EXISTENCE OF C^∞ LOCAL SOLUTIONS OF THE COMPLEX MONGE-AMPÈRE EQUATION

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(Communicated by David S. Tartakoff)

ABSTRACT. We prove the C^∞ local solvability of the n -dimensional complex Monge-Ampère equation $\det(u_{i\bar{j}}) = K(z) f(z, u, \nabla u)$, $f > 0$, in a neighborhood of any point z_0 where $K(z_0) = 0$ but $dK(z_0) \neq 0$.

1. INTRODUCTION

Our motivation to study equations of Monge-Ampère type comes from their close connection with problems in complex analysis (see [1], [5]), in differential geometry (see [2], [6], [7], [10], [11], [12]) and in physics (see [4]).

Here we consider the complex Monge-Ampère equation

$$(1.1) \quad \det(u_{i\bar{j}}) = K(z) f(z, u, \nabla u)$$

in a neighborhood V of $z_0 \in \mathbb{C}^n$, where $f(z, u, p)$ is a positive C^∞ function for $(z, u, p) \in V \times \mathbb{R} \times \mathbb{R}^{2n}$ and $K \in C^\infty(V, \mathbb{R})$. For $z \in \mathbb{C}^n$, we denote $x = (\operatorname{Re} z, \operatorname{Im} z) = (x_1, x_2, \dots, x_n, \dots, x_{2n}) \in \mathbb{R}^{2n}$ and $u_{i\bar{j}} = \frac{\partial^2 u}{\partial z_i \partial \bar{z}_j}$.

We study the existence of a local smooth solution to the problem (1.1) when

$$(1.2) \quad K(z_0) = 0 \text{ and } dK(z_0) \neq 0.$$

The main result of this paper is the following:

Theorem 1. *Under condition (1.2), equation (1.1) in $\mathbb{C}^n$ has a C^∞ local solution in a neighborhood of z_0 .*

2. PROOF OF THEOREM 1

Let us recall the result of G.Nakamura and Y.Maeda ([8], [9]).

Given $x_0 \in \mathbb{R}^m$, U a neighborhood of x_0 , $u_0 \in C^\infty(U, \mathbb{R})$, a nonlinear partial differential equation of order q with C^∞ coefficients in U and

$$(2.1) \quad \Phi(x, D^\alpha u)_{|\alpha| \leq q} = g,$$

we can state

Received by the editors March 6, 2001.

2000 *Mathematics Subject Classification.* Primary 35Mxx, 39B42.

Key words and phrases. Complex Monge-Ampère equation, real principal type symbol.

Theorem 2 ([8], Th. 4). *Assume that the linearized equation of (2.1) is of real principal type at u_0 and x_0 . Then there exist an open neighborhood ω of x_0 , $s_0 \in \mathbb{N}$ and $\eta > 0$ such that for every $g \in C^\infty(\omega)$ satisfying*

$$\left\| g - \Phi(x, D^\alpha u_0)_{|\alpha| \leq q} \right\|_{s_0} \leq \eta,$$

there exists $u \in C^\infty(\omega, \mathbb{R})$, as a solution to equation (2.1). (Here $\|\cdot\|_{s_0}$ is the Sobolev norm. The neighborhood ω and the constants s_0, η are independent from u_0 , where ω can be chosen small enough.)

Recall that a differential operator P is said to be of real principal type at $x_0 \in V$ if its principal symbol p is real and has no null bicharacteristics trapped over x_0 .

Using condition (1.2) and invariance of equation (1.1) under rotation, we may assume, without loss of generality, that $z_0 = 0$ and

$$(2.2) \quad K(z) = x_n + \mathcal{O}(|z|^2).$$

Step 1. Determination of u_0 .

Set $g(x, u, \nabla u) = K(z)f(z, u, \nabla u)$, and let $x = (x', x_n) \in \mathbb{R}^{2n-1} \times \mathbb{R}$.

