SEMIPRIME CROSSED PRODUCTS OVER COPOINTED HOPF ALGEBRAS

DECLAN QUINN AND ȘERBAN RAIANU

(Communicated by Martin Lorenz)

ABSTRACT. We prove a result on the transfer of essentiality of extensions of modules over subnormalizing extensions of rings, then apply it to look at the semiprimeness of Hopf-Galois extensions, in particular that of crossed products.

The following is an important open question in the theory of Hopf algebra actions (see [12, Question 7.4.9, p.121]):

If H is a finite-dimensional and semisimple Hopf algebra over the field k , and A is semiprime, is any crossed product $A \#_{\sigma} H$ semiprime?

This possible extension of the Maschke-type theorem for crossed products [2], was inspired by the fact that it holds in the following cases:

1. when $H = kG$ (G is a finite group, and $|G|^{-1} \in k$) (this is the Fisher-Montgomery theorem, [9]), or
2. when $H = kG^*$ [6].

The first case was extended in [3] to the case where H is semisimple, pointed and cocommutative. The aim of the present note is to try to extend the second case, exploiting the fact that the dual Hopf algebra H^* is pointed in this case. The result of this effort is Corollary 5. The technique that we use involves the so-called essential form of Maschke's theorem. This approach was first used to provide another proof for the Fisher-Montgomery theorem in [15]. Various partial answers to the above question were also given in [1], [17], [13].

Throughout, H will denote a finite-dimensional Hopf algebra over the field k . For all unexplained notation or definitions, the reader is referred to [12].

Recall that if R is a subring of S , the extension $R \subset S$ is called a *subnormalizing* (or *triangular*) extension if there exist elements $x_1, x_2, x_3, \dots \in S$ such that $S = \sum_{i=1}^{\infty} Rx_i$ and for any j we have $\sum_{i=1}^j Rx_i = \sum_{i=1}^j x_i R$. A subnormalizing extension is called *finite* if the sequence $x_1, x_2, x_3, \dots \in S$ is finite (see [18] or [11]). Our first result is

Theorem 1. *Let $R \subset S$ be a subnormalizing extension of rings, such that the elements $x_1, x_2, x_3, \dots \in S$ from the definition form a basis for S as a left and a right R -module. If $M \subset N$ is an essential extension of left R -modules, and N is nonsingular, then $S \otimes_R M \subset S \otimes_R N$ is an essential extension of R -modules.*

Received by the editors May 23, 2001 and, in revised form, August 8, 2001.

2000 *Mathematics Subject Classification.* Primary 16W30.

The second author is on leave from University of Bucharest, Faculty of Mathematics.

Proof. Suppose the extension $S \otimes_R M \subset S \otimes_R N$ is not essential. We say $w \in S \otimes_R N$, $w \neq 0$, is a *nasty* element if any R -multiple of it is either 0 or not in $S \otimes_R M$. It is clear that a nonzero R -multiple of a nasty element is again nasty. Among all nasty elements choose $x_1 \otimes n_1 + \dots + x_t \otimes n_t \in S \otimes_R N$ with j minimal subject to $n_j \notin M$, $n_{j+1} \in M, \dots, n_t \in M$.

For any i there exists an automorphism $\sigma_i : R \rightarrow R$ such that $ax_i = x_i\sigma_i(a) + \text{lower terms}$ for all $a \in R$. If $a \in R$, then

$$aw = \dots + x_j \otimes (\sigma_j(a)n_j + n') + x_{j+1} \otimes n'_{j+1} + \dots + x_t \otimes n'_t$$

where $n', n'_{j+1}, \dots, n'_t \in M$, and by the minimality of j we have that

$$\text{l.ann}_R(w) = \sigma_j^{-1}(\text{l.ann}_R(\widehat{n_j})),$$

where $\widehat{n_j} = n_j + M \in M/N$. But $\text{l.ann}_R(\widehat{n_j})$ is an essential ideal of R , because the extension $M \subset N$ is essential. Thus $\text{l.ann}_R(w)$ is also an essential ideal.

