FERENC LUKÁCS TYPE THEOREMS IN TERMS OF THE ABEL-POISSON MEAN OF CONJUGATE SERIES

FERENC MÓRICZ

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ABSTRACT. A theorem of Ferenc Lukács determines the generalized jumps of a periodic, Lebesgue integrable function f in terms of the partial sum of the conjugate series to the Fourier series of f . The main aim of this paper is to prove an analogous theorem in terms of the Abel-Poisson mean. We also prove an estimate of the partial derivative (with respect to the angle) of the Abel-Poisson mean of an integrable function F at those points at which F is smooth. Finally, we reveal the intimate relation between these two results.

1. A THEOREM OF FERENC LUKÁCS

Let f be a periodic, Lebesgue integrable function, in symbol: $f \in L^1(\mathbb{T})$, $\mathbb{T} := [-\pi, \pi)$, with Fourier series

$$(1.1) \quad f(x) \sim \frac{1}{2}a_0 + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx),$$

where

$$(1.2) \quad a_k := \frac{1}{\pi} \int_{\mathbb{T}} f(x) \cos kx dx \quad \text{and} \quad b_k := \frac{1}{\pi} \int_{\mathbb{T}} f(x) \sin kx dx$$

are the Fourier coefficients of f . We recall that the conjugate series to (1.1) is defined by

$$(1.3) \quad \sum_{k=1}^{\infty} (a_k \sin kx - b_k \cos kx).$$

Denote by $\tilde{s}_n(f, x)$ the n th partial sum of series (1.3). The following theorem was proved by Ferenc Lukács [2].

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Theorem 1. *Let $f \in L^1(\mathbb{T})$ and $x \in \mathbb{T}$. If there exists a number $d_x(f)$ such that*

$$(1.4) \quad \lim_{h \rightarrow 0+} \frac{1}{h} \int_0^h |[f(x+t) - f(x-t)] - d_x(f)| dt = 0,$$

then

$$(1.5) \quad \lim_{n \rightarrow \infty} \frac{\tilde{s}_n(f, x)}{\log n} = -\frac{1}{\pi} d_x(f).$$

By ‘log’ we mean the natural logarithm.

It is clear that if the finite limit

$$(1.6) \quad d_x(f) := \lim_{t \rightarrow 0+} [f(x+t) - f(x-t)]$$

exists, then condition (1.4) is also satisfied with the same $d_x(f)$. In particular, if a periodic function f is of bounded variation over $[-\pi, \pi]$, then (1.5) is satisfied at every point $x \in \mathbb{T}$ with

$$(1.7) \quad d_x(f) := f(x+0) - f(x-0).$$

This means that the terms of the Fourier series of f determine the (ordinary) jumps of f at any point $x \in \mathbb{T}$ of discontinuity of first kind. Or equivalently, the terms of the Fourier series of f determine the atoms of the finite Borel measure induced by f on $\mathbb{T}$.

We note that Fejér [1] achieved the first result in the subject of determining the jumps of a function of bounded variation in terms of the partial sum of its Fourier series. Then Zygmund [4, p. 108] proved a Fejér type theorem via the Abel-Poisson mean of the Fourier series in question.

2. MAIN RESULTS

One of the aims of the present paper is to prove a Ferenc Lukács type theorem in terms of the Abel-Poisson mean of the conjugate series (1.3). We recall that the Abel-Poisson mean of (1.3) is defined by

$$(2.1) \quad \tilde{f}(r, x) := \sum_{k=1}^{\infty} (a_k \sin kx - b_k \cos kx) r^k, \quad 0 \leq r < 1.$$

The analogue of Theorem 1 reads as follows.

Theorem 2. *If $f \in L^1(\mathbb{T})$ and the finite limit*

$$(2.2) \quad \delta_x(f) := \lim_{h \rightarrow 0+} \frac{1}{h} \int_0^h [f(x+t) - f(x-t)] dt$$

exists at some point $x \in \mathbb{T}$, then

$$(2.3) \quad \lim_{r \rightarrow 1-} \frac{\tilde{f}(r, x)}{\log(1-r)} = \frac{1}{\pi} \delta_x(f).$$

The quantity $\delta_x(f)$ defined in (2.2) may be called the generalized jump of the function f at the point x . The existence of $d_x(f)$ in (1.6) or even in (1.4) implies that of $\delta_x(f)$, and both numbers are equal. Observe the lack of the absolute value bars in (2.2). Loosely speaking, condition (2.2) expresses a differentiability property of an integral, while (1.4) resembles the definition of a Lebesgue point.

