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TIGHT FRAME OVERSAMPLING AND ITS EQUIVALENCE TO SHIFT-INVARIANCE OF AFFINE FRAME OPERATORS

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Abstract. Let $\Psi = \{\psi_1, \dots, \psi_L\} \subset L^2 := L^2(-\infty, \infty)$ generate a tight affine frame with dilation factor M, where $2 \le M \in \mathbf{Z}$, and sampling constant $b = 1$ (for the zeroth scale level). Then for $1 \le N \in \mathbf{Z}$, $N\times$oversampling (or oversampling by N) means replacing the sampling constant 1 by $1/N$. The Second Oversampling Theorem asserts that $N\times$oversampling of the given tight affine frame generated by Ψ preserves a tight affine frame, provided that $N = N_0$ is relatively prime to M (i.e., $gcd(N_0, M) = 1$). In this paper, we discuss the preservation of tightness in $mN_0\times$oversampling, where $1 \le m|M$ (i.e., $1 \le m \le M$ and $gcd(m, M) = m$). We also show that tight affine frame preservation in $mN_0\times$oversampling is equivalent to the property of shift-invariance with respect to $\frac{1}{mN_0}\mathbf{Z}$ of the affine frame operator Q_{0,N_0} defined on the zeroth scale level.

1. Introduction and results

A family $\Psi = \{\psi_1, \dots, \psi_L\} \subset L^2 := L^2(-\infty, \infty)$ is said to generate a *tight affine frame*

$$\mathcal{F}_1 = \{\psi_{l;j,k}(x) := M^{j/2}\psi_l(M^j x - k) : \ j, k \in \mathbf{Z}, \ l = 1, \dots, L\} \tag{1}$$

of L^2 with dilation factor M where $2 \le M \in \mathbf{Z}$ (or for simplicity, we say that Ψ is a tight affine frame of L^2), if there exists a positive constant A, called *frame (bound) constant*, such that

$$\sum_{l=1}^{L} \sum_{j,k\in\mathbf{Z}} |\langle f, \psi_{l;j,k}\rangle|^2 = A\|f\|_2^2 \quad \text{for all } f \in L^2. \tag{2}$$

Here, the standard notation for L^2-inner product and L^2-norm is used. In addition, the definition

$$\hat{f}(\omega) := \int_{-\infty}^{\infty} f(x)e^{-ix\omega}dx, \ f \in L^1,$$

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of the Fourier transform will be used throughout this paper. Also, for any $\Psi = \{\psi_1, \dots, \psi_L\} \subset L^2$ that satisfies the property

$$H(\omega) := \sum_{l=1}^{L} \sum_{k\in\mathbf{Z}} |\widehat{\psi}_l(\cdot + 2k\pi)|^2 \in L^\infty(-\infty, \infty), \tag{3}$$

we consider the affine frame operators $Q_{0,n}, 1 \le n \in \mathbf{Z}$, defined by

$$(Q_{0,n}f)(x) := \sum_{l=1}^{L} \sum_{k\in\mathbf{Z}} \langle f, \psi_l(\cdot - k/n)\rangle \psi_l(x - k/n), \quad f \in L^2, \tag{4}$$

on the zeroth scale level, and denote $Q_0 := Q_{0,1}$. Clearly, we have

$$\|Q_{0,n}f\|_2 \le n\|H\|_\infty \|f\|_2. \tag{5}$$

The reason for the terminology of "affine frame operators" is that, by introducing the dilation operator

$$D_j f := f(M^j \cdot), \quad j \in \mathbf{Z}, \tag{6}$$

$\Psi \subset L^2$ is a tight affine frame of L^2 (in the sense that the family $\mathcal{F}_1$ in (1) is a tight frame of L^2), if and only if both (3) and, for some constant $A > 0$,

$$\sum_{j\in\mathbf{Z}} D_{-j} Q_0 D_j = AI \tag{7}$$

are satisfied.

For $M = 2$, a complete characterization of tight affine frames (more precisely, orthonormal wavelets) is discussed in [12], and a generalization from $M = 2$ to arbitrary real dilation $a > 1$ is given in [10]. Generalizations to matrix dilation have been studied in [3, 4, 5] for matrices with integer entries, and most recently in [6] for arbitrary real matrices. Of course, all the eigenvalues of the dilation matrices must have magnitudes greater than one. For the univariate setting with dilation factor $a > 1$, where $a^\gamma =: n_a \in \mathbf{Z}$ for some $1 \le \gamma \in \mathbf{Z}$, and γ being the smallest such integer exponent, the full characterization in [10] reduces to the following.