Given an integer $N \geq s_0 + n + 1$, we consider the analytic Cauchy problem

$$(2.3) \quad \begin{cases} \det(u_{i\bar{j}}) - \sum_{|\alpha|+|\beta|+|\gamma| < N} \frac{1}{(\alpha, \beta, \gamma)!} \partial_x^\alpha \partial_u^\beta \partial_p^\gamma g(0) x^\alpha u^\beta (\nabla u)^\gamma = 0, \\ u|_{x_n=0} = \frac{1}{2} |x'|^2, \\ \frac{\partial u}{\partial x_n}|_{x_n=0} = h(x') \end{cases}$$

where h is an analytic function such that $h(0) = 0$.

The coefficient of $\partial_{x_n}^2 u$ in the left-hand side of equation (2.3) is equal to

$$\det(u_{i\bar{j}})_{1 \leq i, j \leq n-1}$$

which is equal to 1 at the origin (from initial data). Then (2.3) is a noncharacteristic Cauchy problem in a neighborhood of zero, and the existence of an analytic solution u° in a neighborhood of $x_0 = 0$ follows from the Cauchy-Kowalevsky theorem.

From the Taylor formula and using (2.3), we get

$$\begin{aligned} \det(u_{i\bar{j}}^\circ) - g(x, u^\circ, \nabla u^\circ) &= \det(u_{i\bar{j}}^\circ) - \sum_{|\delta| < N} \frac{1}{\delta!} \partial_{(x, u, p)}^\delta g(0) (x, u^\circ, \nabla u^\circ)^\delta \\ &\quad - \left(g(x, u^\circ, \nabla u^\circ) - \sum_{|\delta| < N} \frac{1}{\delta!} \partial_{(x, u, p)}^\delta g(0) (x, u^\circ, \nabla u^\circ)^\delta \right) = \mathcal{O}(|x|^{N-1}). \end{aligned}$$

Then we can deduce for $|\alpha| < N$,

$$(2.4) \quad \partial_x^\alpha \left[\det(u_{i\bar{j}}^\circ) - g(x, u^\circ, \nabla u^\circ) \right] = \mathcal{O}(1), \quad x \rightarrow 0.$$

Step 2. One technical lemma.

Let us denote

$$(2.5) \quad F[u] = F(x, u, \nabla u, D^2 u) = \det(u_{i\bar{j}}).$$

Lemma 1. For $1 \leq i, j, a, b \leq n$, we have

$$(2.6) \quad F \frac{\partial^2 F}{\partial u_{a\bar{b}} \partial u_{i\bar{j}}} = \frac{\partial F}{\partial u_{a\bar{b}}} \frac{\partial F}{\partial u_{i\bar{j}}} - \frac{\partial F}{\partial u_{i\bar{b}}} \frac{\partial F}{\partial u_{a\bar{j}}}.$$

Proof. The proof of this lemma is similar to that given by C. Zuily in [13].

If we denote by H the matrix $(u_{i\bar{j}})_{1 \leq i, j \leq n}$, the matrix of cofactors is then $\tilde{H} = \left(\frac{\partial F}{\partial u_{i\bar{j}}} \right)$. By differentiating the identity

$${}^t\tilde{H}H = \det H \cdot Id = F \cdot Id$$

with respect to $u_{a\bar{b}}$, we get

$$({}^t\tilde{H})'H + {}^t\tilde{H}H' = (F \cdot Id)'.$$

Multiplying this equality on the right by ${}^t\tilde{H}$ and using that $H{}^t\tilde{H} = F \cdot Id$, we find

$$\underbrace{F({}^t\tilde{H})'}_{(1)} = \underbrace{(F \cdot Id)' {}^t\tilde{H}}_{(2)} - \underbrace{{}^t\tilde{H}H''\tilde{H}}_{(3)}.$$

The (i, j) term of the matrix (1) is given by

$$F \frac{\partial}{\partial u_{a\bar{b}}} \left(\frac{\partial F}{\partial u_{j\bar{i}}} \right) = F \frac{\partial^2 F}{\partial u_{j\bar{i}} \partial u_{a\bar{b}}}.$$

On the other hand, since $(F \cdot Id)'$ is the diagonal matrix $\frac{\partial F}{\partial u_{a\bar{b}}} \cdot Id$, the general term of the matrix (2) is given by $\frac{\partial F}{\partial u_{a\bar{b}}} \cdot \frac{\partial F}{\partial u_{j\bar{i}}}$.

