ON MODULES OF FINITE PROJECTIVE DIMENSION OVER COMPLETE INTERSECTIONS

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ABSTRACT. Recently Avramov and Miller proved that over a local complete intersection ring (R, m, k) in characteristic $p > 0$, a finitely generated module M has finite projective dimension if for some $i > 0$ and for some $n > 0$, $\text{Tor}_i^R(M, f_R^n) = 0$ — f^n being the Frobenius map repeated n times. They used the notion of “complexity” and several related theorems. Here we offer a very simple proof of the above theorem without using “complexity” at all.

Recently Avramov and Miller [A–M] gave an important characterization for modules of finite projective dimension over complete intersections in positive characteristic. To describe their result we need the following set-up:

Let (R, m) be a local ring of characteristic $p > 0$ with residue field $k = R/m$. Let $f : R \rightarrow R$, $f(x) = x^p$, denote the Frobenius map and let f^n denote the map f repeated n -times. We denote by fR the bi- R -algebra R , having the structure of an R -algebra from the left by f^n and from the right by identity. For any R -module M , $F_R^n(M)$ will stand for $M \otimes_R {}^fR$ and fN will denote the module N viewed as an R -module via $f^n[P - S]$. Avramov and Miller proved the following:

Theorem ([A–M]). *Let M be a finitely generated module over a local complete intersection ring (R, m, k) . If for some $i > 0$ and for some $n > 0$, $\text{Tor}_i^R(M, {}^fR) = 0$, then M is of finite projective dimension.*

Their proof used the notion of “complexity” and several related theorems. Here, we intend to provide a much simpler proof which does not use “complexity” at all. The only well-known theorems we exploit are the following: the theorem on flatness of the Frobenius due to Kunz [K, 3.3] and Herzog’s theorem on characterization of finite projective dimension [H, 3.1].

Historically, work on such problems started in 1969 with Kunz’s theorem [K, 3.3] on equivalence of flatness of f^n for all $n \geq 1$ with the regularity of the local ring R . Next, Peskine and Szpiro established the following [P–S, 1.7]: If a finitely generated R -module M has finite projective dimension over R , $\text{Tor}_i^R(M, f_R^n) = 0$ for all $i, n \geq 1$. Then Herzog [H, 3.1] proved the converse: If M is finitely generated and $\text{Tor}_i^R(M, f_R^n) = 0$ for all $i > 1$ and infinitely many n , M has finite projective dimension over R . The theorem of Avramov and Miller stated above

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is a much desired extension of Herzog's theorem to complete intersections. For modules M of finite length over a complete intersection this author showed [D, 1.9] that $\ell(F_R^n(M)) \geq p^{nd}\ell(M)$ where d denotes dimension of R and sign of equality holds when M is of finite projective dimension; in [M, 1.1] Miller proved the converse and deduced the following: If for some $i > 0$, $\lim_{n \rightarrow \infty} \ell(\text{Tor}_i^R(M, f_R^n))/p^{nd} = 0$, then M has finite projective dimension.

We accomplish our proof in the following steps.

Step 0. Without any loss of generality, we can assume that R is complete and $R = S/\underline{x}$ where S is a complete regular local ring of characteristic $p > 0$ and $\underline{x} = (x_1, \dots, x_r)$ is the ideal generated by an S -sequence $x_1, \dots, x_r$. Let d be dimension of R (henceforth $\dim$); then $r + d = \dim S$. We know by Kunz's theorem that $f^n : S \rightarrow S : f^n(x) = x^{p^n}$ is a flat map $\forall n > 0$.

Step 1. Since $S \xrightarrow{f^n S} S$ is flat, $S/\underline{x} \xrightarrow{\tilde{f}^n} S/\underline{x}^{p^n}$ is flat (base change). Hence $R \xrightarrow{f^n R} R$ can be factored as

$$R = S/\underline{x} \xrightarrow{\tilde{f}^n} S/\underline{x}^{p^n} \xrightarrow{\eta_n} R = S/\underline{x}$$

where η_n is the natural surjection.

Thus $f^n R = \eta_n \cdot \tilde{f}^n$.

Let M be a finitely generated R -module. Consider an exact sequence (a presentation of M)

$$(1) \quad R^{t_1} \xrightarrow{\phi} R^{t_0} \rightarrow M \rightarrow 0.$$

Apply $\otimes^{f^n S}$ and obtain an exact sequence

$$(2) \quad (S/\underline{x}^{p^n})^{t_1} \xrightarrow{\phi^{[p^n]}} (S/\underline{x}^{p^n})^{t_0} \rightarrow F_S^n(M) \rightarrow 0.$$

(1) and (2) imply that

$$(3) \quad M \otimes_R \tilde{f}^n(S/\underline{x}^{p^n}) \simeq F_S^n(M).$$

Since $\tilde{f}^n$ is flat, we have

$$(4) \quad \text{Tor}_i^R(M, f^n R) = \text{Tor}_i^{R_n}(F_S^n(M), S/\underline{x}),$$

where $R_n = S/\underline{x}^{p^n}$

Step 2. We want to show that $\text{Tor}_1^R(M, f^n R) = 0$ implies that $\text{Tor}_i^R(M, f^n R) = 0$ for $i \geq 1$. It is enough to show that $\text{Tor}_2^R(M, f^n R) = 0$. By flatness of $\tilde{f}^n$ we will be done by showing that

$$\text{Tor}_1^{R_n}(F_S^n(M), S/\underline{x}) = 0 \quad \text{implies} \quad \text{Tor}_2^{R_n}(F_S^n(M), S/\underline{x}) = 0.$$