Theorem A. *Let $\Psi = \{\psi_1, \dots, \psi_L\} \subset L^2$, $b \ne 0$ and $a > 1$ with*

$$\begin{cases} a^\gamma =: n_a \in \mathbf{Z}; \\ a^j \notin \mathbf{Z} \quad \text{for } \gamma > j \in \mathbf{Z}\backslash\{0\}. \end{cases} \tag{8}$$

Then $\{a^{j/2}\psi_l(a^j \cdot - kb) : \ j, k \in \mathbf{Z}, \ l = 1, \dots, L\}$ is a tight frame of L^2 with frame constant A, if and only if

$$\sum_{l=1}^{L} \sum_{j\in\mathbf{Z}} |\widehat{\psi}_l(a^j\omega)|^2 = A \quad a.e.\ \omega \in \mathbf{R} \tag{9}$$

and

$$\sum_{l=1}^{L} \sum_{j=0}^{\infty} \overline{\widehat{\psi}_l(n_a^j\omega)} \widehat{\psi}_l(n_a^j(\omega + 2b^{-1}d\pi)) = 0 \quad a.e.\ \omega \in \mathbf{R} \tag{10}$$

for any $d \in \mathbf{Z}\backslash n_a\mathbf{Z}$.

Returning to the special case $a = M \in \mathbf{Z}$ and $b = 1$, let us also recall the following Second Oversampling Theorem established in [7].

Theorem B. *Let $2 \le M \in \mathbf{Z}$ and assume that $\Psi = \{\psi_1, \dots, \psi_L\} \subset L^2$ generates a tight affine frame $\mathcal{F}_1$ in (1) of L^2 with frame constant $A > 0$. Then for any $2 \le m \in \mathbf{Z}$ with $gcd(m, M) = 1$,*

$$\mathcal{F}_m = \{M^{j/2}\psi_l(M^j x - k/m) : \ j,k \in \mathbf{Z}, \ l = 1, \dots, L\} \tag{11}$$

is also a tight frame of L^2, with frame constant mA.

The notion of oversampling affine frames was first introduced in [8], where the result for $M = 2$ was obtained, although [8] has a later publication date than [7], which deals with the theory of affine frames in general. Generalizations to matrix dilation was studied in [9] and, in full generality, in [6]. In addition, oversampling by M^k was discussed in [11] for dilation $M = 2$ for tight affine frames, and arbitrary $2 \le M \in \mathbf{Z}$ in [13, Chapter 5] for bi-orthogonal wavelets, where $0 < k \in \mathbf{Z}$. A generalization of [11] to matrix dilation was also mentioned in [6]. Observe, however, that since the assumption $gcd(m, M) = 1$ is violated for $m = M^k, 0 < k \in \mathbf{Z}$, it is necessary to derive additional characterization equations, besides those in (10) for $1 < a = n_a = M \in \mathbf{Z}$ and $b = 1$. For instance, by reducing the matrix consideration in [6] to the scalar setting, a necessary and sufficient condition for tight affine frame preservation in $M^{n_0}\times$oversampling, with $0 < n_0 \in \mathbf{Z}$, is that

$$\sum_{l=1}^{L}\sum_{j=0}^{n_0-1} \overline{\widehat{\psi}_l(M^j\omega)}\widehat{\psi}_l(M^j(\omega + 2d\pi)) = 0 \quad \text{a.e. } \omega \in \mathbf{R} \tag{12}$$

for all $d \in \mathbf{Z}\backslash M\mathbf{Z}$.

The objective of this paper is to establish necessary and sufficient conditions for tight affine frame preservation in oversampling by mN_0, where $gcd(N_0, M) = 1$, $1 \le m \le M$, and $gcd(m, M) = m$. One of the equivalent conditions, stated in Theorem 2, is that the affine frame operator Q_{0,N_0} in (4), with $n = N_0$, is shift-invariant with respect to $\frac{1}{mN_0}\mathbf{Z}$. To facilitate the statement of our results, we need the notation of the shift operator

$$\tau_y : f \longmapsto f(\cdot - y), \ y \in \mathbf{R}. \tag{13}$$

Theorem 1. *Let $2 \le M \in \mathbf{Z}$ and assume that $\Psi = \{\psi_1, \dots, \psi_L\} \subset L^2$ generates a tight affine frame $\mathcal{F}_1$ in (1) of L^2 with frame constant $A > 0$. Consider $2 \le m \le M$ with $gcd(m, M) = m$. Then the following statements are equivalent:*

(i) *$\mathcal{F}_m$ in (11) is a tight frame of L^2;*

(ii) *$\sum_{l=1}^{L} \overline{\widehat{\psi}_l(\omega)}\widehat{\psi}_l(\omega + 2\pi d) = 0$ a.e. $\omega \in \mathbf{R}$, for all $d \in \mathbf{Z}\backslash m\mathbf{Z}$;*

(iii) *Q_0 is shift-invariant with respect to $\frac{1}{m}\mathbf{Z}$, i.e.,*

$$\tau_{k/m}Q_0 = Q_0\tau_{k/m} \quad \text{for all } k \in \mathbf{Z}; \tag{14}$$

(iv) *$Q_{0,m} = mQ_0$;*

(v) *there exists some 2π-periodic unitary matrix $A(\omega)$ of dimension L such that*

$$e^{-i\omega/m}\widehat{\Phi}(\omega) = A(\omega)\widehat{\Phi}(\omega) \quad a.e.\ \omega \in \mathbf{R}, \tag{15}$$

where $\Phi = [\psi_1 \ \dots \ \psi_L]^T$ is the column vector of the functions in Ψ.

