ON APPROXIMATELY CONVEX FUNCTIONS

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ABSTRACT. A real valued function f defined on a real interval I is called (ε, δ) -convex if it satisfies

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) + \varepsilon t(1-t)|x-y| + \delta \quad \text{for } x, y \in I, t \in [0, 1].$$

The main results of the paper offer various characterizations for (ε, δ) -convexity. One of the main results states that f is (ε, δ) -convex for some positive ε and δ if and only if f can be decomposed into the sum of a convex function, a function with bounded supremum norm, and a function with bounded Lipschitz-modulus. In the special case $\varepsilon = 0$, the results reduce to that of Hyers, Ulam, and Green obtained in 1952 concerning the so-called δ -convexity.

1. INTRODUCTION

The stability theory of functional inequalities started with the paper [HU52] of Hyers and Ulam who introduced the notion of δ -convex function: If D is a convex subset of a real linear space X and δ is a nonnegative number, then a function $f : D \rightarrow \mathbb{R}$ is called δ -convex if

$$(1) \quad f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) + \delta$$

for all $x, y \in D, t \in [0, 1]$. The basic result obtained by Hyers and Ulam states that if the underlying space X is of finite dimension, then f can be written as $f = g + h$, where g is a convex function and h is a bounded function whose supremum norm is not larger than $k_n \delta$, where the positive constant k_n depends only on the dimension n of the underlying space X . Hyers and Ulam proved that $k_n \leq (n(n+3))/(4(n+1))$. Green [Gre52], Cholewa [Cho84] obtained much better estimations of k_n showing that asymptotically k_n is not bigger than $(\log_2(n))/2$. Laczkovich [Lac99] compared this constant to several other dimension-depending stability constants and proved that it is not less than $(\log_2(n/2))/4$. This result shows that there is no stability results for infinite dimensional spaces X . A counterexample in this direction was earlier constructed by Casini and Papini [CP93]. The stability aspects of δ -convexity are discussed by Ger [Ger94].

If $t = 1/2$ and (1) holds for all $x, y \in D$, then f is called a δ -Jensen-convex function. There is no analogous decomposition for δ -Jensen-convex functions by

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the counterexample given by Cholewa [Cho84]. However, one can get Bernstein-Doetsch type regularity theorems which show that δ -Jensen-convexity and local upper boundedness imply 2δ -convexity. This result is due to Bernstein and Doetsch [BD15] for $\delta = 0$, and to Ng and Nikodem [NN93] in the case $\delta \geq 0$. For some recent extensions of these results to more general convexity concepts, see [Pál00]. For locally upper bounded δ -Jensen-convex functions one can obtain the existence of an analogous stability constant j_n (defined similarly as k_n above). The sharp value of this stability constant has recently been found by Dilworth, Howard, and Roberts [DHR99] who have shown that

$$j_n = \frac{1}{2} \left(\lfloor \log_2(n) \rfloor + 1 + \frac{n}{2^{\lfloor \log_2(n) \rfloor}} \right) \leq 1 + \frac{1}{2} \log_2(n)$$

is the best possible value for j_n . (Here $\lfloor \cdot \rfloor$ denotes the integer-part function.) The connection between δ -Jensen-convexity and δ - $\mathbb{Q}$ -convexity has been investigated by Mrowiec [Mro01].

If $D \subset \mathbb{R}$ and (1) is supposed to be valid for all $x, y \in D$ except a set of 2-dimensional Lebesgue measure zero, then one can speak about *almost δ -convexity*. Results in this direction are due to Kuczma [Kuc70] (the case $\delta = 0$) and Ger [Ger88] (the case $\delta \geq 0$).

