WEIGHTED HOLOMORPHIC SPACES WITH TRIVIAL CLOSED RANGE MULTIPLICATION OPERATORS

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ABSTRACT. We deal with the space H_v^∞ consisting of those analytic functions f on the unit disc $\mathbb{D}$ such that $\|f\|_v := \sup_{z \in \mathbb{D}} v(z)|f(z)| < \infty$, with $v(z) = v(|z|)$. We determine the critical rate of decay of v such that the pointwise multiplication operator M_φ , $M_\varphi(f)(z) = \varphi(z)f(z)$ and φ analytic, has closed range in H_v^∞ only in the trivial case that φ is the product of an invertible function in H^∞ and a finite Blaschke product.

In this note, we deal with pointwise multiplication operators M_φ , $M_\varphi(f)(z) := \varphi(z)f(z)$, where $\varphi : \mathbb{D} \rightarrow \mathbb{C}$ denotes a bounded non-constant analytic function on the unit disc $\mathbb{D}$. We let M_φ act on weighted Banach spaces of the following form:

$$H_v^\infty := H_v^\infty(\mathbb{D}) := \{f \in H(\mathbb{D}) : \|f\|_v := \sup_{z \in \mathbb{D}} v(z)|f(z)| < \infty\}.$$

Here $H(\mathbb{D})$ denotes the space of analytic functions on $\mathbb{D}$ and $v : \mathbb{D} \rightarrow \mathbb{R}^+$ is an arbitrary *weight*, i.e., a continuous strictly positive function such that H_v^∞ contains a non-zero function. We will consider only *radial* weights v , i.e., we assume $v(z) = v(|z|)$.

We are interested in knowing when M_φ has closed range in H_v^∞ , i.e., when there exists a positive constant C such that $\|M_\varphi f\|_v \geq C\|f\|_v$ for all $f \in H_v^\infty$. It is clear that no matter what v is, if $\varphi = hb$ with h an invertible function in H^∞ and b a finite Blaschke product, then trivially M_φ has closed range. The purpose of this note is to determine when this trivial case is the only one allowed by v . As one might expect, this amounts to finding a certain critical rate of decay of v .

A few words on the background of this problem are in order. Pointwise multiplication operators between different Bergman spaces have been studied by many authors; see for example, [A], [L1], [L2], [MS] and [V]. In particular, McDonald and Sundberg [MS] were interested in the exact behavior of the multiplication operator when φ is an inner function. Luecking [L1] determined when the pointwise multiplication operators M_φ have closed range when acting on the weighted Bergman spaces $A_v^p(\mathbb{D}) := \{f \in H(\mathbb{D}) : \|f\|_v^p := \int_{\mathbb{D}} |f(z)|^p v(z) dA(z) < \infty\}$, if $1 \leq p < \infty$, where $dA(z)$ is the Lebesgue area measure and $v(z) := (1 - |z|^2)^\alpha$, $0 < \alpha < \infty$, or $v(z) \equiv 1$. Luecking's result was extended to the weighted Bergman spaces $A_v^\infty(\mathbb{D}) := H_v^\infty(\mathbb{D})$ by Bonet, Domański and Lindström in [BDL2], with more

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general weights v satisfying

$$\frac{A}{(1 - |z|^2)^2} \leq -\Delta \log v(z) \leq \frac{B}{(1 - |z|^2)^2};$$

here Δ denotes the Laplacian and A, B are positive constants. This condition on the Laplacian appears in [BO] and [S]. By an approximation theorem in the latter paper, the condition ensures that the spaces H_v^∞ have function theoretic properties that essentially coincide with those of the Bergman spaces studied by Luecking. For such weights, M_φ has closed range if and only if $\varphi = hb$, with h an invertible function in H^∞ and b a finite union of interpolating Blaschke products [BDL2]. It was conjectured in [BDL2] that if $-(1 - |z|^2)^2 \Delta \log v(z) \rightarrow +\infty$ when $|z| \rightarrow 1^-$, then M_φ has closed range only in the trivial case $\varphi = hb$ with h an invertible function in H^∞ and b a finite Blaschke product. The present note confirms this conjecture:

Theorem 1. *Let $v \in C^2(\mathbb{D})$ be a radial weight such that*

$$-(1 - |z|^2)^2 \Delta \log v(z) \rightarrow +\infty \quad \text{as } |z| \rightarrow 1^-.$$

Then $M_\varphi : H_v^\infty \rightarrow H_v^\infty$ has closed range if and only if $\varphi = hb$, where h is invertible in H^∞ and b is a finite Blaschke product.

