PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 131, Number 5, Pages 1453–1457
S 0002-9939(02)06675-3
Article electronically published on September 19, 2002

DIRECTIONAL CONVEXITY OF LEVEL LINES FOR FUNCTIONS CONVEX IN A GIVEN DIRECTION

DMITRI V. PROKHOROV AND JAN SZYNAL

(Communicated by Juha M. Heinonen)

ABSTRACT. Let $K(\varphi)$ be the class of functions $f(z) = z + a_2z^2 + \dots$ which are holomorphic and convex in direction $e^{i\varphi}$ in the unit disk D, i.e. the domain $f(D)$ is such that the intersection of $f(D)$ and any straight line $\{w : w = w_0 + te^{i\varphi}, t \in \mathbb{R}\}$ is a connected or empty set. In this note we determine the radius $r_{\psi,\varphi}$ of the biggest disk $|z| \leq r_{\psi,\varphi}$ with the property that each function $f \in K(\psi)$ maps this disk onto the convex domain in the direction $e^{i\varphi}$.

1.

The class of holomorphic functions in the unit disk $D = \{z : |z| < 1\}$ which are convex in some direction plays an important role in the extremal problems of univalent holomorphic and harmonic functions (e.g. Goodman [2], Clunie and Sheil-Small [1], Ruscheweyh and Salinas [7]).

The class of holomorphic functions

$$f(z) = z + a_2z^2 + \dots, \quad z \in D, \tag{1}$$

which map D onto a domain $f(D)$ such that the intersection of $f(D)$ with any straight line $\{w : w = w_0 + te^{i\varphi}, t \in \mathbb{R}\}$ is a connected or empty set for all $w_0 \in \mathbb{C}$, is called the class of convex functions in direction $e^{i\varphi}$, $\varphi \in \mathbb{R}$, and is denoted by $K(\varphi)$. In particular, $K(\pi/2) = K(-\pi/2) = CIA$ is known as the class of convex functions in the direction of the imaginary axis.

Many properties of this class have been studied and different normalizations were considered (Goodman [2] and Hengartner and Schober [4], [5]). In the problem under consideration we can restrict ourselves to the classical normalization (1). Of course, the class $K(\varphi)$ consists only of univalent functions [2, p. 196].

It was observed by Hengartner and Schober [5] that $f(z) \in CIA$ does not imply $f(rz)/r \in CIA$ for all $r \in (0, 1)$, contrary to the other classes of univalent functions like convex, starlike and close-to-convex. It was shown by Prokhorov [6] and by a completely different method by Ruscheweyh and Salinas [7] that only the circles $\{z : |z| = r \leq \sqrt{2} - 1\}$ are mapped by $f(z) \in CIA$ onto the domains convex in

Received by the editors August 21, 2001 and, in revised form, December 7, 2001.

2000 *Mathematics Subject Classification.* Primary 30C20; Secondary 30C45.

Key words and phrases. Level curves of holomorphic functions, functions convex in one direction.

The first author was partially supported by the RFBR Grant No. 01-01-00123 and the INTAS Grant No. 99-00089.

©2002 American Mathematical Society

the direction of the imaginary axis, and this result is sharp (see also Goodman and Saff [3]).

During the Conference on Computational Methods and Function Theory, Aveiro, 2001, Suffridge posed the problem of finding the radius of the biggest disk $|z| \le r_{\psi,\varphi}$ with the property that each function $f \in K(\psi)$ maps this disk onto the convex domain in the direction φ. In this note we find the exact value of $r_{\psi,\varphi}$.

2.

In order to state our Theorem we observe that if $f(z) \in K(\psi)$, then $h(z) = e^{i\delta} f(e^{-i\delta} z) \in K(\psi+\delta)$ and therefore we can assume without loss of generality that $\psi = \pi/2$ and $\varphi \in [-\pi/2, \pi/2]$.

Denote $r_\varphi := r_{\pi/2,\varphi}$. We have

Theorem. *If $f(z) \in CIA$, then $f(r_\varphi z)/r_\varphi \in K(\varphi)$ for all $\varphi \in [-\pi/2, \pi/2]$ where*

$$r_\varphi = 2\cos\frac{\varphi}{2} - \sqrt{4\cos^2\frac{\varphi}{2} - 1}. \tag{2}$$

Proof. Let $f(z) \in CIA$. We have to prove that $f(r_\varphi z)/r_\varphi \in K(\varphi)$. Observe that if $\alpha = \pi/2 - \varphi$, then the function

$$F_\alpha(z) = e^{i\alpha} f(e^{-i\alpha} z) \tag{3}$$

possesses the property that $F_\alpha(r_\varphi z)/r_\varphi \in CIA$.