Roughly speaking, the content of the Hyers–Ulam theorem can be expressed in the following way: A function f is a perturbation of a convex function by a bounded function if and only if, for some δ , it satisfies (1). Therefore, the presence of the error term δ refers to a bounded perturbation. In what follows, we consider more general perturbations of convex functions. We assume that X is a normed space and we take, as a perturbation, functions of the form $h + \ell$, where h is bounded and ℓ has a bounded Lipschitz-modulus. The investigation of such functions leads to the following stability notion: Let $\varepsilon, \delta \geq 0$ be constants and let D be a subset of X . A function $f : D \rightarrow \mathbb{R}$ is called (ε, δ) -convex on D if

$$(2) \quad f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) + \varepsilon t(1-t)\|x - y\| + \delta$$

for $x, y \in D$, $t \in [0, 1]$. It is not difficult to see that if X is a normed space, f is of the form $f = g + h + \ell$, where g is convex, h is bounded with supremum norm not greater than $\delta/2$, and the Lipschitz-modulus of ℓ defined by

$$\text{Lip}(\ell) := \inf \{ L \mid |\ell(x) - \ell(y)| \leq L\|x - y\| \text{ for } x, y \in D \}$$

is not greater than $\varepsilon/2$, then f satisfies (2). The main result of this paper (see Corollary 4 below) is the more surprising statement that if D is one dimensional (i.e., D is a real interval) and f satisfies (2), then f admits a decomposition described above. Thus, one can observe, that the error terms $\varepsilon t(1-t)\|x - y\|$ and δ in (2) correspond to perturbations by Lipschitz and by bounded functions, respectively.

Most of the results obtained in this paper essentially use the fact that D is one dimensional. Thus, the analogous problem for the higher dimensional setting remains unsolved.

2. MAIN RESULTS

Throughout the rest of this paper, we assume that I is a nonempty open real interval of $\mathbb{R}$. Our first result offers a number of equivalent conditions for (ε, δ) -convexity.

Theorem 1. Let $f : I \rightarrow \mathbb{R}$ and ε, δ be nonnegative numbers. Then the following conditions are pairwise equivalent:

(i) f is (ε, δ) -convex on I , i.e.,

$$(3) \quad f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) + \varepsilon t(1-t)|x-y| + \delta$$

for $x, y \in I, t \in [0, 1]$.

(ii) For $x, u, y \in I$ with $x < u < y$,

$$(4) \quad \frac{f(x) + \delta - f(u)}{x - u} \leq \frac{f(y) + \delta - f(u)}{y - u} + \varepsilon.$$

(iii) There exists a function $p : I \rightarrow \mathbb{R}$ such that, for $x, u \in I$,

$$(5) \quad f(u) + p(u)(x - u) \leq f(x) + \frac{\varepsilon}{2}|x - u| + \delta.$$

(iv) If $x_1, \dots, x_n \in I, t_1, \dots, t_n \geq 0, t_1 + \dots + t_n = 1$ and $u := t_1x_1 + \dots + t_nx_n$, then

$$(6) \quad f(u) \leq t_1f(x_1) + \dots + t_nf(x_n) + \frac{\varepsilon}{2}(t_1|x_1 - u| + \dots + t_n|x_n - u|) + \delta.$$

Proof. (i) $\Rightarrow$ (ii): Assume that f is (ε, δ) -convex and let $x < u < y$ be arbitrary elements of I . Choose $t \in [0, 1]$ such that $u = tx + (1-t)y$, that is let $t = (y - u)/(y - x)$. Then applying the (ε, δ) -convexity of f , we get

$$f(u) \leq \frac{y - u}{y - x}f(x) + \frac{u - x}{y - x}f(y) + \varepsilon \frac{(y - u)(u - x)}{y - x} + \delta,$$

which is equivalent to (4).

(ii) $\Rightarrow$ (iii): Assume that (ii) holds and define

$$p(u) := \sup_{x \in I, x < u} \left(\frac{f(x) + \delta - f(u)}{x - u} - \frac{\varepsilon}{2} \right) \quad \text{for } u \in I.$$

Then, due to (ii), we have

$$(7) \quad \frac{f(x) + \delta - f(u)}{x - u} - \frac{\varepsilon}{2} \leq p(u) \leq \frac{f(y) + \delta - f(u)}{y - u} + \frac{\varepsilon}{2}$$

for all $x < u < y$ in I . The left-hand side inequality in (7) yields (5) in the case $x < u$, and analogously, the right-hand side inequality reduces to (5) in the case $x > u$. The case $x = u$ is obvious.