An immediate corollary of Theorem 2 is that if a periodic function f is of bounded variation over $[-\pi, \pi]$, then at every point $x \in \mathbb{T}$ we have

$$\lim_{r \rightarrow 1-} \frac{\tilde{f}(r, x)}{\log(1-r)} = \frac{1}{\pi} [f(x+0) - f(x-0)].$$

We remind the reader that a function F is said to be smooth at some inner point x of the domain of F if the following limit relation holds:

$$(2.4) \quad \Delta(F, x, h) := \frac{1}{h} [F(x+h) + F(x-h) - 2F(x)] \rightarrow 0 \quad \text{as } h \rightarrow 0.$$

The function class $\lambda_*(\mathbb{T})$ is defined to consist of all periodic, continuous functions F such that (2.4) holds uniformly in $x \in \mathbb{T}$. The class $\Lambda_*(\mathbb{T})$ is defined by the requirement that the ratio $\Delta(F, x, h)$ is uniformly bounded in $x \in \mathbb{T}$ and $h > 0$. These function classes (not only in the periodic case) were introduced by Zygmund [3] (see also [4, p. 43]).

Our second main result is the following

Theorem 3. (i) *If a function $F \in L^1(\mathbb{T})$ is smooth at some point $x \in \mathbb{T}$, then for the Abel-Poisson mean $\tilde{F}(r, x)$ of the conjugate series to the Fourier series of F we have*

$$(2.5) \quad \frac{\partial \tilde{F}(r, x)}{\partial x} = o\left\{\log \frac{1}{1-r}\right\} \quad \text{as } r \rightarrow 1-.$$

(ii) *If $F \in \lambda_*(\mathbb{T})$, then (2.5) holds uniformly in $x \in \mathbb{T}$. If $F \in \Lambda_*(\mathbb{T})$, then (2.5) holds with 'O' in place of 'o', uniformly in $0 \leq r < 1$ and $x \in \mathbb{T}$.*

We note that Theorem 3 can be considered as a counterpart of a theorem of Zygmund [4, p. 109] which provides an estimate of $\partial^2 F(r, x)/\partial x^2$, where by $F(r, x)$ we denote the Abel-Poisson mean of the Fourier series of F .

3. AUXILIARY RESULTS ON THE CONJUGATE ABEL-POISSON KERNEL

We start with the representation

$$(3.1) \quad \tilde{f}(r, x) = \frac{1}{\pi} \int_{\mathbb{T}} f(x-t) Q(r, t) dt,$$

where

$$(3.2) \quad Q(r, t) := \sum_{k=1}^{\infty} r^k \sin kt = \frac{r \sin t}{1 - 2r \cos t + r^2}, \quad 0 \leq r < 1,$$

is the conjugate Abel-Poisson kernel. (See, for example, [4, p. 96].)

Since $Q(r, t)$ is odd in t , from (3.1) it follows that

$$(3.3) \quad \tilde{f}(r, x) = \frac{1}{\pi} \int_0^{\pi} [f(x-t) - f(x+t)] Q(r, t) dt, \quad 0 \leq r < 1.$$

The other crucial property is that $Q(r, t)$ is positive for $0 < t < \pi$. From (3.2) it also follows that

$$(3.4) \quad Q(r, t) = \frac{r \sin t}{(1-r)^2 + 4r \sin^2 \frac{t}{2}} \leq \frac{1}{2 \tan \frac{t}{2}}, \quad 0 \leq r < 1 \quad \text{and} \quad 0 < t < \pi.$$

Lemma 1. *For $0 \leq r < 1$, we have*

$$(3.5) \quad Q_r := \int_0^{\pi} Q(r, t) dt = \log \frac{1+r}{1-r}.$$

Proof. By elementary calculus, we have

$$Q_r = \left[\frac{1}{2} \log \left\{ (1-r)^2 + 4r \sin^2 \frac{t}{2} \right\} \right]_{t=0}^{\pi},$$

which gives the right-hand side in (3.5). $\square$

Next, we examine the partial derivative

$$(3.6) \quad Q'(r, t) := \frac{\partial Q(r, t)}{\partial t} = \frac{r(1+r^2) \cos t - 2r^2}{(1-2r \cos t + r^2)^2}.$$