Now, for $a \in \text{l.ann}_R(w)$ we have

$$0 = aw = x_1 \otimes n'_1 + \dots + x_{t-1} \otimes n'_{t-1} + x_t \otimes \sigma_t(a)n_t,$$

so $\text{l.ann}_R(w) \subseteq \sigma_t^{-1}(\text{l.ann}_R(n_t))$, and hence $\text{l.ann}_R(n_t)$ is also an essential ideal, contradicting the fact that N is nonsingular.

This implies that $S \otimes_R M$ has no nasty elements and the result follows. $\square$

The next lemma is an application of the Taft-Wilson theorem, and was proved in [7] (see also [8, Exercise 7.7.9, p.338]).

Lemma 2. *Let H be a finite-dimensional pointed Hopf algebra acting on the algebra A . Then $A \# H$ is a finite subnormalizing extension of A , and the elements $x_1, \dots, x_m \in A \# H$ may be chosen to form a basis of $H \subset A \# H$.*

We are now in a position to state and prove our main result.

Theorem 3. *Let H be a finite-dimensional pointed Hopf algebra, and A a left H -module algebra with subalgebra of invariants A^H , such that A/A^H is a faithfully flat right H^* -Galois extension. Then the following hold:*

- a) *If $M \subset_A N$ is an essential extension of left A -modules, and N is left nonsingular, then N is an essential extension of M as left A^H -modules.*
- b) *If A is H -semiprime and A is a left nonsingular ring, then A is semiprime.*
- c) *If A^H is semiprime and A is a left nonsingular ring, then A is semiprime.*

Proof. a) The proof uses the duality approach of [6] as in [16] and [4], in the form for Hopf algebra actions which was also used in [7]: since the induced functor $A \# H \otimes_A - : A\text{-mod} \rightarrow (A \# H) \# H^* - \text{mod}$ is an equivalence, it follows that

$$(1) \quad A \# H \otimes_A M \subset A \# H \otimes_A N$$

is an essential extension of left $(A \# H) \# H^*$ -modules.

By Lemma 2, we can apply Theorem 1 for the subnormalizing extension $A \subset A \# H$ to obtain that (1) is an essential extension of left A -modules, and so it is also an essential extension of left $A \# H$ -modules.

Now we have the functorial isomorphisms of left $A \# H$ -modules $A \# H \otimes_A M \simeq A \otimes_{A^H} M$, and $A \# H \otimes_A N \simeq A \otimes_{A^H} N$, so $A \otimes_{A^H} M \subset A \otimes_{A^H} N$ is an essential extension of left $A \# H$ -modules. But the induced functor $A \otimes_{A^H} - : A^H\text{-mod} \rightarrow A \# H\text{-mod}$ is also an equivalence, since A/A^H is a faithfully flat Galois extension, and therefore we have that $M \subset_{A^H} N$ is an essential extension.

b) The proof is similar to the proof of the Fisher-Montgomery theorem, as given in [15], but we sketch it for the convenience of the reader. Let $0 \neq N$ be an ideal of A such that $N^2 = 0$. Then $I = \text{r.ann}_A(N)$ is an essential left ideal of A . By a), we get that I is an essential A^H -submodule of A , so $I^H = I \cap A^H$ is an essential left ideal of A^H . Let

$$J = (I : H) = \{a \in I \mid h \cdot a \in I \ \forall h \in H\}.$$

Since $I^H \subseteq J \subseteq I$, it follows that $J^H = I^H$, and since the extension is Galois, we get that $J = I^H A$, by [12, Corollary 8.3.10, p.138]. But now

$$\text{l.ann}_A(J) = \text{l.ann}_A(I^H) \supseteq \text{l.ann}_A(I) \supseteq N \neq 0.$$

Thus $L = \text{l.ann}_A(J)$ is a nonzero H -stable ideal of A , which is again generated (as a left ideal) by its intersection with A^H . Hence we have that $L^H \cap I^H \neq 0$. Finally, $J \cap L$ is a nonzero H -stable ideal of A with $(J \cap L)^2 = 0$, a contradiction to the fact that A is H -semiprime.

c) follows from b), because H -stable ideals of A are generated by their intersection with A^H . $\square$

The following is a corollary of the proof of Theorem 3, and is presumably known.