The weights covered by Theorem 1 are typically weights tending to 0 faster than any of the weights $v(z) = (1 - |z|^2)^\alpha$, $\alpha > 0$, when $|z| \rightarrow 1^-$. The conclusion of Theorem 1 has previously been obtained under the stronger assumption that v tends exponentially to zero at the boundary [B]. (A typical case is $v(z) = \exp\left(-\frac{1}{(1 - |z|^2)^\delta}\right)$, $\delta > 0$.)

It was shown in [BDL2] that the problem of determining those φ such that M_φ has closed range in H_v^∞ has a nice link to the theory of uniform algebras. Let $M(H^\infty)$ be the maximal ideal space of H^∞ , and let $\Gamma(H^\infty)$ denote the Shilov boundary of H^∞ . It was proved in [BDL2] that for any fixed weight v there is a closed subset A_v of $M(H^\infty)$, $\Gamma(H^\infty) \subset A_v \subset M(H^\infty) \setminus \mathbb{D}$, such that M_φ has closed range in H_v^∞ if and only if φ does not vanish on A_v . Theorem 1 identifies the extreme case $A_v = M(H^\infty) \setminus \mathbb{D}$, while [BDL2] deals with the case that φ does not vanish on any trivial Gleason part. An interesting question is whether there exists a weight v for any closed set A , $\Gamma(H^\infty) \subset A \subset M(H^\infty) \setminus \mathbb{D}$, such that $A_v = A$.

The remainder of this note is devoted to the proof of Theorem 1. We begin with some explicit calculations.

Proposition 2. *Let $f : \mathbb{D} \rightarrow \mathbb{R}_+$ be a continuous radial function. Then the only radial solution $\psi \in C^2(\mathbb{D})$ of the equation*

$$\Delta \psi(z) = f(z)$$

satisfying $\psi(0) = 0$ is the following:

$$\psi(z) = \int_0^{|z|} r f(r) (\log |z| - \log r) dr.$$

Proof. We recall that the Laplacian has the following representation in polar coordinates (t, θ) :

$$\Delta \psi(t, \theta) = \frac{\partial^2 \psi}{\partial t^2} + \frac{1}{t} \frac{\partial \psi}{\partial t} + \frac{1}{t^2} \frac{\partial^2 \psi}{\partial \theta^2}.$$

Since ψ is a radial function, we have to find the solution of the non-homogenous Cauchy-Euler differential equation:

$$\left(\frac{d^2}{dt^2} + \frac{1}{t} \frac{d}{dt}\right) \psi(t) = f(t).$$

It is plain that for $z \neq 0$

$$\psi(z) = \int_0^{|z|} r f(r) (\log |z| - \log r) dr$$

satisfies the above equation. Hence,

$$\lim_{|z| \rightarrow 0} \psi(z) = 0.$$

A calculation shows that $\Delta\psi(0) = f(0)$. Thus a general solution of the equation is

$$\psi_g(z) = \psi(z) + u(z),$$

where u is a harmonic function. But the mean value property of harmonic functions shows that the only radial harmonic functions are constant functions, and so $u(z) \equiv 0$. $\square$

Given a continuous function f on $[0, 1)$ and some $\eta \in (0, 1]$, we define $p(t) := \frac{1-\eta t}{t} \int_0^t r f(r) dr$ for $0 \leq t < 1$.

