WEYL FORMULA FOR HYPOELLIPTIC OPERATORS OF SCHRÖDINGER TYPE

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ABSTRACT. In this work we consider a general class of hypoelliptic operators, for which we give an estimate of the remainder of the so-called Weyl asymptotic formula for the eigenvalues.

1. INTRODUCTION

In this paper we deal with the problem of the asymptotic estimate of the counting function $\mathcal{N}(\tau)$ of a hypoelliptic operator in $\mathbb{R}^n$, with symbol $h(x, \xi)$ diverging at infinity, in terms of the volume $\mathcal{W}(\tau)$ of the set

$$\{(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n \mid h(x, \xi) \leq \tau\}.$$

Our starting point is the very general estimate of the relative remainder

$$\frac{\mathcal{N}(\tau) - \mathcal{W}(\tau)}{\mathcal{W}(\tau)}$$

obtained by Dencker in [5], who generalized Hörmander's papers [6] and [7] to locally temperate metrics. As already remarked in [2], Dencker's estimate is not always infinitesimal, as $\tau \rightarrow +\infty$.

In this work we identify a wide class of hypoelliptic operators, we call of Schrödinger type, for which we manage to give an estimate of the kind

$$\mathcal{N}(\tau) = \mathcal{W}(\tau)(1 + O(\tau^{-\epsilon})), \quad \text{as } \tau \rightarrow \infty,$$

for a suitable $\epsilon > 0$.

We employ the following notation: given two functions $f, g : X \rightarrow \mathbb{R}$, and a subset $A \subset X$, we write

$$f(x) \prec g(x), \quad \forall x \in A,$$

if there exists a constant C such that

$$f(x) \leq Cg(x), \quad \forall x \in A.$$

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2. ASYMPTOTIC BEHAVIOR OF THE SPECTRUM OF FORMALLY HYPOELLIPTIC OPERATORS ON $\mathbb{R}^n$

We recall some results from [5] and [6].

Let

$$\phi(x, \xi; y, \eta) = \sum_{j=1}^n \xi_j y_j - x_j \eta_j$$

be the standard symplectic form on $\mathbb{R}^n \times \mathbb{R}^n$. A Riemannian metric $g_{x,\xi}(y, \eta)$ on $\mathbb{R}^n \times \mathbb{R}^n$ is *locally ϕ temperate* if it is slowly varying and there exist two positive real numbers r and N and a slowly varying metric $G_x(y)$ on $\mathbb{R}^n$ such that

$$G_x(y) \leq g_{x,\xi}(y, \eta),$$

for all $(x, \xi), (y, \eta) \in \mathbb{R}^n \times \mathbb{R}^n$, and

$$(1) \quad g_{x,\xi}(z, \zeta) \prec g_{y,\eta}(z, \zeta) (1 + g_{x,\xi}^\phi(x - y, \xi - \eta))^N,$$

for all $(x, \xi), (y, \eta), (z, \zeta) \in \mathbb{R}^n \times \mathbb{R}^n$ such that

$$G_x(x - y) \leq r.$$

The quadratic form $g_{x,\xi}^\phi(y, \eta)$ appearing in (1) is the *dual metric*

$$g_{x,\xi}^\phi(y, \eta) = \sup \{ (\phi(y, \eta; z, \zeta))^2 \mid g_{x,\xi}(z, \zeta) = 1 \}.$$

Similarly we say that a positive function $m(x, \xi)$ is *locally ϕ , g temperate* if it is g continuous (see [6]) and

$$m(x, \xi) \prec m(y, \eta) (1 + g_{x,\xi}^\phi(x - y, \xi - \eta))^N,$$

for all $(x, \xi), (y, \eta) \in \mathbb{R}^n \times \mathbb{R}^n$ such that

$$G_x(x - y) \leq r.$$

Definition 1. A differential operator h^w is *formally hypoelliptic*¹ if its Weyl symbol $h(x, \xi)$ satisfies the following hypotheses:

1) h is a smooth function such that

$$(2) \quad h(x, \xi) \neq 0, \quad \forall (x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n;$$

2) there exists a locally ϕ temperate metric $g_{x,\xi}(y, \eta)$ on $\mathbb{R}^n \times \mathbb{R}^n$ such that $|h|$ is locally ϕ , g temperate and

$$h \in S(|h|, g),$$

where $S(|h|, g)$ denotes the class of symbols of Weyl-Hörmander (see [6] and [5]).