(iii) $\Rightarrow$ (iv): To deduce (iv) from (iii), let $x_1, \dots, x_n \in I, t_1, \dots, t_n \geq 0, t_1 + \dots + t_n = 1$ and $u := t_1x_1 + \dots + t_nx_n$. Then, substituting x by x_i in (5), multiplying this inequality by t_i , and adding up the inequalities so obtained, we get

$$\begin{aligned} f(u) &= \sum_{i=1}^n t_i [f(u) + p(u)(x_i - u)] \\ &\leq \sum_{i=1}^n t_i \left(f(x_i) + \frac{\varepsilon}{2}|x_i - u| + \delta \right) = \sum_{i=1}^n t_i f(x_i) + \frac{\varepsilon}{2} \sum_{i=1}^n t_i |x_i - u| + \delta, \end{aligned}$$

which is the desired inequality (6).

(iv) $\Rightarrow$ (i): Taking $x_1 = x, x_2 = y, t_1 = t$, and $t_2 = 1 - t$ in condition (iv), one can immediately see that the inequality (6) reduces to (3). $\square$

Remark 1. The equivalence of the conditions (i)-(iv) established in Theorem 1 reduces to well-known characterizations of convexity (cf. Hardy-Littlewood-Pólya [HLP34], Roberts-Varberg [RV73], and Kuczma [Kuc85]).

Motivated by condition (iii) of Theorem 1, a function $f : I \rightarrow \mathbb{R}$ is called (ε, δ) -subdifferentiable at a point $u \in I$, if there exists a real number $p = p(u)$ such that (5) holds for all $x \in I$. The (ε, δ) -subgradient of f at u is defined as the set of all values $p(u)$ such that (5) is valid for $x \in I$ and is denoted by $\partial_{\varepsilon, \delta} f(u)$. Clearly,

$$\begin{aligned} \partial_{\varepsilon, \delta} f(u) &= \left\{ p \in \mathbb{R} \mid \frac{f(x) + \delta - f(u)}{x - u} - \frac{\varepsilon}{2} \leq p \leq \frac{f(y) + \delta - f(u)}{y - u} + \frac{\varepsilon}{2}, x < u < y \right\} \\ &= \left[\sup_{x \in I, x < u} \frac{f(x) + \delta - f(u)}{x - u} - \frac{\varepsilon}{2}, \inf_{y \in I, y > u} \frac{f(y) + \delta - f(u)}{y - u} + \frac{\varepsilon}{2} \right]. \end{aligned}$$

In the case if f is (ε, δ) -subdifferentiable at each point of I and if a nondecreasing function $p : I \rightarrow \mathbb{R}$ can be chosen such that (5) holds, then we say that f is nondecreasingly (ε, δ) -subdifferentiable on I .

The equivalence of conditions (i) and (iii) in Theorem 1 now can be expressed via the following result.

Corollary 1. *Let $f : I \rightarrow \mathbb{R}$ and ε, δ be nonnegative numbers. Then f is (ε, δ) -convex on I if and only if it is (ε, δ) -subdifferentiable on I .*

In our next result we obtain the decomposition of (ε, δ) -convex functions in terms of an $(\varepsilon, 0)$ -convex function and a function whose supremum norm is not larger than $\delta/2$.

Theorem 2. *Let $f : I \rightarrow \mathbb{R}$ and ε, δ be nonnegative numbers. Then f is (ε, δ) -convex on I if and only if there exists an $(\varepsilon, 0)$ -convex function $\varphi : I \rightarrow \mathbb{R}$ such that $\|f - \varphi\| = \sup_I |f - \varphi| \leq \delta/2$.*

Proof. Assume that there exists an $(\varepsilon, 0)$ -convex function φ such that $\|f - \varphi\| \leq \delta/2$. Denote by h the difference $f - \varphi$. Then we have

$$\varphi(tx + (1-t)y) \leq t\varphi(x) + (1-t)\varphi(y) + \varepsilon t(1-t)|x - y|$$

and

$$\begin{aligned} h(tx + (1-t)y) &\leq th(x) + (1-t)h(y) \\ &\quad + |h(tx + (1-t)y)| + t|h(x)| + (1-t)|h(y)| \\ &\leq th(x) + (1-t)h(y) + \delta \end{aligned}$$

for all $x, y \in I$ and $t \in [0, 1]$. Summing up these two inequalities, we get that f is (ε, δ) -convex.