Corollary 4. *If G is a finite group acting as automorphisms on the k -algebra R , R^G denotes the subalgebra of invariants, and R/R^G is a Galois extension having an element of trace 1, then the following hold:*

- a) *If $M \subset_R N$ is an essential extension of left R -modules, then $M \subset N$ is also an essential extension as left R^G -modules.*
- b) *If R^G is semiprime, then R is semiprime.*

Proof. As in the proof of part a) of Theorem 3, we want to prove that

$$R * kG \otimes_R M \subset R * kG \otimes_R N$$

is an essential extension of $R * kG$ -modules. But since this is an essential extension of left $(R * kG) \# kG^*$ -modules, i.e. a gr-essential extension of graded $R * kG$ -modules, the claim follows easily from [14, I.2.8, p. 9]. The rest of the proof is the same. $\square$

A direct consequence of Theorem 3 is the following.

Corollary 5. *Let H be a finite-dimensional Hopf algebra such that H^* is pointed, and $A \#_\sigma H$ a crossed product with invertible cocycle σ . If A is H -semiprime and $A \#_\sigma H$ is a left nonsingular ring, then $A \#_\sigma H$ is also semiprime.*

Proof. Denote the left weak action of H on A by $h \cdot a$ for $a \in A$ and $h \in H$.

We remark first that if A is H -semiprime, then $A \#_\sigma H$ is H^* -semiprime. Indeed, if N is a nonzero H^* -stable ideal of $A \#_\sigma H$ with $N^2 = 0$, then $N \cap A$ generates N as a left ideal (again by [12, Corollary 8.3.10, p.138]), so $N \cap A \neq 0$, and clearly $(N \cap A)^2 = 0$. Moreover, $N \cap A$ is an H -stable ideal of A , since for $n \in N \cap A$ and $h \in H$, we have by [12, 7.2.3 and 7.2.7] that

$$h \cdot n = \sum (1 \# h_1)(n \# 1)(\sigma^{-1}(S(h_3), h_4) \# S(h_2)).$$

This provides the desired contradiction.

Now apply part b) of Theorem 3 to finish the proof. $\square$

Remark 6. The relationship between the above Corollary and the question mentioned in the beginning of this note is as follows: the condition “ $A\#_{\sigma}H$ is nonsingular” is needed as a replacement for the condition “ H is semisimple” (the latter would imply that $H = kG$ for some finite group G , and this would bring us back to the graded case). This can be seen by taking A to be trivial. Then H nonsingular is equivalent to H semisimple, or H semiprime, because H is Frobenius (see [10, 13.2, p.362]).

A simple example where $A\#H$ is nonsingular (in fact, even simple) without H being semisimple is as follows: let $\mathbf{F}_2$ be the field with two elements, $E = \mathbf{F}_2(X)$ the field of rational fractions in the indeterminate X , and let k be the subfield $\mathbf{F}_2(X^2)$. Let $\delta = \frac{d}{dX} \in \text{Der}(E)$, and denote by L the 1-dimensional Lie algebra generated by δ . Consider the action of L on E as derivations. Then the invariants of this action are just k itself, and the Hopf algebra is $H = U(L) \simeq H^*$. The extension is Galois. This follows immediately from a result in [5] which may also be found in [12, 8.3.7, p. 137], or it can be seen directly after a short computation. Now, since the extension is Galois, the smash product $E\#H$ is isomorphic to $\text{End}_k(E)$, which is a simple ring.