From the geometric properties of the convexity in the direction of the imaginary axis it follows that the function

$$u'(\theta) := \frac{\partial}{\partial\theta} \Re[F_\alpha(r_\varphi e^{i\theta})] = -\Im[r_\varphi e^{i\theta} F'_\alpha(r_\varphi e^{i\theta})] \tag{4}$$

has to be non-negative and non-positive respectively for θ corresponding to two complementary arcs of the unit circle $z = e^{i\theta}$ [2, p. 195]. This is equivalent to the fact that the function

$$v(\theta) := \arg[r_\varphi e^{i\theta} F'_\alpha(r_\varphi e^{i\theta})] \tag{5}$$

attains its extremal values on some intervals $\Delta_1 = [\theta_1, \theta_2]$ and $\Delta_2 = [\theta_2, \theta_1 + 2\pi]$, $\theta_1 < \theta_2 < \theta_1 + 2\pi$, only at the end points.

Suppose r_φ is the critical radius for $f(z)$, therefore there exists a critical point θ_0 such that

$$u'(\theta_0) = u''(\theta_0) = 0,$$

which can be expressed under the denotation $z_0 = r_\varphi e^{i\theta_0}$ as

$$\Im[z_0 F'_\alpha(z_0)] = 0 \tag{6}$$

and

$$\Re\left[1 + \frac{z_0 F''_\alpha(z_0)}{F'_\alpha(z_0)}\right] = 0. \tag{7}$$

According to (6) $e^{i\theta_0} F'_\alpha(z_0)$ is real. Hence the function

$$g(z) = \frac{e^{-i\theta_0}\left[F_\alpha\left(\frac{e^{i\theta_0} z + z_0}{1 + \overline{z}_0 e^{i\theta_0} z}\right) - F_\alpha(z_0)\right]}{F'_\alpha(z_0)(1 - r_\varphi^2)} = z + b_2 z^2 + \dots \tag{8}$$

belongs to $K(\pi - \varphi)$ together with $F_\alpha(z)$.

Substituting (8) into (6) and (7) we obtain

$$\Im[e^{-i\theta_0}g'(-r_\varphi)] = 0 \tag{9}$$

and

$$\Re[1 + r_\varphi^2 + 2r_\varphi b_2] = 0. \tag{10}$$

The function

$$G_{-\alpha}(z) = e^{-i\alpha}g(e^{i\alpha}z) = z + c_2z^2 + \dots \tag{11}$$

belongs to CIA and $c_2 = e^{i\alpha}b_2$. Equations (9) and (10) are now equivalent to

$$\Im[e^{-i\theta_0}G'_{-\alpha}(-e^{-i\alpha}r_\varphi)] = 0 \tag{12}$$

and

$$\Re[1 + r_\varphi^2 + 2e^{-i\alpha}r_\varphi c_2] = 0. \tag{13}$$

To estimate $\Re[e^{-i\alpha}c_2]$ for $G_{-\alpha}(z) \in CIA$ consider the integral representation [2]

$$G_{-\alpha}(z) = \int_0^z \frac{p(t)dt}{(1-\overline{z}_1t)(1-\overline{z}_2t)}, \tag{14}$$

where $p(z)$, $p(0) = 1$, is holomorphic in D and satisfies $\Re[e^{-i\gamma}p(z)] > 0$, $z \in D$, for a certain $\gamma \in (-\pi/2, \pi/2)$. Points z_1 and z_2, $|z_1| = |z_2| = 1$, are such that the function

$$w(z) = \frac{z}{(1-\overline{z}_1z)(1-\overline{z}_2z)}$$

maps D onto the complex plane minus a continuum on the straight line which passes through the origin and has the slope $(\pi/2 - \gamma)$. This means that $z_1 = e^{i\theta_1}$ and $z_2 = e^{i\theta_2}$ where $(\theta_1 + \theta_2)/2 = -(\pi/2 + \gamma)$. From (14) we obtain ($\alpha = \frac{\pi}{2} - \varphi$):

$$\begin{aligned}\Re[e^{-i\alpha}c_2] &= \frac{1}{2}\Re\{e^{-i\alpha}[p'(0) + \overline{z}_1 + \overline{z}_2]\} \geq -\frac{|p'(0)| - \cos(\theta_1+\alpha) - \cos(\theta_2+\alpha)}{2} \\ &\geq -\cos\gamma - \left|\cos\left(\frac{\theta_1+\theta_2}{2} + \alpha\right)\right| = -\cos\gamma - |\cos(\varphi+\gamma)| \geq -2\cos\frac{\varphi}{2}.\end{aligned} \tag{15}$$

Inequality (15) and equation (13) imply the estimate

$$r_\varphi \geq 2\cos\frac{\varphi}{2} - \sqrt{4\cos^2\frac{\varphi}{2} - 1}. \tag{16}$$

Notice that all the inequalities in (15) are sharp. To show that estimate (16) cannot be improved we should determine a function $G_{-\alpha}(z) \in CIA$ for which the inequalities (15) become equalities. Going back to $g(z)$, $F_\alpha(z)$ and $f(z) \in CIA$ by (11), (8) and (3) we find the extremal function of the above problem.