It is clear that $Q'(r, t)$ changes sign in the interval $(0, \pi)$ only once, namely, for $t = \tau = \tau(r)$ satisfying

$$\cos \tau = \frac{2r}{1+r^2},$$

so that $\tau \rightarrow 0+$ as $r \rightarrow 1-$. Since $0 < \tau \leq \pi/2$ for $0 \leq r < 1$, we have

$$1 - \frac{\tau^2}{2} \leq \frac{2r}{1+r^2},$$

whence

$$(3.7) \quad \tau \geq \sqrt{2} \frac{1-r}{\sqrt{1+r^2}} \geq 1-r.$$

Lemma 2. For $0 \leq r < 1$, we have

$$(3.8) \quad \int_0^{\pi} t |Q'(r, t)| dt \leq 2 + \log \frac{\pi}{1-r}.$$

Proof. By integration by parts, we obtain

$$\begin{aligned} \int_0^{\pi} t |Q'(r, t)| dt &= \int_0^{\tau} t Q'(r, t) dt - \int_{\tau}^{\pi} t Q'(r, t) dt \\ &= 2\tau Q(r, \tau) - \int_0^{\tau} Q(r, t) dt + \int_{\tau}^{\pi} Q(r, t) dt. \end{aligned}$$

By virtue of (3.4) and (3.7), we conclude that

$$\begin{aligned} \int_0^{\pi} t |Q'(r, t)| dt &\leq \frac{2\tau \sin \tau}{4 \sin^2 \frac{\tau}{2}} + \int_{\tau}^{\pi} \frac{\sin t}{4 \sin^2 \frac{t}{2}} dt \\ &= \frac{\tau}{\tan \frac{\tau}{2}} + \int_{\tau}^{\pi} \frac{dt}{2 \tan \frac{t}{2}}, \end{aligned}$$

whence (3.8) follows immediately. $\square$

4. PROOF OF THEOREM 2

By Lemma 1, we have

$$\lim_{r \rightarrow 1-} \frac{Q_r}{-\log(1-r)} = 1.$$

Thus, it is enough to prove that

$$(4.1) \quad \lim_{r \rightarrow 1-} \left(\frac{\tilde{f}(r, x)}{Q_r} + \frac{1}{\pi} \delta_x(f) \right) = 0.$$

By (2.2), given any $\varepsilon > 0$ we can choose $0 < \eta < \pi/2$ so that for the function

$$I(h) := \int_0^h [f(x-t) - f(x+t) + \delta_x(f)] dt$$

we have

$$(4.2) \quad |I(h)| \leq \varepsilon h \quad \text{if } 0 \leq h \leq \eta.$$

Keeping (3.3), the notation in (3.5), and (4.2) in mind, we may write that

$$(4.3) \quad \begin{aligned} \frac{\tilde{f}(r, x)}{Q_r} + \frac{1}{\pi} \delta_x(f) &= \frac{1}{\pi Q_r} \int_0^\pi [f(x-t) - f(x+t) + \delta_x(f)] Q(r, t) dt \\ &= \frac{1}{\pi Q_r} \left(\int_0^\eta + \int_\eta^\pi \right) =: J_1 + J_2, \quad \text{say.} \end{aligned}$$

First, integrating by parts and making use of (4.2), (3.4), (3.5) and (3.8) yield

$$(4.4) \quad \begin{aligned} |J_1| &\leq \frac{1}{\pi Q_r} \left(I(\eta) Q(r, \eta) + \int_0^\eta |I(t) Q'(r, t)| dt \right) \\ &\leq \frac{\varepsilon}{\pi Q_r} \left(\eta Q(r, \eta) + \int_0^\eta t |Q'(r, t)| dt \right) \\ &\leq \frac{\varepsilon}{-\pi \log(1-r)} \left\{ \frac{\eta}{2 \tan \frac{\eta}{2}} + 2 + \log \pi - \log(1-r) \right\} \\ &\leq \frac{\varepsilon}{-\pi \log(1-r)} \left\{ \frac{9}{2} - \log(1-r) \right\} \leq \frac{11\varepsilon}{2\pi} \quad \text{if } r \geq \frac{e-1}{e}, \end{aligned}$$

since $\log \pi \leq 3/2$ and $-\log(1-r) \geq 1$ if $0 < (1-r) \leq 1/e$.