The 2π-periodic matrix $A(\omega)$ in (15) of Theorem 1 acts like a bases transform on the shift-invariant space generated by $\psi_l(\cdot - k), l = 1, \dots, L, k \in \mathbf{Z}$. Such an invertible transform of frames was discussed in [1, 13]. In particular, the implication of (v)$\Longrightarrow$(i) was given in [13, Theorem 6.1].

In addition, the equivalence of (ii) and (v) was proved in [14] for the special case where $L = 1$, $M = 2$, and that $\mathcal{F}_1$ is an orthonormal basis of L^2. But the proof in [14] does not seem to have a simple generalization to the study of frames for $L \ge 1$ and $M \ge 2$. In this regard, our proof of (ii)$\Longrightarrow$(v) is fairly technical.

From (iii) of Theorem 1, we see that the range $W_0 := Q_0 L^2$ of the operator Q_0 is shift-invariant over $\frac{1}{m}\mathbf{Z}$. If $\mathcal{F}_1$ in (1) is an orthonormal basis of L^2, then as in [14], W_0 is generated by ℓ^2 linear combinations of integer shifts of $\psi_1, \dots, \psi_L$; namely,

$$W_0 = \left\{ \sum_{l=1}^{L} \sum_{j\in\mathbf{Z}} d_l(j)\psi_l(\cdot - j) : \ \{d_l(j)\}_{j\in\mathbf{Z}} \in \ell^2 \right\},$$

and thus the affine frame operator Q_0 on the zeroth scale level is the projection operator on W_0, i.e., $Q_0^2 = Q_0$. The interested reader is referred to [2] for a study of closedness of the space generated by ℓ^p linear combinations, $1 \le p \le \infty$, of integer shifts in general.

For functions $\Psi = \{\psi_1, \dots, \psi_L\} \subset L^2$ and $b \ne 0$, it is easy to check that $\mathcal{F}_{b^{-1}} = \{M^{j/2}\psi_l(M^j \cdot -kb) : j,k \in \mathbf{Z}, l = 1, \dots, L\}$ is a tight frame of L^2 if and only if $\{M^{j/2}\psi_{l,b}(M^j \cdot -k) : j,k \in \mathbf{Z}, l = 1, \dots, L\}$ is a tight frame of L^2, where $\psi_{l,b} = b^{1/2}\psi_l(b\cdot)$. Let N_0 be a positive integer with $gcd(N_0, M) = 1$. Then for any tight affine frame $\Psi = \{\psi_1, \dots, \psi_L\}$ of L^2, it follows from Theorem B that $\{M^{j/2}\psi_{l,N_0^{-1}}(M^j \cdot -k) : j,k \in \mathbf{Z}, l = 1, \dots, L\}$ is also a tight frame of L^2. Therefore this, together with Theorem 1, give the following extension of Theorem 1.

Theorem 2. *Let $2 \le M \in \mathbf{Z}$ and assume that $\Psi = \{\psi_1, \dots, \psi_L\} \subset L^2$ generates a tight affine frame $\mathcal{F}_1$ in* (1) *of L^2 with frame constant $A > 0$. Consider $1 \le N_0 \in \mathbf{Z}$ with $gcd(N_0, M) = 1$, and $2 \le m \le M$ with $gcd(m, M) = m$. Then the following statements are equivalent:*

(i) $\mathcal{F}_{mN_0}$ *in* (11) *is a tight frame of L^2;*

(ii) $\sum_{l=1}^{L} \overline{\widehat{\psi}_l(\omega)}\widehat{\psi}_l(\omega + 2dN_0\pi) = 0$ *a.e.* $\omega \in \mathbf{R}$, *for all* $d \in \mathbf{Z}\backslash m\mathbf{Z}$*;*

(iii) *the operator Q_{0,N_0} in* (4)*, with $n = N_0$, is shift-invariant with respect to $(mN_0)^{-1}\mathbf{Z}$, i.e.,*

$$\tau_{k/(mN_0)} Q_{0,N_0} = Q_{0,N_0}\tau_{k/(mN_0)} \quad \text{for all } k \in \mathbf{Z}; \tag{16}$$

(iv) $Q_{0,mN_0} = mQ_{0,N_0}$*;*

(v) *there exists some 2π-periodic unitary matrix $A(\omega)$ of dimension L such that*

$$e^{-i\omega/(mN_0)}\widehat{\Phi}(\omega) = A(\omega/N_0)\widehat{\Phi}(\omega) \quad a.e.\ \omega \in \mathbf{R}, \tag{17}$$

where $\Phi = [\psi_1 \ \dots \ \psi_L]^T$ is the column vector of the functions in Ψ.

By applying (7) and (iii) of Theorem 2, we also have the following result for tightness of shifted affine frames.