Remark. Under suitable hypotheses on h (see [10], [1] and [4]) one can show that h^w is globally hypoelliptic in the following sense:

$$u \in \mathcal{S}' \text{ \& } h^w(u) \in \mathcal{S} \implies u \in \mathcal{S},$$

where $\mathcal{S}$ is the Schwartz class and $\mathcal{S}'$ are the tempered distributions. This justifies the term formally hypoelliptic in the definition above.

¹ This definition is not to be confused with Definition 2.3 of Chapter III of [11].

Theorem 1. Consider a real positive and formally hypoelliptic symbol $h \in S(h, g)$ and assume that there exists a positive real number γ such that

$$(3) \quad g_{x,\xi}(y, \eta) \prec h(x, \xi)^{-\gamma} g_{x,\xi}^\phi(y, \eta),$$

for all $(x, \xi), (y, \eta) \in \mathbb{R}^n \times \mathbb{R}^n$. Then h^w is semi-bounded from below and essentially self-adjoint in $L^2(\mathbb{R}^n)$.

Moreover, if

$$(4) \quad h(x, \xi) \rightarrow +\infty, \quad \text{as } |x| + |\xi| \rightarrow +\infty,$$

then the closure H of h^w in $L^2(\mathbb{R}^n)$ has discrete spectrum diverging to $+\infty$.

Proof. This is Proposition 6.1 of [5]. $\square$

Thanks to this theorem, we can define the *counting function* of the operator H :

$$(5) \quad \mathcal{N}(\tau) = \text{number of eigenfunctions of } H, \text{ corresponding to eigenvalues less than or equal to } \tau.$$

Theorem 2. Under the hypotheses of Theorem 1 assume there exists $\kappa > 0$ such that

$$(6) \quad h^{-\kappa} \in L^1.$$

Then for all $0 < \delta < \gamma/3$ we have

$$(7) \quad \mathcal{N}(\tau) = \mathcal{W}(\tau) \{1 + O(\mathcal{R}(\tau))\}, \quad \text{as } \tau \rightarrow +\infty,$$

where

$$(8) \quad \mathcal{W}(\tau) = (2\pi)^{-n} \int_{h \leq \tau} dx d\xi$$

and

$$(9) \quad \mathcal{R}(\tau) = \frac{\mathcal{W}(\tau + \tau^{1-\delta}) - \mathcal{W}(\tau - \tau^{1-\delta})}{\mathcal{W}(\tau)}.$$

Remark. Estimate (7) is known as the *Weyl formula*.

Proof. If we change the hypothesis (6) with

$$1 + |x| + |\xi| \prec h(x, \xi)^\kappa, \quad \forall (x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n,$$

this theorem becomes Theorem 6.2 of [5].

A careful inspection of Dencker's argument shows that its proof continues to hold also under our more general hypothesis. $\square$

It is not always true that the remainder $\mathcal{R}(\tau)$ vanishes as $\tau \rightarrow +\infty$. In [2] there is an example of a Schrödinger operator with potential with algebraic growth at infinity, thus meeting in particular hypothesis (6), for which $\mathcal{R}(\tau)$ is *unbounded* as $\tau \rightarrow +\infty$. In the section which follows we estimate the remainder $\mathcal{R}(\tau)$ under rather general conditions.

3. HYPOELLIPTIC DIFFERENTIAL OPERATORS OF SCHRÖDINGER TYPE AND ESTIMATE OF THE REMAINDER IN THE WEYL FORMULA

Definition 2. A formally hypoelliptic differential operator h^w is called of *Schrödinger type* if

$$h(x, \xi) = p(\xi) + q(x)$$

where p is a real polynomial such that

$$(10) \quad p(0) = 0,$$

and q is a real-valued function, called a *potential*.

Theorem 3. Under the hypotheses of Theorem 1, consider a formally hypoelliptic operator of Schrödinger type $h^w = p^w + q$, and set

$$\mathcal{V}(\tau) = \int_{q \leq \tau} dx.$$

Assume that there exists $\tau_0 > 0$, such that

$$(11) \quad \mathcal{V}(2\tau) \prec \mathcal{V}(\tau), \quad \forall \tau \geq \tau_0.$$

Then there exists $a > 0$ such that for all $0 < \epsilon < \frac{a}{1+a} \min \{1, \frac{\gamma}{3}\}$ we have

$$(12) \quad \mathcal{N}(\tau) = \mathcal{W}(\tau) \{1 + O(\tau^{-\epsilon})\}, \quad \text{as } \tau \rightarrow +\infty.$$

Remark. When p is a multi-quasi-elliptic polynomial, one can compute the constant a ; see [3].