ACKNOWLEDGMENT

We thank Susan Montgomery for helpful conversations, and the referee for suggesting improvements in the exposition.

REFERENCES

- [1] J. Bergen, S. Montgomery, Ideals and quotients in crossed products of Hopf Algebras, *J. Algebra* **152** (1992), 374-439. MR **94a**:16054
- [2] R.J. Blattner, S. Montgomery, Crossed products and Galois extension of Hopf algebras, *Pacific J. Math.* **137** (1989), 37-54. MR **90a**:16007
- [3] W. Chin, Crossed products of semisimple cocommutative Hopf algebras, *Proc. AMS* **116** (1992), 321-327. MR **92m**:16059
- [4] W. Chin, D. Quinn, Rings graded by polycyclic-by-finite groups, *Proc. AMS* **102** (1988), 235-241. MR **89a**:16001
- [5] M. Cohen, D. Fischman, S. Montgomery, Hopf Galois extensions, smash products, and Morita equivalence, *J. Algebra* **133** (1990), 351-372. MR **91i**:16068
- [6] M. Cohen, S. Montgomery, Group-graded rings, smash products, and group actions, *Trans. AMS* **282** (1984), 237-258. MR **85i**:16002
- [7] M. Cohen, Ș. Raianu, S. Westreich, Semi-invariants for Hopf algebra actions, *Israel J. Math.* **88** (1994), 279-306. MR **95j**:16042
- [8] S. Dăscălescu, C. Năstăsescu, Ș. Raianu, Hopf Algebras: an Introduction, *Pure and Applied Mathematics, A series of Monographs and Textbooks*, vol. 235, Marcel Dekker Inc., New York-Basel, 2001. MR **2001j**:16056
- [9] J.W. Fisher, S. Montgomery, Semiprime skew group rings, *J. Algebra* **52** (1978), 241-247. MR **58**:772
- [10] T.Y. Lam, *Lectures on Modules and Rings*, GTM **189**, Springer, 1999. MR **99i**:16001
- [11] B. Lemonnier, Dimension de Krull et dualité de Morita dans les extensions triangulaires, *Comm. Algebra* **12** (1984), 3071-3110. MR **86g**:16034
- [12] S. Montgomery, Hopf algebras and their actions on rings, *CBMS Reg. Conf. Series* **82**, Providence, R.I., 1993. MR **94i**:16019
- [13] S. Montgomery, H.-J. Schneider, Prime ideals in Hopf Galois extensions, *Israel J. Math.* **112** (1999), 187-235. MR **2001e**:16075
- [14] C. Năstăsescu, F. Van Oystaeyen, *Graded ring theory*, Math. Library **82**, North Holland, 1982. MR **84i**:16002
- [15] D. Passman, It's essentially Maschke's theorem, *Rocky Mountain J. Math.* **13** (1983), 37-54. MR **84e**:16023

- [16] D. Quinn, Group-graded rings and duality, Trans. AMS **292** (1985), 155-167. MR **87d**:16002
- [17] H.-J. Schneider, On inner actions of Hopf algebras and stabilizers of representations, J. Algebra **165** (1994), 138-163. MR **95d**:16055
- [18] E.A. Whelan, Finite subnormalizing extensions of rings, J. Algebra **101** (1986), 418-432. MR **88e**:16042

DEPARTMENT OF MATHEMATICS, SYRACUSE UNIVERSITY, SYRACUSE, NEW YORK 13244

E-mail address: `dpquinn@syr.edu`

DEPARTMENT OF MATHEMATICS, SYRACUSE UNIVERSITY, SYRACUSE, NEW YORK 13244

E-mail address: `sraianu@syr.edu`

Current address: Department of Mathematics, California State University Dominguez Hills,
1000 E Victoria Street, Carson, California 90747

E-mail address: `sraianu@csudh.edu`