Corollary 1. *Let M, m and N_0 be as in Theorem 2. If both $\mathcal{F}_1$ and $\mathcal{F}_{mN_0}$ are tight frames of L^2, then for any integer s, $\{M^{j/2}\psi_l(M^j \cdot -k/N_0 - s/(mN_0)) : j,k \in \mathbf{Z}, l = 1, \dots, L\}$ is also a tight frame of L^2.*

Note that for the case where $b^{-1} \in \mathbf{Z}$ is a factor of n_a, condition (10) is equivalent to

$$\begin{aligned}\sum_{l=1}^{L}\sum_{j=1}^{\infty}\overline{\widehat{\psi}_l(n_a^j\omega)}\widehat{\psi}_l(n_a^j(\omega+2\pi d))\\ +\sum_{l=1}^{L}\overline{\widehat{\psi}_l(\omega)}\widehat{\psi}_l(\omega+2\pi d)\delta_{d,b^{-1}\mathbf{Z}}=0 \quad \text{a.e. } \omega\in\mathbf{R}\end{aligned} \tag{18}$$

for all $d \in \mathbf{Z}\backslash n_a\mathbf{Z}$, where we set $\delta_{d,b^{-1}\mathbf{Z}} = 1$ for $d \in b^{-1}\mathbf{Z}$ and $\delta_{d,b^{-1}\mathbf{Z}} = 0$ otherwise. Then by (18), Theorem A, and Theorem 1, we have the following result about preservation of tightness for oversampling affine frames with dilation a that satisfies (8) and with m being a factor of n_a.

Theorem 3. *Let $a > 1$ satisfy (8), $m|n_a$, and assume that $\{a^{j/2}\psi_l(a^j\cdot -k) :\ j,k \in \mathbf{Z},\ l = 1,\dots,L\}$ is a tight frame of L^2. Set $\Phi = [\psi_1\ \dots\ \psi_L]^T$. Then the following statements are equivalent:*

(i) *$\{a^{j/2}\psi_l(a^j\cdot -k/m) :\ j,k \in \mathbf{Z}, l = 1,\dots,L\}$ is a tight frame of L^2;*
(ii)

$$\sum_{l=1}^{L}\overline{\widehat{\psi}_l(\omega)}\widehat{\psi}_l(\omega+2\pi d)=0 \quad a.e.\ \omega\in\mathbf{R},$$

for all $d \in \mathbf{Z}\backslash m\mathbf{Z}$;
(iii) *Q_0 is shift-invariant with respect to $\frac{1}{m}\mathbf{Z}$;*
(iv) *$Q_{0,m} = mQ_0$;*
(v) *there exists some 2π-periodic unitary matrix $A(\omega)$ of dimension L such that*

$$e^{-i\omega/m}\widehat{\Phi}(\omega) = A(\omega)\widehat{\Phi}(\omega) \quad a.e.\ \omega\in\mathbf{R}.$$

For the case $b^{-1} = n_a^{\gamma}m \in \mathbf{Z}$ with $0 \le \gamma \in \mathbf{Z}$ and $m|n_a$, it is easy to check that condition (10) is equivalent to

$$\begin{aligned}\sum_{l=1}^{L}\sum_{j=\gamma+1}^{\infty}\overline{\widehat{\psi}_l(n_a^j\omega)}\widehat{\psi}_l(n_a^j(\omega+2\pi d))\\ +\sum_{l=1}^{L}\overline{\widehat{\psi}_l(n_a^{\gamma}\omega)}\widehat{\psi}_l(n_a^{\gamma}(\omega+2\pi d))\delta_{d,m\mathbf{Z}}=0 \quad \text{a.e.}\end{aligned} \tag{19}$$

for all $d \in \mathbf{Z}\backslash n_a\mathbf{Z}$. This together with (7) give the following extension of the equivalence of (i) and (iv) in Theorem 1.

Theorem 4. *Let $2 \le M \in \mathbf{Z}, 0 \le \gamma \in \mathbf{Z}$, $2 \le m \le M$ with $\gcd(m,M) = m$, and assume that $\Psi = \{\psi_1,\dots,\psi_L\} \subset L^2$ generates a tight affine frame $\mathcal{F}_1$ in (1) of L^2. Then $\mathcal{F}_{M^{\gamma}m}$ in (11) is a tight frame of L^2 if and only if*

$$\begin{aligned}Q_{0,M^{\gamma}m} &= M^{\gamma}mD_{\gamma}Q_0D_{-\gamma}\\ &\quad +m\sum_{j=0}^{\gamma-1}M^j(D_jQ_{0,M^{\gamma-j}}D_{-j}-D_{j+1}Q_{0,M^{\gamma-j}}D_{-j-1}).\end{aligned} \tag{20}$$

2. Preliminary results

Given a measurable set E, we let $M(E,\ell_0)$ be the space of all sequences $\{\alpha_n(\omega)\}_{n=-\infty}^{\infty}$ of measurable functions on E, such that for almost all $\omega\in E$, $\alpha_n(\omega)=0$ for all but finitely many $n\in\mathbf{Z}$, i.e.,

$$\bigcup_{N\ge 1}\Big(\bigcap_{|n|\ge N}\{\omega\in E:\ \alpha_n(\omega)=0\}\Big)=E.$$

Here, we say that two measurable sets A and B are equal, denoted by $A=B$, if both $A\backslash B$ and $B\backslash A$ have zero Lebesgue measure. For a sequence $X=\{x_n(\omega)\in\mathbf{C}^L\}_{n=-\infty}^{\infty}$ of vector-valued measurable functions on a measurable set E, let

$$S(E,X):=\Big\{\sum_{n=-\infty}^{\infty}\alpha_n(\omega)x_n(\omega):\ \{\alpha_n(\omega)\}_{n=-\infty}^{\infty}\in M(E,\ell_0)\Big\}.$$

We remark that $S(E,X)$ is well defined since the summation for n in the definition is taken over a finite set for almost all $\omega\in E$.