To prove the converse, assume that f is (ε, δ) -convex and apply Theorem 1. Then there exists a function $p : I \rightarrow \mathbb{R}$ such that (5) holds for all $x, u \in I$. Define

$$\varphi(x) := \sup_{u \in I} \left(f(u) + p(u)(x - u) - \frac{\varepsilon}{2}|x - u| - \frac{\delta}{2} \right) \quad \text{for } x \in I.$$

Then, by (5), we have that $\varphi(x) \leq f(x) + \delta/2$ for all $x \in I$. On the other hand, clearly $f(x) - \delta/2 \leq \varphi(x)$. Thus, we get that $\|f - \varphi\| \leq \delta/2$.

To complete the proof, it remains to show that φ is $(\varepsilon, 0)$ -convex. For, let $x, y \in I$ and $t \in [0, 1]$ be fixed. Let $c > 0$ be an arbitrary (small) positive number. Then,

by the definition of $\varphi(tx + (1-t)y)$, there exists $u \in I$ such that

$$\varphi(tx + (1-t)y) - c < f(u) + p(u)(tx + (1-t)y - u) - \frac{\varepsilon}{2}|tx + (1-t)y - u| - \frac{\delta}{2}.$$

Thus,

$$\begin{aligned} & \varphi(tx + (1-t)y) - c \\ & < t\left(f(u) + p(u)(x - u) - \frac{\varepsilon}{2}|x - u| - \frac{\delta}{2}\right) \\ & \quad + (1-t)\left(f(u) + p(u)(y - u) - \frac{\varepsilon}{2}|y - u| - \frac{\delta}{2}\right) \\ & \quad + \frac{\varepsilon}{2}t(|x - u| - |tx + (1-t)y - u|) \\ & \quad + \frac{\varepsilon}{2}(1-t)(|y - u| - |tx + (1-t)y - u|) \\ & \leq t\varphi(x) + (1-t)\varphi(y) + \frac{\varepsilon}{2}t(1-t)|x - y| + \frac{\varepsilon}{2}(1-t)t|x - y|. \end{aligned}$$

Taking the limit $c \rightarrow 0$, it follows from the above inequality that φ is $(\varepsilon, 0)$ -convex. $\square$

Remark 2. The case $\varepsilon = 0$ of the above theorem is the stability theorem of convex functions obtained by Hyers and Ulam [HU52] (see also Green [Gre52]). Their results, however, concern also the higher but finite dimensional case.

Corollary 2. *Let $f : I \rightarrow \mathbb{R}$ and ε, δ be nonnegative numbers. Then f is (ε, δ) -subdifferentiable at each point of I if and only if there exists an $(\varepsilon, 0)$ -subdifferentiable function $\varphi : I \rightarrow \mathbb{R}$ such that $\|f - \varphi\| \leq \delta/2$. In addition, for all $u \in I$,*

$$\partial_{\varepsilon, 0}\varphi(u) \subset \partial_{\varepsilon, \delta}f(u).$$

Proof. By Corollary 1, f is (ε, δ) -subdifferentiable on I if and only if it is (ε, δ) -convex on I . This property, by the previous theorem, is equivalent to the condition that there exists an $(\varepsilon, 0)$ -convex function φ such that $\|f - \varphi\| \leq \delta/2$. Applying Corollary 1 again, we see that this latter property is equivalent to the condition that there exists an $(\varepsilon, 0)$ -subdifferentiable function φ such that $\|f - \varphi\| \leq \delta/2$.

To see the inclusion between the subdifferentials, let $u \in I$ and $p \in \partial_{\varepsilon, 0}\varphi(u)$. Then

$$\varphi(u) + p(x - u) \leq \varphi(x) + \frac{\varepsilon}{2}|x - u| \quad \text{for } x \in I.$$

Using $\|f - \varphi\| \leq \delta/2$, we get, for all $x \in I$, that

$$f(u) + p(x - u) \leq \varphi(u) + p(x - u) + \frac{\delta}{2} \leq \varphi(x) + \frac{\varepsilon}{2}|x - u| + \frac{\delta}{2} \leq f(x) + \frac{\varepsilon}{2}|x - u| + \delta,$$

i.e., $p \in \partial_{\varepsilon, \delta}f(u)$. $\square$

Now we intend to give characterizations of $(\varepsilon, 0)$ -subdifferentiable functions. First we investigate the properties of the $(\varepsilon, 0)$ -subdifferential. For, introduce the following notion: A function $p : I \rightarrow \mathbb{R}$ is called ε -nondecreasing on I if

$$p(x) \leq p(y) + \varepsilon$$

holds for all $x \leq y$ in I . A set-valued function $P : I \rightarrow 2^{\mathbb{R}}$ is called ε -nondecreasing if any selection function p of P is ε -nondecreasing, which is equivalent to requiring that $u \leq v + \varepsilon$ whenever $u \in P(x)$, $v \in P(y)$, and $x \leq y$.