The general term of the matrix H' is $\alpha_{rs} = \delta_r^a \cdot \delta_b^s$, where $\delta_j^i = \begin{cases} 1, & \text{for } i = j \\ 0, & \text{for } i \neq j \end{cases}$.

Then, the general term of $H'{}^t\tilde{H}$ is

$$\beta_{kl} = \sum_{s=1}^n \alpha_{ks} \frac{\partial F}{\partial u_{l\bar{s}}} = \frac{\partial F}{\partial u_{l\bar{b}}} \cdot \delta_k^a.$$

Finally, the (i, j) term of the matrix (3) is given by

$$\sum_{l=1}^n \frac{\partial F}{\partial u_{l\bar{i}}} \beta_{lj} = \frac{\partial F}{\partial u_{a\bar{i}}} \cdot \frac{\partial F}{\partial u_{j\bar{b}}}.$$

□

Step 3. The linearized equation of (1.1) at u° is of real principal type at the origin.

The principal symbol of this linearized equation is

$$(2.7) \quad p(x, \xi) = \sum_{1 \leq i, j \leq n} \frac{\partial F}{\partial u_{i\bar{j}}} [u^\circ] \xi_i \bar{\xi}_j.$$

When $\xi = (\xi', \xi'') = \eta + i\zeta$, $(\xi', \xi'') \in \mathbb{C}^{n-1} \times \mathbb{C}$, $(\eta, \zeta) \in \mathbb{R}^n \times \mathbb{R}^n$. $p(x, \xi)$ is a real symbol, and we have seen in Step 1 that

$$(2.8) \quad \frac{\partial F}{\partial u_{n\bar{n}}} (0, u^\circ(0), \nabla u^\circ(0), D^2 u^\circ(0)) = \det \left(u_{i\bar{j}}^\circ \right)_{1 \leq i, j \leq n-1} = 1.$$

Thus, if we denote the matrix $\left(u_{i\bar{j}}^\circ\right)_{1 \leq i, j \leq n-1}$ by A , the matrix of cofactors is then

$$\tilde{A} = \left(\frac{\partial^2 F}{\partial u_{n\bar{n}} \partial u_{i\bar{j}}} \right)_{1 \leq i, j \leq n-1}.$$

Using the identity $\tilde{A} = (\det A) A^{-1}$ and the initial condition $A(0) = Id$, we can deduce

$$(2.9) \quad \sum_{1 \leq i, j \leq n-1} \frac{\partial^2 F}{\partial u_{i\bar{j}} \partial u_{n\bar{n}}} (0, u^\circ(0), \nabla u^\circ(0), D^2 u^\circ(0)) \xi_i \bar{\xi}_j > 0, \quad \forall \xi' \neq 0.$$

Then using (2.6) we get
(2.7')

$$p(x, \xi) = \frac{\partial F}{\partial u_{n\bar{n}}} [u^\circ] \left\{ \left| \xi_n + \frac{\sum_{j=1}^{n-1} \frac{\partial F}{\partial u_{j\bar{n}}} \xi_j}{\frac{\partial F}{\partial u_{n\bar{n}}} [u^\circ]} \right|^2 + \frac{F[u^\circ]}{\left(\frac{\partial F}{\partial u_{n\bar{n}}} [u^\circ] \right)^2} \sum_{i,j=1}^{n-1} \frac{\partial^2 F}{\partial u_{n\bar{n}} \partial u_{i\bar{j}}} [u^\circ] \xi_i \bar{\xi}_j \right\}.$$