For the proof of Theorem 1, we need the following technical lemma.

Lemma 1. *Let $1\le L\in\mathbf{Z}$, let E be a measurable set, and let $X=\{x_n(\omega)\}_{n=-\infty}^{\infty}$ be a sequence of vector-valued measurable functions $x_n(\omega)\in\mathbf{C}^L, n\in\mathbf{Z}$, on E. Then there exists an L-dimensional square matrix $P(\omega)$ of measurable functions on E such that*

$$P(\omega)v\in S(E,X)\quad \textit{for all } v\in\mathbf{C}^L, \tag{21}$$

$$\overline{P(\omega)^T}=P(\omega),\ P(\omega)^2=P(\omega)\quad \textit{a.e. } \omega\in E, \tag{22}$$

and

$$x_n(\omega)=P(\omega)x_n(\omega)\quad \textit{a.e. } \omega\in E\quad \textit{for all}\ \ n\in\mathbf{Z}. \tag{23}$$

In order to prove Lemma 1, we need the following result.

Lemma 2. *Let $Y=\{y_n(\omega)\}_{n=-\infty}^{\infty}$ and $Z=\{z_n(\omega)\}_{n=-\infty}^{\infty}$ be sequences of vector-valued measurable functions on a measurable set E. If $z_n(\omega)\in S(E,Y)$ for all $n\in\mathbf{Z}$, then $S(E,Z)\subset S(E,Y)$.*

Proof. Write

$$z_n(\omega)=\sum_{k=-\infty}^{\infty}\beta_{n,k}(\omega)y_k(\omega)$$

with $\{\beta_{n,k}(\omega)\}_{k=-\infty}^{\infty}\in M(E,\ell_0)$ for all $n\in\mathbf{Z}$. Then for any sequence $\{\alpha_n(\omega)\}_{n=-\infty}^{\infty}$ of measurable functions in $M(E,\ell_0)$, we have

$$\sum_{n=-\infty}^{\infty}\alpha_n(\omega)z_n(\omega)=\sum_{k=-\infty}^{\infty}\gamma_k(\omega)y_k(\omega)$$

where $\gamma_k(\omega)=\sum_{n=-\infty}^{\infty}\alpha_n(\omega)\beta_{n,k}(\omega)$. Therefore, it suffices to show that

$$\{\gamma_k(\omega)\}_{k=-\infty}^{\infty}\in M(E,\ell_0). \tag{24}$$

For any $0<C\in\mathbf{R}, n\in\mathbf{Z}$, and $1\le N,K\in\mathbf{Z}$, set

$$E_{C,K}:=\{\omega\in E:\ |\omega|\le C\quad\text{and}\quad \gamma_k(\omega)=0\quad\text{for all } |k|\ge K\},$$

$$F_{C,N}:=\{\omega\in E:\ |\omega|\le C\quad\text{and}\quad \alpha_n(\omega)=0\quad\text{for all } |n|\ge N\},$$

and

$$G_{C,n,K} := \{\omega \in E : \ |\omega| \le C \quad \text{and} \quad \beta_{n,k}(\omega) = 0 \quad \text{for all } |k| \ge K\}.$$

It is easy to see that for any $1 \le K, N \in \mathbf{Z}$ and $0 < C \in \mathbf{R}$,

$$F_{C,N} \cap \Big(\bigcap_{|n| \le N-1} G_{C,n,K} \Big) \subset E_{C,K}. \tag{25}$$

By the definition of $M(E, \ell_0)$, the proof of (24) reduces to the existence of an integer $1 \le K_1 = K_1(\epsilon, C)$ for any pre-assigned positive constants ϵ and C, such that

$$\text{meas}((E \cap \{\omega : |\omega| \le C\}) \backslash E_{C,K_1}) < \epsilon, \tag{26}$$

where $\text{meas}(A)$ denotes the Lebesgue measure of a measurable set A. Since

$$\{\alpha_n(\omega)\}_{n=-\infty}^{\infty} \in M(E, \ell_0),$$

there exists an integer $N_1 \ge 1$ such that

$$\text{meas}((E \cap \{\omega : |\omega| \le C\}) \backslash F_{C,N_1}) < \epsilon/3. \tag{27}$$

From the assumption $\{\beta_{n,k}(\omega)\}_{k=-\infty}^{\infty} \in M(E, \ell_0)$ for any $n \in \mathbf{Z}$, we may find an integer $K_1 \ge 1$ so that

$$\text{meas}((E \cap \{\omega : |\omega| \le C\}) \backslash G_{C,n,K_1}) < \epsilon/(3N_1) \tag{28}$$

for all $n \in \mathbf{Z}$ with $|n| \le N_1 - 1$. Hence, (26) follows from (25), (27), and (28). $\square$