The connection between ε -nondecreasing and nondecreasing functions is described in the next result.

Theorem 3. *Let I be an open interval of $\mathbb{R}$, $p : I \rightarrow \mathbb{R}$, and ε be a nonnegative number. Then p is ε -nondecreasing if and only if there exists a nondecreasing function $q : I \rightarrow \mathbb{R}$ such that $\|p - q\| \leq \varepsilon/2$.*

Proof. Assume that q is nondecreasing such that $\|p - q\| \leq \varepsilon/2$. Then for $x \leq y$, we have

$$p(x) \leq q(x) + |p(x) - q(x)| \leq q(y) + \frac{\varepsilon}{2} \leq p(y) + \frac{\varepsilon}{2} + |p(y) - q(y)| \leq p(y) + \varepsilon.$$

Thus, p is ε -nondecreasing.

Conversely, assume that p is ε -nondecreasing and define

$$q(x) := \sup_{v \in I, v \leq x} \left(p(v) - \frac{\varepsilon}{2} \right) \quad \text{for } x \in I.$$

Then q is obviously nondecreasing. By its definition, we have that

$$p(x) - \frac{\varepsilon}{2} \leq q(x).$$

On the other hand, using that p is ε -nondecreasing, $p(v) \leq p(x) + \varepsilon$ for all $v \leq x$, whence

$$q(x) = \sup_{v \in I, v \leq x} \left(p(v) - \frac{\varepsilon}{2} \right) \leq p(x) + \frac{\varepsilon}{2}.$$

The two inequalities obtained yield that $\|p - q\| \leq \varepsilon/2$. $\square$

Corollary 3. *Let $\varphi : I \rightarrow \mathbb{R}$ be an $(\varepsilon, 0)$ -convex function on I , where ε is a nonnegative number. Then φ is nondecreasingly $(2\varepsilon, 0)$ -subdifferentiable on I .*

Proof. By Corollary 1, φ is $(\varepsilon, 0)$ -subdifferentiable on I , that is there exists a function $p : I \rightarrow \mathbb{R}$ such that

$$(8) \quad \varphi(u) + p(u)(x - u) \leq \varphi(x) + \frac{\varepsilon}{2}|x - u| \quad \text{for } x, u \in I.$$

Interchanging x and u and adding up the two inequalities, we get

$$(p(u) - p(x))(x - u) \leq \varepsilon|x - u| \quad \text{for } x, u \in I.$$

If $x < u$, then this inequality yields

$$p(x) - p(u) \leq \varepsilon,$$

whence it follows that p is ε -nondecreasing. By the previous result, there exists a nondecreasing function $q : I \rightarrow \mathbb{R}$ such that $\|p - q\| \leq \varepsilon/2$. Thus, using (8), we get

$$\begin{aligned} \varphi(u) + q(u)(x - u) &\leq \varphi(u) + p(u)(x - u) + |(p(u) - q(u))(x - u)| \\ &\leq \varphi(u) + p(u)(x - u) + \frac{\varepsilon}{2}|x - u| \\ &\leq \varphi(x) + \varepsilon|x - u| \end{aligned}$$

for all $x, u \in I$. Hence φ is nondecreasingly $(2\varepsilon, 0)$ -subdifferentiable. $\square$

Now we characterize nondecreasingly $(\varepsilon, 0)$ -subdifferentiable functions.