Lemma 3. *Let $f : [0, 1) \rightarrow \mathbb{R}_+$ be a continuous function. If $(1 - t^2)^2 f(t) \nearrow +\infty$ as $t \rightarrow 1^-$, then $p(t) \nearrow +\infty$ as $t \rightarrow 1^-$.*

Proof. Since

$$\begin{aligned} p(t) &\geq (1-t) \int_{2t-1}^t r f(r) dr \\ &\geq \frac{1}{4} (1-t)(3t-1)(1-(2t-1)^2) f(2t-1) \quad \text{for } t \text{ near } 1, \end{aligned}$$

it is immediate that $p(t) \rightarrow \infty$ as $t \rightarrow 1^-$. So we need only check that $p'(t) \geq 0$ for all t . We compute

$$p'(t) = -\frac{1}{t^2} \int_0^t r f(r) dr + (1-\eta t) f(t).$$

By the assumption on f , we have

$$f(r) \leq f(t) \frac{(1-t^2)^2}{(1-r^2)^2}$$

for each $r \leq t$. Hence,

$$\frac{1}{t^2} \int_0^t r f(r) dr \leq \frac{(1-t^2)^2 f(t)}{t^2} \int_0^t \frac{r dr}{(1-r^2)^2} \leq (1-t) f(t) \leq (1-\eta t) f(t).$$

It follows that

$$(1) \quad p'(t) \geq (1-\eta t) f(t) - (1-t) f(t) = t(1-\eta) f(t).$$

In particular, $p'(t) \geq 0$. $\square$

We define the lower norm of the operator $T : X \rightarrow Y$, X, Y Banach spaces as

$$L(T) := \inf\{\|Tf\| : \|f\| = 1\}.$$

The operator T has closed range if and only if $L(T) > 0$. We set $\varphi_\eta(z) := \frac{\eta-z}{1-\bar{\eta}z}$ for $\eta, z \in \mathbb{D}$ and $\rho(z, \eta) = |\varphi_\eta(z)|$. We now establish the main ingredient in the proof of Theorem 1.

Lemma 4. *Let $w \in C^2(\mathbb{D})$ be a radial weight and set $-\Delta \log w(z) = f(z)$. If*

$$(1-t^2)^2 f(t) \nearrow +\infty \quad \text{as } t \rightarrow 1^-,$$

then the lower norm of $M_{\varphi_\eta} : H_w^\infty \rightarrow H_w^\infty$ tends to 0 as $|\eta| \rightarrow 1^-$.

Proof. We make some preliminary estimates. By Proposition 2, we have that

$$w(z) = \exp \left\{ \int_0^{|z|} r f(r) (\log r - \log |z|) dr \right\}.$$

Since the weight w is radial, all rotations are isometries, so without loss of generality we may assume that η is a positive real number.

We fix an $\eta \in (0, 1)$, and define $F_\eta(z) := \frac{a(\eta)}{(1-\eta z)^{\alpha(\eta)}}$, where $\alpha(\eta) > 1$ and $a(\eta)$ are chosen in such a way that the function

$$H(z) = \log |F_\eta(z)w(z)|$$

satisfies $H(\eta) = H'(\eta) = 0$. Since

$$H'(t) = \frac{\alpha(\eta)\eta}{1-\eta t} - \frac{1}{t} \int_0^t r f(r) dr,$$

we obtain

$$\alpha(\eta) = \frac{1-\eta^2}{\eta^2} \int_0^\eta r f(r) dr.$$

Thus

$$H'(t) = \frac{1}{(1-\eta t)} \left[\frac{1-\eta^2}{\eta} \int_0^\eta r f(r) dr - \frac{1-\eta t}{t} \int_0^t r f(r) dr \right]$$

which may be written

$$(2) \quad H'(t) = \frac{1}{(1-\eta t)} [p(\eta) - p(t)].$$

From Lemma 3, it is seen that $H'(t) > 0$ for $t < \eta$ and $H'(t) < 0$ for $t > \eta$. Hence the function $H(t)$ has only one maximum η , increases for $t < \eta$ and decreases for $t > \eta$. Thus $\|F_\eta\|_w = 1$.