We can deduce from (2.9) that $p(x, \xi) = 0$ if and only if one of the following two conditions is satisfied:

$$(2.10) \quad \begin{cases} F[u^\circ](x) < 0, \\ \left| \xi_n + \frac{\sum_{j=1}^{n-1} \frac{\partial F}{\partial u_{j\bar{n}}} \xi_j}{\frac{\partial F}{\partial u_{n\bar{n}}} [u^\circ]} \right|^2 = \frac{-F[u^\circ]}{\left(\frac{\partial F}{\partial u_{n\bar{n}}} [u^\circ] \right)^2} \sum_{i,j=1}^{n-1} \frac{\partial^2 F}{\partial u_{n\bar{n}} \partial u_{i\bar{j}}} [u^\circ] \xi_i \bar{\xi}_j \end{cases}$$

or

$$(2.11) \quad F[u^\circ](x) = 0 \text{ and } \xi_n = - \left(\frac{\partial F}{\partial u_{n\bar{n}}} [u^\circ] \right)^{-1} \sum_{j=1}^{n-1} \frac{\partial F}{\partial u_{j\bar{n}}} [u^\circ] \xi_j.$$

* First case. When (2.10) holds, then we have $\frac{\partial p}{\partial \xi_n}(x, \xi) \neq 0$.

* Second case. Consider now the case when (2.11) holds.

For $1 \leq k \leq n$, using (2.11), we get by differentiating (2.7') with respect to x_k

$$\frac{\partial F}{\partial u_{n\bar{n}}} [u^\circ](0) \frac{\partial p}{\partial x_k}(0, \xi) = \frac{\partial}{\partial x_k} \left[\det \left(u_{i\bar{j}}^\circ \right) \right]_{|x=0} \sum_{i,j=1}^{n-1} \frac{\partial^2 F}{\partial u_{n\bar{n}} \partial u_{i\bar{j}}} [u^\circ](0) \xi_i \bar{\xi}_j.$$

From (2.9), $q(0, \xi) = \sum_{1 \leq i, j \leq n-1} \frac{\partial^2 F}{\partial u_{n\bar{n}} \partial u_{i\bar{j}}} [u^\circ] \xi_i \bar{\xi}_j > 0$.

Using (2.4) and the definition of $g(x, u, \nabla u)$, we have

$$\det \left(u_{i\bar{j}}^\circ \right) = K(z) f(z, u^\circ, \nabla u^\circ) + \mathcal{O}(|z|^2), \quad z \rightarrow 0,$$

where $K(z) = x_n + \mathcal{O}(|z|^2)$, $f(z, u^\circ, \nabla u^\circ) > 0$ and $K(0) = 0$. Then

$$(2.12) \quad \begin{cases} \frac{\partial F}{\partial u_{n\bar{n}}} [u^\circ](0) \frac{\partial p}{\partial x_k}(0, \xi) = 0, \quad \forall k \in [1, 2n] \setminus \{n\} \cap \mathbb{N}, \\ \frac{\partial F}{\partial u_{n\bar{n}}} [u^\circ](0) \frac{\partial p}{\partial x_n}(0, \xi) = f(0, u^\circ(0), \nabla u^\circ(0)) q(0, \xi). \end{cases}$$

Consider the bicharacteristic $(x(t), \eta(t), \zeta(t))$ descended from the point $(0, \xi_0)$. We have $x(0) = 0$, $\xi(0) = \xi_0$ and $\dot{x}(0) = 0$. Using (2.12), we get

$$\begin{cases} \dot{\zeta}_k(0) = 0, \quad \forall 1 \leq k \leq n, \\ \dot{\eta}_k(0) = 0, \quad \forall 1 \leq k \leq n-1, \\ \dot{\eta}_n(0) = - \left[\frac{\partial F}{\partial u_{n\bar{n}}} [u^\circ](0) \right]^{-1} f(0) q(0, \xi_0). \end{cases}$$

We can deduce that

$$\begin{aligned} \ddot{x}_n(0) &= \frac{d}{dt} \frac{\partial p}{\partial \eta_n} (x(t), \eta(t), \zeta(t))|_{t=0} \\ &= \dot{\eta}_n(0) \frac{\partial^2 p}{\partial \eta_n^2} (0, \xi_0) \\ &= 2\dot{\eta}_n(0) \frac{\partial F}{\partial u_{n\bar{n}}} [u^\circ](0). \end{aligned}$$

Finally

$$\ddot{x}_n(0) = -2f(0) q(0, \xi_0) < 0 \quad \text{when } \xi_0 \neq 0.$$

Thus, p is a real principal type symbol at 0.

ACKNOWLEDGEMENT

This paper is the result of conversations the author had with Professor Claude Zuily.

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