Proof. First we show that hypothesis (6) is satisfied, so that Weyl formula (7) holds; then we estimate the remainder $\mathcal{R}(\tau)$.

Lemma 1. There exists $\kappa > 0$ such that (6) is satisfied.

Proof. Since $h > 0$, we have

$$p(\xi) + q(x) > 0, \quad \forall x, \xi \in \mathbb{R}^n.$$

Because $p(0) = 0$ and $q(x) \rightarrow +\infty$, as $|x| \rightarrow +\infty$, by (4), it follows that

$$(13) \quad L = \inf_{x \in \mathbb{R}^n} q(x) > 0.$$

Hence there exists $R > 0$ such that

$$p(\xi) + q(x) \geq \sqrt{p(\xi)q(x)}, \quad \forall \xi \notin B(R) \text{ \& } \forall x \in \mathbb{R}^n,$$

where $B(R)$ is the open ball with center at the origin and radius R . Therefore it suffices to show that there exists $\kappa_1 > 0$ and $\kappa_2 > 0$ such that $p^{-\kappa_1} \in L^1(\mathbb{R}^n \setminus B(R))$ and $q^{-\kappa_2} \in L^1(\mathbb{R}^n)$.

From (2) and (4) we have that $p \rightarrow +\infty$, as $|\xi| \rightarrow +\infty$. Then, because p is a polynomial, it follows from the Tarski-Seidenberg Theorem that there exists $k_0 > 0$ such that

$$(14) \quad p(\xi) \succ |\xi|^{k_0}, \quad \forall |\xi| \geq R.$$

In fact,

$$E = \{(s, t, \xi) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n \mid s = |\xi|^2 \text{ \& } p(\xi) = t\}$$

is algebraic in $\mathbb{R}^{2+n}$, and therefore

$$f(s) = \inf_{|\xi|^2=s} p(\xi) = \inf\{t \mid \exists s \exists \xi ((s, t, \xi) \in E)\} < +\infty$$

is semi-algebraic. Moreover $f(s) \rightarrow +\infty$, as $s \rightarrow +\infty$; hence (14) follows from Corollary A.2.6 of [8].

It is clear that estimate (14) implies the existence of κ_1 .

Now we prove the existence of κ_2 . First we show that there exists $c_0 > 0$ such that

$$(15) \quad \mathcal{V}(\tau) \prec \tau^{c_0}, \quad \forall \tau > 0.$$

From (13) we have that $\mathcal{V}(\tau) = 0$ for $\tau \leq L$, hence it suffices to prove (15) for $\tau \geq \tau_0 > 0$. From (11) we have that there exists $c > 0$ such that

$$(16) \quad \mathcal{V}(2\tau) \leq c\mathcal{V}(\tau), \quad \forall \tau \geq \tau_0.$$

Let $k_1 \in \mathbb{N}$ be such that

$$\tau_0 \leq 2^{-k_1} \tau \leq 2\tau_0.$$

By applying k_1 times inequality (16), we have

$$\begin{aligned} \mathcal{V}(\tau) &\leq c^{k_1} \mathcal{V}(2^{-k_1} \tau) \\ &\leq c^{\log_2(\tau/\tau_0)} \mathcal{V}(2\tau_0) \\ &= \left(\frac{\tau}{\tau_0}\right)^{\log_2 c} \mathcal{V}(2\tau_0), \end{aligned}$$

which is (15). Now we prove the integrability of $q^{-\kappa_2}$ for $\kappa_2 > c_0 + 1$. In fact, from (15) we have

$$\begin{aligned} \int q(x)^{-\kappa_2} dx &= \sum_{k=0}^{+\infty} \int_{k \leq q \leq k+1} q(x)^{-\kappa_2} dx \\ &\leq \int_{q \leq 1} q(x)^{-\kappa_2} dx + \sum_{k=1}^{+\infty} k^{-\kappa_2} \mathcal{V}(k+1) \\ &\prec \sum_{k=1}^{+\infty} k^{-\kappa_2} (k+1)^{c_0} < +\infty, \end{aligned}$$

for $\kappa_2 > c_0 + 1$. □

Thanks to Lemma 1 and Theorem 2 we have that (9) holds.