By (2), we may write

$$H'(t) = \frac{1}{(1-\eta t)} \int_t^\eta p'(x) dx.$$

It then follows from inequality (1) that

$$|H'(t)| = \frac{1}{(1-\eta t)} \left| \int_t^\eta p'(x) dx \right| \geq \frac{(1-\eta)}{(1-\eta t)} \left| \int_t^\eta x f(x) dx \right|.$$

We set $\tau = \min(t, \eta)$, so that we obtain

$$(3) \quad |H'(t)| \geq \frac{(1-\eta)}{(1-\eta t)} f(\tau) \left| \int_t^\eta x dx \right| = \frac{(1-\eta)}{(1-\eta t)} f(\tau) \frac{|t^2 - \eta^2|}{2}.$$

We now use (3) and the assumption on f to prove that there exists a function $\epsilon(\eta) > 0$ such that $\epsilon(\eta) \rightarrow 0$ and $\sup_{\rho(z,\eta) \geq \epsilon(\eta)} H(z) \rightarrow -\infty$ when $\eta \rightarrow 1^-$. We will prove this statement in the following form. Suppose $\delta(\eta)$ is such that $\delta(\eta)/(1-\eta) \rightarrow 0$, and define

$$Q_{\eta,\delta} = \{z = te^{i\theta} : |t - \eta| < \delta(\eta), |\theta| \leq \delta(\eta)\}.$$

We claim that $\sup_{z \notin Q_{\eta,\delta}} H(z) \rightarrow -\infty$ when $\eta \rightarrow 1^-$ for a suitable function δ , depending on f . It is clear that this statement proves the lemma, since $\sup_{z \in Q_{\eta,\delta}} \varphi_\eta(z) \rightarrow 0$ when $\eta \rightarrow 1^-$.

To prove the claim, we now pick δ . We may assume that $\eta > 1/2$. By monotonicity of $t \mapsto H(t)$ for (respectively) $t < \eta$ and $t > \eta$, and of $\theta \mapsto H(te^{i\theta})$ for (respectively) $\theta > 0$ and $\theta < 0$, the supremum of H outside $Q_{\eta,\delta}$ will be attained on the boundary of $Q_{\eta,\delta}$. In particular, we may thus assume that $|t - \eta| < (1 - \eta)/2$. Then (3) gives

$$H(t) = \int_\eta^t H'(\xi) d\xi \leq -C(\eta) \frac{(t - \eta)^2}{(1 - \eta)^2},$$

with

$$C(\eta) = \frac{1}{20}(1 - \eta)^2 f(\eta - (1 - \eta)/2).$$

By the growth assumption on f , $C(\eta) \rightarrow +\infty$ when $\eta \rightarrow 1^-$. Thus we may require

$$(4) \quad C(\eta) \frac{[\delta(\eta)]^2}{(1 - \eta)^2} \rightarrow \infty.$$

In addition, we will require that for $|t - \eta| \leq \delta(\eta)$ and $|\theta| \geq \delta(\eta)$, we have

$$\log |F_\eta(t)| - \log |F_\eta(te^{i\theta})| = -\alpha(\eta) \log \left| \frac{1 - \eta t}{1 - \eta te^{i\theta}} \right| \rightarrow \infty$$

when $\eta \rightarrow 1^-$. Since $\alpha(\eta) \rightarrow \infty$ when $\eta \rightarrow 1^-$, this is trivial for $|\theta| \geq (1 - \eta)$, so we may assume that $|\theta| < 1 - \eta$. In this case, we have

$$-\log \left| \frac{1 - \eta t}{1 - \eta te^{i\theta}} \right| \geq \log \left| 1 + i \frac{\eta t \sin \theta}{1 - \eta t} \right| = \frac{1}{2} \log \left(1 + \frac{\eta^2 t^2 \sin^2 \theta}{(1 - \eta t)^2} \right) \geq \frac{\log 2}{4} \frac{\eta^2 t^2 \sin^2 \theta}{(1 - \eta t)^2}$$

by convexity of the logarithm, and so

$$\log |F_\eta(t)| - \log |F_\eta(te^{i\theta})| \geq C\alpha(\eta) \frac{\theta^2}{(1 - \eta)^2},$$

with C an absolute positive constant. Since $\alpha(\eta) \rightarrow \infty$, we may require that

$$(5) \quad \alpha(\eta) \frac{[\delta(\eta)]^2}{(1 - \eta)^2} \rightarrow \infty.$$

It is clear that the conditions $\delta(\eta)/(1 - \eta) \rightarrow 0$, (4), and (5) are compatible, i.e., that we can find a δ meeting all three of them. It is also clear that with such a δ , $Q_{\eta,\delta}$ has the required property. $\square$

Set

$$\tilde{v}(z) = (\sup\{|f(z)| : f \in H_v^\infty, \|f\| \leq 1\})^{-1}.$$

Following [T], we say that a weight v is essential if $v \sim \tilde{v}$, i.e., if there exists a positive constant C such that $v(z) \leq \tilde{v}(z) \leq Cv(z)$ for all $z \in \mathbb{D}$. In order to apply the previous lemma to general weights, we need the following lemma.