Proof of Lemma 1. Set $x_{0,n}(\omega) = x_n(\omega), n \in \mathbf{Z}$, and $X_0 = X$, and define inductively, $e_l(\omega)$ and $X_l = \{x_{l,n}(\omega)\}_{n=-\infty}^{\infty}, 1 \le l \le L$, by

$$e_l(\omega) = \begin{cases} x_{l-1,I(X_{l-1})(\omega)}(\omega)/|x_{l-1,I(X_{l-1})(\omega)}(\omega)| & \text{if} \quad \omega \in E \backslash E_{l-1}, \\ 0 & \text{if} \quad \omega \in E_{l-1}, \end{cases} \tag{29}$$

and

$$x_{l,n}(\omega) = x_{l-1,n}(\omega) - e_l(\omega)\overline{e_l(\omega)}^T x_{l-1,n}(\omega), \ n \in \mathbf{Z}, \tag{30}$$

where

$$E_{l-1} = \{\omega \in E : \ x_{l-1,n}(\omega) = 0 \quad \text{for all } n \in \mathbf{Z}\}, \tag{31}$$

and where $I(X_{l-1})(\omega)$ is an integer-valued measurable function of $\omega \in E$, so chosen that

$$x_{l-1,I(X_{l-1})(\omega)}(\omega) \ne 0 \quad \text{for any} \quad \omega \in E \backslash E_{l-1}. \tag{32}$$

For instance, given a sequence $Y = \{y_n(\omega)\}_{-\infty}^{\infty}$ of measurable functions, for any $\omega \in \bigcup_{n=-\infty}^{\infty} \text{supp}\, y_n$, we may choose $I(Y)(\omega)$ to be the smallest integer n such that $y_n(\omega) \ne 0$ and $|y_{n'}(\omega)| = 0$ for all $n' \in \mathbf{Z}$ with $|n'| < |n|$.

Now by (29), (30), and (31), we see that for any $1 \le l \le L$ and $n \in \mathbf{Z}$, $e_l(\omega)$ and $x_{l,n}(\omega)$ are measurable functions on E, $e_l(\omega), x_{l,n}(\omega) \in S(E, X_{l-1})$, and

$\overline{x_{l,n}(\omega)}^T e_l(\omega) = 0$. This together with Lemma 2 leads to

$$S(E, X_L) \subset S(E, X_{L-1}) \subset \ldots \subset S(E, X_1) \subset S(E, X), \tag{33}$$

and for all $1 \le l \le L$,

$$e_l(\omega) \in S(E, X) \tag{34}$$

and

$$\overline{y(\omega)}^T e_l(\omega) = 0 \quad \text{for all } y(\omega) \in S(E, X_l). \tag{35}$$

By (29), (32), (33), and (35), we obtain

$$\overline{e_l(\omega)}^T e_{l'}(\omega) = \begin{cases} \chi_{E\setminus E_{l-1}}(\omega) & \text{if } l = l', \\ 0 & \text{if } l \ne l'. \end{cases} \tag{36}$$

Hence from (34) and (36), we see that the choice of

$$P(\omega) = \sum_{l=1}^{L} e_l(\omega)\overline{e_l(\omega)}^T$$

gives (21) and (22). To verify (23), we set

$$E_L = \{\omega \in E : \ x_{L,n}(\omega) = 0 \quad \text{for all } n \in \mathbf{Z}\},$$

and observe that this set contains the sets introduced in (31), namely,

$$E_0 \subset E_1 \subset \cdots \subset E_{L-1} \subset E_L.$$

Therefore by (36), we have

$$\overline{Q(\omega)}^T Q(\omega) = I_L \quad \text{a.e. } \omega \in E\setminus E_L, \tag{37}$$

where the L-dimensional square matrix $Q(\omega)$ of measurable functions is defined by $Q(\omega) = [e_1(\omega) \ldots e_L(\omega)]$. It is easy to see that $P(\omega) = Q(\omega)\overline{Q(\omega)}^T$ which, together with (37), leads to

$$P(\omega) = I_L \quad \text{a.e. } \omega \in E\setminus E_L. \tag{38}$$

By (30) and (36), we obtain

$$\begin{aligned} x_{L,n}(\omega) &= x_n(\omega) - \sum_{l=1}^{L} e_l(\omega)\overline{e_l(\omega)}^T x_{l-1,n}(\omega) \\ &= x_n(\omega) - P(\omega)x_n(\omega) \quad \text{a.e. } \omega \in E. \end{aligned} \tag{39}$$

Thus, combining (38) and (39), we have

$$x_{L,n}(\omega) \equiv 0 \quad \text{a.e. } \omega \in E\setminus E_L \quad \text{for all } n \in \mathbf{Z}.$$

This, together with (39) and the definition of the set E_L, imply (23). □

3. Proof of the main results

We only give the proof of Theorem 1, since the other three theorems and Corollary 1 follow accordingly. We divide the proof of Theorem 1 into the following steps: (i)$\Longrightarrow$ (ii)$\Longrightarrow$(iii)$\Longrightarrow$(iv)$\Longrightarrow$(i), and (v)$\Longrightarrow$(ii)$\Longrightarrow$(v). The proof of (ii)$\Longrightarrow$(v) is the most technical part in our proof, and will be dealt with last.