Now we recall the following result of Nilsson:

Theorem 4. *The function*

$$\Phi(\lambda) = \int_{p(\xi) \leq \lambda} d\xi$$

is a real analytic for λ real and large and there exist a positive rational number a and a non-negative integer b , such that for all $k \in \mathbb{N}$ there exists $A_k \in \mathbb{R}$ such that the following estimate holds:

$$(17) \quad \Phi^{(k)}(\lambda) = A_k \lambda^{a-k} (\log \lambda)^b (1 + o(1)), \quad \text{as } \lambda \rightarrow +\infty.$$

Proof. This is a special case of Theorem 3 of [9], where the author only assumes that p is a polynomial such that $p(\xi) \rightarrow +\infty$, as $|\xi| \rightarrow +\infty$. $\square$

Let us draw some consequences from estimate (17):

Corollary 1. *There exists $\lambda_0 > 0$ and $A > 0$ such that*

$$\Phi(\lambda_2) \leq \Phi(\lambda_1) \left(\frac{\lambda_2}{\lambda_1} \right)^A, \quad \forall \lambda_2 \geq \lambda_1 \geq \lambda_0.$$

Proof. Because $\Phi(\lambda)$ is positive and increasing, it is clear that in the estimate (17) we must have $A_0, A_1 > 0$. In particular, from (17) there exists $\lambda_0 > 0$ such that

$$\Phi(\lambda) \geq \frac{A_0}{2} \lambda^a (\log \lambda)^b$$

and

$$\Phi'(\lambda) \leq \frac{3A_1}{2} \lambda^{a-1} (\log \lambda)^b,$$

for all $\lambda \geq \lambda_0$. It follows that

$$(18) \quad \frac{\Phi'(\lambda)}{\Phi(\lambda)} \leq \frac{3A_1}{A_0} \frac{1}{\lambda}, \quad \forall \lambda \geq \lambda_0.$$

Let

$$\phi(\lambda) = \frac{\Phi'(\lambda)}{\Phi(\lambda)} = \log(\Phi(\lambda))';$$

then we have

$$\log \frac{\Phi(\lambda_2)}{\Phi(\lambda_1)} = \int_{\lambda_1}^{\lambda_2} \phi(t) dt.$$

From (18) it follows that

$$\begin{aligned} \frac{\Phi(\lambda_2)}{\Phi(\lambda_1)} &\leq \exp \left(\int_{\lambda_1}^{\lambda_2} \frac{3A_1}{A_0} \frac{1}{t} dt \right) \\ &= \left(\frac{\lambda_2}{\lambda_1} \right)^{\frac{3A_1}{A_0}}, \quad \forall \lambda_2 \geq \lambda_1 \geq \lambda_0. \end{aligned}$$

$\square$

Now we estimate the remainder $\mathcal{R}(\tau)$ with $0 < \delta < 1$. Let $0 < s < 1$ be a number to be chosen later and write

$$\mathcal{W}(\tau + \tau^{1-\delta}) - \mathcal{W}(\tau - \tau^{1-\delta}) = \mathcal{W}_1(\tau) + \mathcal{W}_2(\tau),$$

where

$$(19) \quad \mathcal{W}_1(\tau) = (2\pi)^{-n} \int_{\substack{\tau - \tau^{1-\delta} < p+q \leq \tau + \tau^{1-\delta} \\ q \leq \tau - \tau^{1-s\delta}}} dx d\xi$$

and

$$(20) \quad \mathcal{W}_2(\tau) = (2\pi)^{-n} \int_{\substack{\tau - \tau^{1-\delta} < p+q \leq \tau + \tau^{1-\delta} \\ q > \tau - \tau^{1-s\delta}}} dx d\xi.$$

Let us estimate $\mathcal{W}_1$. On the domain of integration of $\mathcal{W}_1$ we have

$$\tau + \tau^{1-\delta} - q \geq p > \tau - \tau^{1-\delta} - q \geq \tau^{1-s\delta} - \tau^{1-\delta}.$$

Because

$$\tau^{1-s\delta} - \tau^{1-\delta} = \tau^{1-s\delta} \left(1 - \tau^{(s-1)\delta}\right) \rightarrow +\infty, \quad \text{as } \tau \rightarrow +\infty,$$

there exists $\tau_1 > 0$ such that

$$\tau^{1-s\delta} - \tau^{1-\delta} \geq \lambda_0, \quad \forall \tau \geq \tau_1.$$