Theorem 4. *Let $\varphi : I \rightarrow \mathbb{R}$ and ε be a nonnegative number. Then φ is nondecreasingly $(\varepsilon, 0)$ -subdifferentiable on I if and only if there exists a convex function $g : I \rightarrow \mathbb{R}$ such that $\text{Lip}(\varphi - g) \leq \varepsilon/2$.*

Proof. Assume that $\varphi = g + \ell$, where g is convex and $\text{Lip}(\ell) \leq \varepsilon/2$. Then, by Corollary 3 (with $\varepsilon = 0$) applied to g , there exists a nondecreasing function $q : I \rightarrow \mathbb{R}$ such that

$$g(u) + q(u)(x - u) \leq g(x) \quad \text{for } x, u \in I.$$

The function ℓ also satisfies

$$\ell(u) \leq \ell(x) + \frac{\varepsilon}{2}|x - u| \quad \text{for } x, u \in I.$$

Adding up these inequalities, we get that φ satisfies

$$\varphi(u) + q(u)(x - u) \leq \varphi(x) + \frac{\varepsilon}{2}|x - u| \quad \text{for } x, u \in I,$$

i.e., it is nondecreasingly $(\varepsilon, 0)$ -subdifferentiable.

Conversely, assume that φ is nondecreasingly $(\varepsilon, 0)$ -subdifferentiable. Then, we have that

$$(9) \quad q(u)(x - u) \leq \varphi(x) - \varphi(u) + \frac{\varepsilon}{2}|x - u| \quad \text{for } x, u \in I.$$

Define now $g : I \rightarrow \mathbb{R}$ as the primitive function of q , that is, let $g(x) := \int_{x_0}^x q$, where x_0 is an arbitrarily fixed element of I . Then, q being nondecreasing, we get that g is a convex function. To complete the proof of the theorem, we show that the Lipschitz modulus of the function $\ell := \varphi - g$ is not greater than $\varepsilon/2$.

For, let $x < y$, $x, y \in I$ be arbitrary. Let $t_0 = x < t_1 < \dots < t_n = y$ be an arbitrary division of the interval $[x, y]$. Substituting $x := t_{i-1}$, $u := t_i$ for $i = 1, \dots, n$ into (9) and adding the inequalities obtained, we get

$$\begin{aligned} \sum_{i=1}^n q(t_i)(t_{i-1} - t_i) &\leq \sum_{i=1}^n \left(\varphi(t_{i-1}) - \varphi(t_i) + \frac{\varepsilon}{2}(t_i - t_{i-1}) \right) \\ &= \varphi(t_0) - \varphi(t_n) + \frac{\varepsilon}{2}(t_n - t_0) \\ &= \varphi(x) - \varphi(y) + \frac{\varepsilon}{2}(y - x). \end{aligned}$$

The left-hand side of this inequality is arbitrarily close to $-\int_x^y q = g(x) - g(y)$ if $\max_{1 \leq i \leq n} (t_i - t_{i-1})$ is small enough. Therefore, we obtain

$$g(x) - g(y) \leq \varphi(x) - \varphi(y) + \frac{\varepsilon}{2}(y - x),$$

that is,

$$(10) \quad \ell(y) - \ell(x) \leq \frac{\varepsilon}{2}(y - x).$$

On the other hand, substituting $x := t_i$, $u := t_{i-1}$ for $i = 1, \dots, n$ into (9) and adding these inequalities, we get

$$\begin{aligned} \sum_{i=1}^n q(t_{i-1})(t_i - t_{i-1}) &\leq \sum_{i=1}^n \left(\varphi(t_i) - \varphi(t_{i-1}) + \frac{\varepsilon}{2}(t_i - t_{i-1}) \right) \\ &= \varphi(t_n) - \varphi(t_0) + \frac{\varepsilon}{2}(t_n - t_0) \\ &= \varphi(y) - \varphi(x) + \frac{\varepsilon}{2}(y - x), \end{aligned}$$

whence, by taking the limit, we get that

$$g(y) - g(x) = \int_x^y q \leq \varphi(y) - \varphi(x) + \frac{\varepsilon}{2}(y - x).$$

This inequality yields

$$\ell(x) - \ell(y) \leq \frac{\varepsilon}{2}(y - x),$$

which, together with (10), results that

$$|\ell(x) - \ell(y)| \leq \frac{\varepsilon}{2}(y - x)$$

for all $x < y$ in I . Hence, the Lipschitz modulus of ℓ is not greater than $\varepsilon/2$. $\square$

In view of Theorem 2, in order to obtain the desired decomposition of (ε, δ) -convex functions, it suffices to show that $(\varepsilon, 0)$ -convex functions can be decomposed in terms of a convex function and a function with small Lipschitz modulus.