Set $\mathcal{G} = \{\psi_l(\cdot - n/m) : \ 1 \le l \le L, 0 \le n \le m-1\}$. Then for $\mathcal{F}_m$ in (11), we have

$$\mathcal{F}_m = \{M^{j/2} g(M^j \cdot - k) : \ g \in \mathcal{G}, j, k \in \mathbf{Z}\}. \tag{40}$$

By direct computation, we obtain

$$\sum_{g\in\mathcal{G}} \sum_{j\in\mathbf{Z}} |\widehat{g}(M^j\omega)|^2 = m \sum_{l=1}^{L} \sum_{j\in\mathbf{Z}} |\widehat{\psi}_l(M^j\omega)|^2,$$

and for all integers d,

$$\begin{aligned} &\sum_{g\in\mathcal{G}} \sum_{j=0}^{\infty} \overline{\widehat{g}(M^j\omega)} \widehat{g}(M^j(\omega + 2\pi d)) \\ = \ & \sum_{n=0}^{m-1} \sum_{l=1}^{L} \sum_{j=0}^{\infty} \overline{\widehat{\psi}_l(M^j\omega)} \widehat{\psi}_l(M^j(\omega+2\pi d)) e^{i2\pi d n M^j/m} \\ = \ & m \sum_{l=1}^{L} \sum_{j=1}^{\infty} \overline{\widehat{\psi}_l(M^j\omega)} \widehat{\psi}_l(M^j(\omega+2\pi d)) + m \sum_{l=1}^{L} \overline{\widehat{\psi}_l(\omega)} \widehat{\psi}_l(\omega + 2\pi d) \delta_{d, m\mathbf{Z}}. \end{aligned} \tag{41}$$

Therefore, by (40), (41), the assumption (i), and Theorem A, we have

$$\sum_{l=1}^{L} \sum_{j=1}^{\infty} \overline{\widehat{\psi}_l(M^j\omega)} \widehat{\psi}_l(M^j(\omega+2\pi d)) + \sum_{l=1}^{L} \overline{\widehat{\psi}_l(\omega)} \widehat{\psi}_l(\omega+2\pi d) \delta_{d,m\mathbf{Z}} = 0 \tag{42}$$

for any $d \in \mathbf{Z} \backslash M\mathbf{Z}$. Similarly by Theorem A, (40) and (41) with $m = 1$, and the assumption that $\mathcal{F}_1$ is a tight frame of L^2, we also have

$$\sum_{l=1}^{L} \sum_{j=0}^{\infty} \overline{\widehat{\psi}_l(M^j\omega)} \widehat{\psi}_l(M^j(\omega+2\pi d)) = 0 \tag{43}$$

for any $d \in \mathbf{Z} \backslash M\mathbf{Z}$. Hence (ii) follows from (42) and (43).

Taking the Fourier transform on both sides of (4) leads to

$$(Q_0 f)^\wedge(\omega) = \sum_{l=1}^{L} \sum_{k\in\mathbf{Z}} \widehat{f}(\omega + 2k\pi) \overline{\widehat{\psi}_l(\omega + 2k\pi)} \widehat{\psi}_l(\omega), \ j \ge 0.$$

Hence, for any $f \in L^2$ and $d \in \mathbf{Z}$,

$$\begin{aligned} &(\tau_{d/m} Q_0 \tau_{-d/m} f - Q_0 f)^\wedge(\omega) \\ = \ & \sum_{l=1}^{L} \sum_{k\in\mathbf{Z}} \widehat{f}(\omega + 2k\pi) \overline{\widehat{\psi}_l(\omega+2k\pi)} \widehat{\psi}_l(\omega) (e^{2idk\pi/m} - 1). \end{aligned}$$

This, together with assumption (ii), lead to the shift-invariance of the operator Q_0 with respect to $\frac{1}{m}\mathbf{Z}$, and this establishes (ii)$\Longrightarrow$(iii).

To prove (iii)$\Longrightarrow$(iv), note that

$$Q_{0,m} = \sum_{k=0}^{m-1} \tau_{k/m} Q_0 \tau_{-k/m} \tag{44}$$

from its definition (4). Hence (iv) follows from (14) and (44).

To prove (iv)$\Longrightarrow$(i), note that since $\mathcal{F}_1$ is a tight frame, the function H in (3) is bounded. Therefore by (7), $\mathcal{F}_m$ in (11) is a tight frame if and only if

$$\sum_{j\in\mathbf{Z}} D_{-j} Q_{0,m} D_j = AI \tag{45}$$

for some positive constant A. Hence (i) follows from (45), the tight frame assumption on $\mathcal{F}_1$, and the hypothesis $Q_{0,m} = mQ_0$.