Lemma 5. *Let $v \in C^2(\mathbb{D})$ be a radial weight. If $-(1 - |z|^2)^2 \Delta \log v(z) \rightarrow +\infty$ as $|z| \rightarrow 1^-$, then there exists a radial weight $w \in C^2(\mathbb{D})$ such that $-(1 - |z|^2)^2 \Delta \log w(z) \nearrow +\infty$ as $|z| \rightarrow 1^-$ and $u(z) = \frac{v(z)}{w(z)}$ is an essential weight.*

Proof. Define

$$f(z) = -(1 - |z|^2)^2 \Delta \log v(z), \quad z \in \mathbb{D}.$$

Since v is radial, it follows that $f(z)$ is also a radial function. By assumption, we have $f(t) \rightarrow +\infty$ as $t \rightarrow 1^-$. We set $h(z) = \inf_{|z| \leq t < 1} f(t)$, and see that h is radial, $h(z) \leq f(z)$, $z \in \mathbb{D}$, and $h(z) \nearrow +\infty$ as $|z| \rightarrow 1^-$. We define

$$F(z) = \frac{h(z)}{(1 - |z|^2)^2}, \quad z \in \mathbb{D}.$$

By Proposition 2, there exists a radial weight $w(z) = e^{-\psi(z)}$ such that $\Delta \psi(z) = F(z)$. We have therefore

$$-(1 - |z|^2)^2 \Delta \log v(z) \leq -(1 - |z|^2)^2 \Delta \log w(z), \quad z \in \mathbb{D},$$

or, in other words,

$$-\Delta \log \frac{v(z)}{w(z)} \geq 0.$$

The last inequality means that $\frac{w}{v}$ is a log-convex function. We now invoke [BDL1, Proposition 7], which implies that $u(z) = \frac{v(z)}{w(z)}$ is then an essential weight. $\square$

We are now in a position to prove Theorem 1. We need only to prove the necessity part, as the sufficiency part is trivial. By Lemma 5, the weight $u(z) = \frac{v(z)}{w(z)}$ is an essential weight, with $w(z)$ being radial and $-(1 - |z|^2)^2 \Delta \log w(z) \nearrow +\infty$ as $|z| \rightarrow 1^-$. Since $M_\varphi : H_v^\infty \rightarrow H_v^\infty$ has closed range, by [BDL2, Lemma 3.1], also $M_\varphi : H_w^\infty \rightarrow H_w^\infty$ has closed range. Now observe that $u_1(z) = \frac{w(z)}{1 - |z|^2}$ is also an essential weight. Indeed, by the growth property of Δw , we have $-\Delta \log u_1(t) \rightarrow +\infty$ as $t \rightarrow 1^-$. Hence there exists $t_1 > 0$ such that

$$-\Delta \log u_1(t) > 0 \quad \text{for } t \in (t_1, 1),$$

and so the same argument as above ensures that u_1 is essential. Again using [BDL2, Lemma 3.1], $M_\varphi : H_{w_1}^\infty \rightarrow H_{w_1}^\infty$ with $w_1(z) = 1 - |z|^2$ has closed range. By [BDL2, Theorem 3.6], $\varphi = hb$, where $h \in H^\infty$ is invertible in H^∞ and b is a finite union of interpolating Blaschke products. From Lemma 4 it follows that the lower norm of $M_{\varphi_\eta} : H_w^\infty \rightarrow H_w^\infty$ tends to zero as $|\eta| \rightarrow 1^-$. If b were an infinite Blaschke product, then b would have a factor φ_η with $|\eta|$ arbitrarily close to 1. This leads to a contradiction because $L(M_b) \leq L(M_{\varphi_\eta})$.

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