Theorem 5. *Let $\varphi : I \rightarrow \mathbb{R}$ and ε be a nonnegative number. If there exists a convex function $g : I \rightarrow \mathbb{R}$ such that $\text{Lip}(\varphi - g) \leq \varepsilon/2$, then φ is $(\varepsilon, 0)$ -convex on I .*

Conversely, if φ is $(\varepsilon, 0)$ -convex on I , then there exists a convex function $g : I \rightarrow \mathbb{R}$ such that $\text{Lip}(\varphi - g) \leq \varepsilon$.

Proof. Assume that there exists a convex function $g : I \rightarrow \mathbb{R}$ such that, for $\ell := \varphi - g$, we have $\text{Lip}(\ell) \leq \varepsilon/2$. Then we have

$$g(tx + (1 - t)y) \leq tg(x) + (1 - t)g(y)$$

and

$$\begin{aligned} \ell(tx + (1 - t)y) &\leq t\ell(x) + (1 - t)\ell(y) \\ &\quad + t|\ell(x) - \ell(tx + (1 - t)y)| + (1 - t)|\ell(y) - \ell(tx + (1 - t)y)| \\ &\leq t\ell(x) + (1 - t)\ell(y) + t\frac{\varepsilon}{2}(1 - t)|x - y| + (1 - t)\frac{\varepsilon}{2}t|x - y| \\ &= t\ell(x) + (1 - t)\ell(y) + \varepsilon t(1 - t)|x - y| \end{aligned}$$

for all $x, y \in I$ and $t \in [0, 1]$. Summing up these two inequalities, we get that φ is $(\varepsilon, 0)$ -convex.

Conversely, assume that φ is $(\varepsilon, 0)$ -convex on I . Then, by Corollary 3, it is also nondecreasingly $(2\varepsilon, 0)$ -subdifferentiable. Applying Theorem 4 (with ε replaced by 2ε), we get that there exists a convex function $g : I \rightarrow \mathbb{R}$ satisfying the condition $\text{Lip}(\varphi - g) \leq \varepsilon$. $\square$

Remark 3. There is a discrepancy in the above result, since it is not an if-and-only-if characterization of $(\varepsilon, 0)$ -convex functions. We show, however, that the constants in the result cannot be improved. Without loss of generality, we may assume that $[-1, 1] \subset I$.

First let $\varphi(x) := -\varepsilon|x|/2$ for $x \in I$. Then, obviously, $\text{Lip}(\varphi - 0) = \text{Lip}(\varphi) = \varepsilon/2$. Therefore, by the first part of Theorem 5, φ is $(\varepsilon, 0)$ -convex, i.e., it satisfies

$$\varphi(tx + (1 - t)y) \leq t\varphi(x) + (1 - t)\varphi(y) + \varepsilon t(1 - t)|x - y| \quad (x, y \in I, t \in [0, 1]).$$

Putting $x := t - 1$, $y := t$ into this inequality, we can see that it holds with equality. Thus, φ cannot be $(\bar{\varepsilon}, 0)$ -convex for any $0 \leq \bar{\varepsilon} < \varepsilon$.

On the other hand, we show that there exists an $(\varepsilon, 0)$ -convex function $\varphi : I \rightarrow \mathbb{R}$ such that $\text{Lip}(\varphi - g) \geq \varepsilon$ for all convex functions $g : I \rightarrow \mathbb{R}$. Define $\varphi : I \rightarrow \mathbb{R}$ by

$$\varphi(x) := \begin{cases} \varepsilon(1 - x^2)/2 & \text{if } |x| < 1, \\ 0 & \text{if } |x| \geq 1. \end{cases}$$

In order to prove the $(\varepsilon, 0)$ -convexity of φ , due to Theorem 1, it suffices to check the inequality

$$(11) \quad \frac{\varphi(x) - \varphi(u)}{x - u} \leq \frac{\varphi(y) - \varphi(u)}{y - u} + \varepsilon \quad (x, u, y \in I, x < u < y).$$

We distinguish three cases according to the position of u with respect to the interval $[-1, 1]$.