We know $S/\underline{x}^{p^n}$ has a filtration such that successive quotients are isomorphic to $S/\underline{x}$. We have the following exact sequences:

$$0 \rightarrow K_1 \rightarrow S/\underline{x}^{p^n} \rightarrow S/\underline{x} \rightarrow 0,$$

$$0 \rightarrow K_2 \rightarrow K_1 \rightarrow S/\underline{x} \rightarrow 0,$$

$$0 \rightarrow K_{t_n} \rightarrow K_{t_n-1} \rightarrow S/\underline{x} \rightarrow 0$$

where $K_{t_n} = S/\underline{x}$. Since $\text{Tor}_1^{R_n}(F_S^n(M), S/\underline{x}) = 0$, we obtain by going up along the above exact sequences successively that $\text{Tor}_1^{R_n}(F_S^n(M), K_1) = 0$. This implies that $\text{Tor}_2^{R_n}(F_S^n(M), S/\underline{x}) = 0$. Hence $\text{Tor}_2^R(M, {}^fR) = 0$.

Step 3. We want to show that $\text{Tor}_1^R(M, {}^fR) = 0$ implies that $\text{Tor}_1^R(M, {}^{f^{n+1}}R) = 0$. Hence, we need to show that $\text{Tor}_1^{R_{n+1}}(F_S^{n+1}(M), S/\underline{x}) = 0$, provided that $\text{Tor}_1^{R_n}(F_S^n(M), S/\underline{x}) = 0$. Recall from Step 2 that since $\text{Tor}_1^{R_n}(F_S^n(M), S/\underline{x}) = 0$, $\text{Tor}_i^{R_n}(F_S^n(M), S/\underline{x}) = 0$ for $i \geq 1$ and from Step 1 that $R_n \xrightarrow{f} R_{n+1}$ is flat. Hence (tensoring the above equation by ${}^fR_{n+1}$),

$$(5) \quad \text{Tor}_i^{R_{n+1}}(F_S^{n+1}(M), S/\underline{x}^p) = 0 \quad \text{for } i \geq 1.$$

Consider the short exact sequences

$$(6) \quad 0 \rightarrow S/(x_1, x_2^p, \dots, x_r^p) \rightarrow S/(x_1^{p+1}, x_2^p, \dots, x_r^p) \rightarrow S/\underline{x}^p \rightarrow 0$$

and

$$(7) \quad 0 \rightarrow S/\underline{x}^p \rightarrow S/(x_1^{p+1}, x_2^p, \dots, x_r^p) \rightarrow S/(x_1, x_2^p, \dots, x_r^p) \rightarrow 0.$$

Applying (5), from (6) we obtain

$$(8) \quad \begin{aligned} & \text{Tor}_i^{R_{n+1}}(F_S^{n+1}(M), S/(x_1, x_2^p, \dots, x_r^p)) \\ & \simeq \text{Tor}_i^{R_{n+1}}(F_S^{n+1}(M), S/(x_1^{p+1}, x_2^p, \dots, x_r^p)) \quad \text{for } i \geq 1, \end{aligned}$$

and applying (5), from (7) we obtain

$$(9) \quad \begin{aligned} & \text{Tor}_i^{R_{n+1}}(F_S^{n+1}(M), S/(x_1^{p+1}, x_2^p, \dots, x_r^p)) \\ & \hookrightarrow \text{Tor}_i^{R_{n+1}}(F_S^{n+1}(M), S/(x_1, x_2^p, \dots, x_r^p)) \quad \text{for } i \geq 1 \end{aligned}$$

which is an isomorphism for $i \geq 2$ and an injection for $i = 1$.

Note that the composition of the two maps (8) and (9) is the one induced by multiplication by x_1^p .

Combining (6), (7), (8) and (9) we observe that

$$S/(x_1, x_2^p, \dots, x_r^p) \xrightarrow{x_1^p} S/(x_1, x_2^p, \dots, x_r^p)$$

is the 0-map and it induces an injection between

$$\text{Tor}_i^{R_{n+1}}(F_S^{n+1}(M), S/(x_1, x_2^p, \dots, x_r^p)) \hookrightarrow \text{Tor}_i^{R_{n+1}}(F_S^{n+1}(M), S/(x_1, x_2^p, \dots, x_r^p))$$

for $i \geq 1$.

Thus $\text{Tor}_i^{R_{n+1}}(F_S^{n+1}(M), S/(x_1, x_2^p, \dots, x_r^p)) = 0$. Repeating this process $(r-1)$ times we obtain

$$\text{Tor}_i^{R_{n+1}}(F_S^{n+1}(M), S/\underline{x}) = 0 \quad \text{for } i \geq 1.$$

Step 4. Steps 1, 2, 3 and the well-known theorem due to Herzog [H, 3.1] prove the assertion stated in the theorem at the beginning.

Remark. With notation as above, over a complete intersection $R = S/\underline{x}$, for a finitely generated module M , we have $\text{Pd}_R M < \infty$ if and only if $\text{Pd}_{R_n} F_S^n(M) < \infty$.

The proof follows easily from Step 1.

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