Then from (19) and Corollary 1 we have

$$\begin{aligned} (2\pi)^n \mathcal{W}_1(\tau) &= \int_{q \leq \tau - \tau^{1-s\delta}} \left(\int_{\tau - \tau^{1-\delta} - q < p \leq \tau + \tau^{1-\delta} - q} d\xi \right) dx \\ &= \int_{q \leq \tau - \tau^{1-s\delta}} \left\{ \Phi(\tau + \tau^{1-\delta} - q(x)) - \Phi(\tau - \tau^{1-\delta} - q(x)) \right\} dx \\ &\leq \int_{q \leq \tau - \tau^{1-s\delta}} \Phi(\tau - q) \left\{ \left(1 + \frac{\tau^{1-\delta}}{\tau - q}\right)^A - \left(1 - \frac{\tau^{1-\delta}}{\tau - q}\right)^A \right\} dx, \end{aligned}$$

for all $\tau \geq \tau_1$. This in turn implies that there exists $\tau_2 \geq \tau_1$ such that

$$\begin{aligned} \mathcal{W}_1(\tau) &\prec \int_{q \leq \tau - \tau^{1-s\delta}} \Phi(\tau - q) \frac{\tau^{1-\delta}}{\tau - q} dx \\ (21) \quad &\leq \tau^{(s-1)\delta} \int_{q \leq \tau - \tau^{1-s\delta}} \Phi(\tau - q) dx \\ &= \tau^{(s-1)\delta} \mathcal{W}(\tau), \quad \forall \tau \geq \tau_2. \end{aligned}$$

Now we estimate $\mathcal{W}_2$. Because p is non-negative as $|\xi| \rightarrow +\infty$, there exists $\tau_3 \geq 1$ such that, on the domain of integration, we have

$$\begin{aligned} p &\leq \tau + \tau^{1-\delta} - q < \tau^{1-\delta} + \tau^{1-s\delta} = \tau^{1-s\delta} (\tau^{(s-1)\delta} + 1) \leq 2\tau^{1-s\delta}, \\ q &\leq \tau + \tau^{1-\delta} - p \leq 2\tau, \end{aligned}$$

for $\tau \geq \tau_3$.

It follows that

$$\begin{aligned} \mathcal{W}_2(\tau) &\leq (2\pi)^{-n} \int_{p(\xi) \leq 2\tau^{1-s\delta}} d\xi \int_{q(x) \leq 2\tau} dx \\ &= (2\pi)^{-n} \Phi(2\tau^{1-s\delta}) \mathcal{V}(2\tau), \end{aligned}$$

for all $\tau \geq \tau_3$. Now, from (11) we have

$$\mathcal{V}(2\tau) \prec \mathcal{V}\left(\frac{\tau}{2}\right), \quad \forall \tau \geq \tau_0.$$

On the other side, from (17), we have that

$$\begin{aligned}
 \Phi(2\tau^{1-s\delta}) &= A_0 (2\tau^{1-s\delta})^a (\log(2\tau^{1-s\delta}))^b (1+o(1)) \\
 &= 2^a (1-s\delta)^b A_0 \tau^{(1-s\delta)a} (\log \tau)^b (1+o(1)) \\
 &= 2^{2a} (1-s\delta)^b A_0 \tau^{-s\delta a} \left(\frac{\tau}{2}\right)^a \left(\log\left(\frac{\tau}{2}\right)\right)^b (1+o(1)) \\
 &= 2^{2a} (1-s\delta)^b \tau^{-s\delta a} \Phi\left(\frac{\tau}{2}\right) (1+o(1)), \quad \text{for } \tau \rightarrow +\infty.
 \end{aligned}$$

But these estimates imply that there exists $\tau_4 \geq \tau_0$ such that

$$\begin{aligned}
 (22) \quad \mathcal{W}_2(\tau) &\prec \tau^{-s\delta a} \Phi\left(\frac{\tau}{2}\right) \mathcal{V}\left(\frac{\tau}{2}\right) \\
 &\leq \tau^{-s\delta a} \int_{p \leq \tau/2, q \leq \tau/2} dx d\xi \\
 &\leq \tau^{-s\delta a} \mathcal{W}(\tau), \quad \forall \tau \geq \tau_4.
 \end{aligned}$$