Second, by (3.4) and (3.5) we obtain

$$(4.5) \quad \begin{aligned} |J_2| &\leq \frac{1}{\pi Q_r} \int_\eta^\pi |f(x-t) - f(x+t) + \delta_x(f)| Q(r, t) dt \\ &\leq \frac{1}{\pi Q_r} \frac{1}{2 \tan \frac{\eta}{2}} \left(\int_{\mathbb{T}} |f(t)| dt + 2\pi |\delta_x(f)| \right) \\ &= O \left\{ \frac{1}{-\log(1-r)} \right\} \quad \text{as } r \rightarrow 1-. \end{aligned}$$

Combining (4.3)-(4.5) gives (4.1), which is equivalent to (2.3) which was to be proved. $\square$

5. PROOF OF THEOREM 3

Part (i). We make use of (3.1) with F in place of f and the series representation of $Q(r, t)$ in (3.2) (which allows a term-by-term differentiation) to obtain

$$(5.1) \quad \frac{\partial \tilde{F}(r, x)}{\partial x} = \frac{1}{\pi} \int_{\mathbb{T}} F(t) Q'(r, x-t) dt = \frac{1}{\pi} \int_{\mathbb{T}} F(x-t) Q'(r, t) dt.$$

Since $Q'(r, t)$ is even in t (see (3.6)) and

$$\int_0^\pi Q'(r, t) dt = Q(r, \pi) - Q(r, 0) = 0,$$

from (5.1) we conclude that

$$(5.2) \quad \frac{\partial \tilde{F}(r, x)}{\partial x} = \frac{1}{\pi} \int_0^\pi [F(x+t) + F(x-t) - 2F(x)] Q'(r, t) dt, \quad 0 \leq r < 1.$$

By assumption, F is smooth at x . Thus, given any $\varepsilon > 0$ there exists $\eta > 0$ so that (cf. (2.4) as to the notation)

$$(5.3) \quad |t\Delta(F, x, t)| = |F(x+t) + F(x-t) - 2F(x)| \leq \varepsilon t \quad \text{if } 0 \leq t \leq \eta.$$

Accordingly, we decompose the integral in (5.2) as follows:

$$(5.4) \quad \frac{\partial \tilde{F}(r, x)/\partial x}{-\log(1-r)} = \frac{1}{-\pi \log(1-r)} \left(\int_0^\eta + \int_\eta^\pi \right) t\Delta(F, x, t)Q'(r, t)dt$$

$$=: I_1 + I_2, \quad \text{say.}$$

By (5.3) and (3.8), we have

$$(5.5) \quad |I_1| \leq \frac{1}{-\pi \log(1-r)} \int_0^\eta \varepsilon |tQ'(r, t)| dt = O(\varepsilon) \quad \text{as } r \rightarrow 1-.$$

On the other hand, we estimate $|Q'(r, t)|$ (see (3.6)) in the same way as in the case of inequality (3.4) to obtain

$$|Q'(r, t)| \leq \frac{1}{4r \sin^4 \frac{t}{2}}, \quad 0 \leq r < 1 \quad \text{and} \quad 0 < t \leq \pi.$$

Taking into account this and the first equality in (5.3) gives

$$(5.6) \quad \begin{aligned} |I_2| &\leq \frac{1}{-\pi \log(1-r)} \frac{1}{4r \sin^4 \frac{\eta}{2}} \int_\eta^\pi |F(x+t) + F(x-t) - 2F(x)| dt \\ &\leq \frac{1}{-\pi \log(1-r)} \frac{1}{4r \sin^4 \frac{\eta}{2}} \left(\int_{\mathbb{T}} |F(t)| dt + 2\pi |F(x)| \right) \\ &= O\left\{ \frac{1}{-\log(1-r)} \right\} \quad \text{as } r \rightarrow 1-. \end{aligned}$$

Combining (5.4)-(5.6) yields (2.5) to be proved.