We will give the explicit examples for $\varphi \in [0, \pi/2]$. Let $\beta = (\pi - \varphi)/2 = (\alpha + \pi/2)/2$ and

$$G_{-\alpha}(z) = \int_0^z \frac{1 + e^{i3\beta}t}{(1 + e^{i\beta}t)^3}dt. \tag{17}$$

This is a special case of the formula (14) with $p(z) = (1 + e^{i3\beta}z)/(1 + e^{i\beta}z)$ and $z_1 = z_2 = -e^{-i\beta}$. Therefore $G_{-\alpha}(z) \in CIA$.

From (11) and (17) we have

$$c_2 = \frac{1}{2}(e^{i3\beta} - 3e^{i\beta})$$

and therefore

$$\Re[e^{-i\alpha}c_2] = \frac{1}{2}(\cos(3\beta - \alpha) - 3\cos(\beta - \alpha)) = -2\cos\frac{\varphi}{2}. \tag{18}$$

Equations (2), (13) and (18) imply that if $r = r_\varphi + \epsilon$ for $\epsilon > 0$ small enough, then

$$\Re[1 + r^2 + 2e^{-i\alpha}rc_2] < 0. \tag{19}$$

Equation (13) does not depend on (12), and the critical value θ_0 can be found directly from (12).

Inequality (19) together with (12) mean that there are at least four different points $\theta \in [0, 2\pi]$ where the function

$$u_r'(\theta) := \frac{\partial}{\partial\theta}\Re[F_\alpha(re^{i\theta})] = -\Im[re^{i\theta}F_\alpha'(re^{i\theta})]$$

changes its sign. Therefore the function $F_\alpha(rz)/r$ does not belong to CIA. This completes the proof of the Theorem.

Remark. The radius r_φ increases on $[0, \pi/2]$ from $r_0 = 2 - \sqrt{3}$ which is the radius of convexity in the class S of holomorphic and univalent functions $f(z)$ of form (1) to $r_{\pi/2} = \sqrt{2} - 1$. One can observe that the classes $K(\varphi)$ are different for different $\varphi \in [0, 2\pi]$. Moreover, the class C of convex univalent functions $f(z)$ of the form (1) is the proper subclass of $K(\varphi)$ for any φ.

The next example illustrates the Remark.

Example. Let $B = B_1 \cup B_2$ be the union of two rectangles $B_1 = \{z = x + iy : -2 < x < 2,\ -1 < y < 1\}$ and $B_2 = \{z = x + iy :\ -1 < x < 1,\ -2 < y < 2\}$. The function $f(z) = d_1z + d_2z^2 + \dots$, $d_1 > 0$, mapping D onto B can be represented by the Schwarz-Christoffel integral. Then $f(z)/d_1$ belongs to CIA and $K(0)$ but does not belong to any $K(\varphi)$ for $\varphi \in (0, \pi/2) \cup (\pi/2, \pi)$.

Acknowledgement

The authors thank the referee for the information that the problem considered in this paper was solved numerically (but not in explicit form) by E. Nash, *Untersuchungen ueber DCP–Funktionen*, Diploma–Thesis, Wuerzburg University, 1995, who used the characterization of DCP–functions [7].

References

[1] J. Clunie and T. Sheil-Small, Harmonic univalent functions, Ann. Acad. Sci. Fenn. Ser. A.I Math. **9** (1984), 3-25. MR **85i:**30014

[2] A. W. Goodman, *Univalent functions*, Vol.1. Mariner Publ. Comp.: Tampa, 1983. MR **85j:**30035a

[3] A. W. Goodman and E.B. Saff, On univalent functions convex in one direction, Proc. Amer. Math. Soc. **73** (1979), 183-187. MR **80e:**30006

[4] W. Hengartner and G. Schober, On Schlicht mappings to domains convex in one direction, Comm. Math. Helv. **45** (1970), 303-314. MR **43:**3436

[5] W. Hengartner and G. Schober, A remark on level curves for domains convex in one direction, Appl. Anal. **3** (1973), 101-106. MR **52:**5244

[6] D. V. Prokhorov, Level lines of functions that are convex in the direction of an axis, Mat. Zametki **44** (1988), 523-527. MR **89m:**30030

[7] S. Ruscheweyh and L.C. Salinas, On the preservation of direction-convexity and the Goodman-Saff conjecture, Ann. Acad. Sci. Fenn. Ser. A.I Math. **14** (1989), 63-73. MR **90h:**30041

Department of Mathematics, Saratov State University, 410026 Saratov, Russia
E-mail address: ProkhorovDV@info.sgu.ru

Department of Mathematics, M. Curie-Skłodowska University, 20-031 Lublin, Poland
E-mail address: jsszynal@golem.umcs.lublin.pl