If $u \leq -1$, then $\frac{\varphi(x) - \varphi(u)}{x - u} = 0$ and

$$\frac{\varphi(y) - \varphi(u)}{y - u} \begin{cases} \leq \frac{\varphi(y) - \varphi(-1)}{y + 1} = \frac{\varepsilon(1 - y^2)}{2(y + 1)} = \frac{\varepsilon(1 - y)}{2} \leq \varepsilon & \text{if } |y| < 1, \\ = 0 & \text{if } |y| \geq 1. \end{cases}$$

Thus, (11) trivially holds.

A similar argument shows (11) also in the case when $u \geq 1$.

The last case is when $|u| < 1$. Then, it is elementary to check the following inequalities and equalities:

$$\frac{\varphi(x) - \varphi(u)}{x - u} \leq \frac{\varphi(u) - \varphi(-1)}{u - (-1)} = \frac{\varepsilon(1 - u)}{2} = \frac{\varphi(u) - \varphi(1)}{u - 1} + \varepsilon \leq \frac{\varphi(y) - \varphi(u)}{y - u} + \varepsilon.$$

Therefore, (11) is also valid in this case.

Thus we have proved (11), i.e., the $(\varepsilon, 0)$ -convexity of φ .

Finally, we show that $\text{Lip}(\varphi - g) \geq \varepsilon$ for all convex functions $g : I \rightarrow \mathbb{R}$. Assume on the contrary, that $\lambda = \text{Lip}(\varphi - g) < \varepsilon$ for some convex function g . Then the right (and also left) derivatives of $\ell = \varphi - g$ are between $-\lambda$ and λ , that is,

$$-\lambda \leq \varphi'(t) - g^+(t) \leq \lambda \quad (|t| < 1).$$

Let $-1 < x < y < 1$ be arbitrary. Then, by the convexity of g , we have $g^+(x) \leq g^+(y)$. Substituting $t = x$, and $t = y$ into the above inequality, we get

$$-\lambda \leq \varphi'(y) - g^+(y) \leq \varphi'(y) - g^+(x) \leq \varphi'(y) - \varphi'(x) + \lambda = \varepsilon(x - y) + \lambda.$$

Taking the limits $x \rightarrow -1$ and $y \rightarrow 1$, this inequality reduces to

$$-\lambda \leq -2\varepsilon + \lambda,$$

which yields $\varepsilon \leq \lambda$, a contradiction.

As an immediate consequence of Theorems 2 and 5, we get the main result of this paper.

Corollary 4. *Let $f : I \rightarrow \mathbb{R}$ and ε, δ be nonnegative numbers. If f is of the form $f = g + \ell + h$, where $g : I \rightarrow \mathbb{R}$ is convex, $h : I \rightarrow \mathbb{R}$ is bounded with $\|h\| \leq \delta/2$, and $\ell : I \rightarrow \mathbb{R}$ is Lipschitz with $\text{Lip}(\ell) \leq \varepsilon/2$, then f is (ε, δ) -convex on I .*

Conversely, if f is (ε, δ) -convex on I , then there exist a convex function $g : I \rightarrow \mathbb{R}$, a bounded function $h : I \rightarrow \mathbb{R}$ with $\|h\| \leq \delta/2$, and a Lipschitz function $\ell : I \rightarrow \mathbb{R}$ with $\text{Lip}(\ell) \leq \varepsilon$ such that $f = g + h + \ell$.

Proof. Assume that f is of the form $f = g + h + \ell$. Then, by Theorem 5, $\varphi = g + \ell$ is an $(\varepsilon, 0)$ -convex function. Since $\|f - \varphi\| \leq \delta/2$, hence, by Theorem 2, f is (ε, δ) -convex.

Conversely, suppose that f is (ε, δ) -convex. Then, by Theorem 2, there exists an $(\varepsilon, 0)$ -convex function $\varphi : I \rightarrow \mathbb{R}$ such that, with $h := f - \varphi$, we have $\|h\| \leq \delta/2$.

Now, applying Theorem 5, there exists a convex function $g : I \rightarrow \mathbb{R}$ such that, with the notation $\ell := \varphi - g$, we have $\text{Lip}(\ell) \leq \varepsilon$. Thus we obtain

$$f = \varphi + h = g + \ell + h,$$

which is the desired decomposition. $\square$

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