Eventually, from (21) and (22), we have that it suffices to choose s such that

$$(s-1)\delta = -s\delta a,$$

that is,

$$s = \frac{1}{1+a},$$

in order to obtain

$$\mathcal{W}(\tau + \tau^{1-\delta}) - \mathcal{W}(\tau - \tau^{1-\delta}) = O(\tau^{-\epsilon}) \mathcal{W}(\tau), \quad \text{as } \tau \rightarrow +\infty,$$

with

$$\epsilon = \frac{\delta a}{1+a}.$$

□

4. AN EXAMPLE

We end our paper by giving an example of potential verifying our hypotheses. First we prove the following

Proposition 1. *Let $q : \mathbb{R}^n \rightarrow \mathbb{R}$ be a smooth function such that there exist $\mu \in \mathbb{R}_+^n$, $C_\mu > 1$ and $R_0 \geq 0$ such that*

$$q(\mu \cdot x) \geq C_\mu q(x), \quad \forall |x| \geq R_0,$$

where

$$\mu \cdot x = \sum_{i=1}^n \mu_i x_i.$$

Let N be the smallest integer such that

$$(23) \quad C_\mu^N \geq 2.$$

Then $\mathcal{V}$ satisfies condition (11) with

$$\tau_0 = \frac{C_\mu^N}{2} \max_{|x| \leq R_0} q(x).$$

Proof. Letting $x = \mu \cdot y$ and using (23) yields

$$\begin{aligned}
 \int_{q(x) \leq 2\tau} dx &= \mu_1 \cdots \mu_n \int_{q(\mu \cdot y) \leq 2\tau} dy \\
 &\leq \mu_1 \cdots \mu_n \left\{ \int_{|y| < R_0} dy + \int_{\substack{q(\mu \cdot y) \leq 2\tau \\ |y| \geq R_0}} dy \right\} \\
 &\leq \mu_1 \cdots \mu_n \left\{ \int_{|y| < R_0} dy + \int_{\substack{C_\mu q(y) \leq 2\tau \\ |y| > R_0}} dy \right\} \\
 &\leq \mu_1 \cdots \mu_n \int_{q(y) \leq 2C_\mu^{-1}\tau} dy,
 \end{aligned}$$

for $\tau \geq \tau_0$. By iterating, we obtain

$$\mathcal{V}(2\tau) = \int_{q(x) \leq 2\tau} dx \leq (\mu_1 \cdots \mu_n)^N \int_{q(x) \leq 2C_\mu^{-N}\tau} dx \leq (\mu_1 \cdots \mu_n)^N \mathcal{V}(\tau),$$

for all $\tau \geq \tau_0$. □

Now we can construct a class of operators of Schrödinger type, with potential verifying condition (11).

Proposition 2. *Let us consider*

- a real polynomial $p(\xi)$ vanishing at the origin and hypoelliptic, i.e. such that

$$\lim_{|\xi| \rightarrow +\infty} p(\xi) = +\infty$$

and

$$\nabla p(\xi) \prec p(\xi)^{1-\rho}, \quad \text{for } p(\xi) \geq 1,$$

with $0 < \rho \leq 1$,

- a positive smooth function $q(x)$ such that
 - $p(\xi) + q(x) > 0$, $\forall (x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n$,
 - there exists $0 \leq \nu < \rho$ such that for all $\alpha \in \mathbb{N}^n$ we have $D^\alpha q(x) \prec q(x)^{1+\nu|\alpha|}$, $\forall x \in \mathbb{R}^n$,
 - it satisfies the hypothesis of Proposition 1.

Then $h^w = p^w + q$ satisfies the hypotheses of Theorem 3.

Proof. We give a brief sketch of the proof. By Proposition 1 condition (11) is satisfied.

Then it suffices to verify the hypotheses of Theorem 1. Consider the metric

$$g_{x,\xi}(y, \eta) = h(x, \xi)^{2\nu} |y|^2 + h(x, \xi)^{-2\rho} |\eta|^2.$$

Then one verifies that g and h are locally ϕ temperate; for example see [3]. Moreover one can show that $h \rightarrow +\infty$, as $|x| + |\xi| \rightarrow +\infty$, $h \in S(h, g)$ and (3) is verified. We omit the details. □

For example we can take (see [1], section 1.1)

$$p(\xi) = (\xi_1^{2k} - \xi_2^{2k-1})^2 + \xi_1^{2k} \xi_2^{2k-2},$$

$$q(x) = \exp(x_1^{2m_1} + x_2^{2m_2}),$$

where k is an integer greater than 1 and m_1 and m_2 are positive integers.

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