Part (ii). The obvious extension to uniformity completes the proof of the first statement when $F \in \lambda_*(\mathbb{T})$. The proof of the second statement when $F \in \Lambda_*(\mathbb{T})$ runs along the same lines, the details of which are left to the reader. $\square$

6. CONCLUDING REMARKS

Remark 1. We start with a function $f \in L^1(\mathbb{T})$. Being interested in the Abel-Poisson mean of the conjugate series to the Fourier series of f , there is no loss of generality if we assume that $a_0 = 0$ in (1.2). Then the integral F of f defined by

$$(6.1) \quad F(u) := \int_0^u f(t) dt, \quad u \in \mathbb{R},$$

is also periodic. Furthermore, we consider such a point $x \in \mathbb{T}$ at which $\delta_x(f) = 0$.

We claim that if we apply Theorem 3 to this F at this point x , then conclusion (2.5) coincides with (2.3). In fact, by (6.1), condition (2.2) can be rewritten in the form

$$\delta_x(f) = \lim_{h \rightarrow 0+} \frac{1}{h} [F(x+h) + F(x-h) - 2F(x)],$$

whence we see that $\delta_x(f) = 0$ is the same thing as the smoothness of F at x .

On the other hand, if A_k and B_k are the Fourier coefficients of F , then an integration by parts gives (see, for example, [4, p. 42]) that

$$A_k = -\frac{b_k}{k} \quad \text{and} \quad B_k = \frac{a_k}{k}, \quad k = 1, 2, \dots$$

Hence it follows immediately that

$$\frac{\partial \tilde{F}(r, x)}{\partial x} = \tilde{f}(r, x), \quad 0 \leq r < 1.$$

This justifies our claim made above.

In other words, we have found another proof of Theorem 2 by means of Theorem 3 in the particular case when $\delta_x(f) = 0$.

Remark 2. To prove Theorem 2 in the general case when $\delta_x(f) \neq 0$, we could have proceeded in the following way, as well. Imitating the argument of Fejér in [1], we introduce a new function g by setting

$$(6.2) \quad g(\xi) := f(\xi) - \frac{1}{\pi} \delta_x(f) \phi(\xi - x), \quad \xi \in \mathbb{T},$$

where the auxiliary function ϕ is defined by

$$\phi(t) := \frac{1}{2}(\pi - t) \quad \text{for} \quad 0 < t < 2\pi,$$

$\phi(0) = \phi(2\pi) := 0$, and ϕ is continued periodically for all $t \in \mathbb{R}$. Since the ordinary jump of the function $\phi(\xi - x)$ at the point $\xi := x$ equals π , we have $\delta_x(g) = 0$. By Remark 1, we conclude that

$$(6.3) \quad \lim_{r \rightarrow 1-} \frac{\tilde{s}_n(g, x)}{\log(1-r)} = 0.$$

By (6.2), it remains only to take into account that the Fourier series of $\phi(\xi - x)$ (as a function of ξ) and its conjugate series are given by

$$\phi(\xi - x) \sim \sum_{k=1}^{\infty} \frac{\sin k(\xi - x)}{k} \quad \text{and} \quad - \sum_{k=1}^{\infty} \frac{\cos k(\xi - x)}{k},$$

respectively. Hence it follows immediately that

$$(6.4) \quad \frac{\tilde{\phi}(\cdot - x)(r, x)}{\log(1-r)} = -\frac{1}{\log(1-r)} \sum_{k=1}^{\infty} \frac{r^k}{k} = 1, \quad 0 < r < 1.$$

Combining (6.2)-(6.4) provides (2.3) in the general case when $\delta_x(f) \neq 0$.

Remark 3. In case $x := 0$, (6.4) is of the form

$$\frac{\tilde{\phi}(r, 0)}{\log(1-r)} = 1 = \frac{1}{\pi} d_0(\phi), \quad 0 < r < 1,$$

where $d_0(\phi)$ is defined in (1.6). This shows that relation (2.3) holds true even without “ $\lim_{r \rightarrow 1-}$ ” in the case of the function $\phi(t)$ defined above (observe that this time $\delta_0(\phi) = d_0(\phi)$).

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BOLYAI INSTITUTE, UNIVERSITY OF SZEGED, ARADI VÉRTANÚK TERE 1, 6720 SZEGED, HUNGARY
E-mail address: `moricz@math.u-szeged.hu`