square$

Lemma 3.4. *Suppose that μ is a finite positive measure on a measure space X , $0 < p < +\infty$, and f_n, f are measurable functions such that*

$$\limsup_{n \rightarrow +\infty} \int_X |f_n|^p d\mu \leq \int_X |f|^p d\mu < +\infty,$$

and $f_n \rightarrow f$ almost everywhere with respect to the measure μ . Then

$$\lim_{n \rightarrow +\infty} \int_X |f - f_n|^p d\mu = 0.$$

Proof. This is a well-known statement; for a proof the reader is referred to [3], p. 21, or [7], p. 66. $\square$

We can now state the main result of this section.

Theorem 3.5. *($0 < p < +\infty$) Let w be a superbiharmonic weight function satisfying the condition $0 \leq w(z) \leq C(1 - |z|)$. Then the polynomials are dense in the weighted Bergman space $L_a^p(\mathbb{D}, w)$.*

Proof. Let f be a function in the weighted Bergman space $L_a^p(\mathbb{D}, w)$, $0 < p < +\infty$. We intend to prove that the function f can be approximated by the polynomials in norm. Since the dilation f_r for every $0 < r < 1$ can be approximated in norm by the polynomials, it suffices to show that $f_r \rightarrow f$ in norm as $r \rightarrow 1^-$. It is easy to see that $f_r \rightarrow f$ pointwise, hence what we need to verify is

$$\|f_r\|_{L_a^p(\mathbb{D}, w)} \rightarrow \|f\|_{L_a^p(\mathbb{D}, w)}, \quad \text{as } r \rightarrow 1^-.$$

This together with the standard Lemma 3.4 implies that

$$\|f_r - f\|_{L_a^p(\mathbb{D}, w)} \rightarrow 0, \quad \text{as } r \rightarrow 1^-,$$

from which the result follows. We start by writing

$$\|f_r\|_{L_a^p(\mathbb{D}, w)}^p = \int_{\mathbb{D}} |f_r(z)|^p w(z) dA(z).$$

We now replace z with z/r on the right side of the above formula to obtain

$$\|f_r\|_{L_a^p(\mathbb{D}, w)}^p = \frac{1}{r^3} \int_{r\mathbb{D}} |f(z)|^p r w\left(\frac{z}{r}\right) dA(z).$$

According to Proposition 3.3 the integrand is an increasing function of r , hence we can apply the monotone convergence theorem to conclude that

$$\|f_r\|_{L_a^p(\mathbb{D}, w)}^p \rightarrow \|f\|_{L_a^p(\mathbb{D}, w)}^p, \quad \text{as } r \rightarrow 1^-,$$

from which the theorem follows. $\square$

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