Let (v) be satisfied. Then we have

$$e^{i(\omega+2\pi d)/m}\widehat{\Phi}(\omega+2d\pi) = A(\omega)\widehat{\Phi}(\omega+2d\pi)$$

for any integer d. Thus,

$$\begin{aligned} e^{2di\pi/m}\overline{\widehat{\Phi}(\omega)^T}\widehat{\Phi}(\omega+2\pi d) &= \overline{\widehat{\Phi}(\omega)^T}\ \overline{A(\omega)^T} A(\omega)\widehat{\Phi}(\omega+2\pi d) \\ &= \overline{\widehat{\Phi}(\omega)^T}\widehat{\Phi}(\omega+2\pi d), \end{aligned}$$

which implies (ii).

Finally, we come to the proof of (ii)$\Longrightarrow$(v). Let $P(\omega)$ be the L-dimensional square matrix of measurable functions on $[-m\pi, m\pi)$ in Lemma 1, with $E = [-m\pi, m\pi)$ and $X = \{\widehat{\Phi}(\omega + 2mn\pi)\}_{n=-\infty}^{\infty}$. For notational convenience, we denote the $2m\pi$-periodization of $P(\omega)$ again by $P(\omega)$. Then by Lemma 1, we have

$$P(\omega)v \in S(\mathbf{R}, \mathcal{G}_m) \quad \text{for all } v \in \mathbf{C}^L, \tag{46}$$

$$P(\omega + 2m\pi) = P(\omega),\ \overline{P(\omega)^T} = P(\omega),\ P(\omega)^2 = P(\omega) \quad \text{a.e. } \omega \in \mathbf{R}, \tag{47}$$

and

$$\widehat{\Phi}(\omega) = P(\omega)\widehat{\Phi}(\omega) \quad \text{a.e. } \omega \in \mathbf{R}, \tag{48}$$

where $\mathcal{G}_m = \{\widehat{\Phi}(\omega + 2mn\pi)\}_{n=-\infty}^{\infty}$. By (46) and assumption (ii), we obtain

$$\overline{P(\omega)^T} P(\omega + 2d\pi) = 0 \quad \text{for all } d \in \mathbf{Z}\backslash m\mathbf{Z}. \tag{49}$$

Therefore, the function

$$A(\omega) := I_L + \sum_{d=0}^{m-1} (e^{i(\omega+2\pi d)/m} - 1) P(\omega + 2\pi d)$$

is an L-dimensional square matrix of measurable functions that satisfies $A(\omega+2\pi) = A(\omega)$ by (47), and

$$\begin{aligned}
&\overline{A(\omega)^T}A(\omega)\\
= \;& I_L + \sum_{d=0}^{m-1}(e^{i(\omega+2\pi d)/m}-1)P(\omega+2\pi d) + \sum_{d=0}^{m-1}(e^{-i(\omega+2\pi d)/m}-1)\overline{P(\omega+2\pi d)^T}\\
&+\Big(\sum_{d=0}^{m-1}(e^{-i(\omega+2\pi d)/m}-1)\overline{P(\omega+2\pi d)^T}\Big)\\
&\times\Big(\sum_{d=0}^{m-1}(e^{i(\omega+2\pi d)/m}-1)P(\omega+2\pi d)\Big)\\
= \;& I_L + \sum_{d=0}^{m-1}(e^{i(\omega+2\pi d)/m}-1)P(\omega+2\pi d) + \sum_{d=0}^{m-1}(e^{-i(\omega+2\pi d)/m}-1)P(\omega+2\pi d)\\
&+\sum_{d=0}^{m-1}(e^{-i(\omega+2\pi d)/m}-1)\times(e^{i(\omega+2\pi d)/m}-1)P(\omega+2d\pi)\\
= \;& I_L,
\end{aligned}$$

where we have used (47) and (49) to obtain the second equality. Moreover, by (47), (48), and (49), we get

$$\begin{aligned}
A(\omega)\widehat{\Phi}(\omega) &= \widehat{\Phi}(\omega) + \sum_{d=0}^{m-1}(e^{i(\omega+2\pi d)/m}-1)P(\omega+2\pi d)\widehat{\Phi}(\omega)\\
&= \widehat{\Phi}(\omega) + \sum_{d=0}^{m-1}(e^{i(\omega+2\pi d)/m}-1)\overline{P(\omega+2\pi d)^T}P(\omega)\widehat{\Phi}(\omega)\\
&= e^{i\omega/m}\widehat{\Phi}(\omega).
\end{aligned}$$

This completes the proof of (v), and hence Theorem 1. □

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References

1. A. Aldroubi, *Portraits of frames*, Proc. Amer. Math. Soc. **123** (1995), 1661–1668. MR **95g:**46037
2. A. Aldroubi, Q. Sun and W.-S. Tang, *p-frames and shift invariant subspaces of L^p*, J. Fourier Anal. Appl. **7** (2001), 1–21. MR **2002c:**42046
3. M. Bownik, *A characterization of affine dual frames in $L^2(\mathbf{R}^n)$*, Appl. Comp. Harmonic Anal. **8** (2000), 203–221. MR **2001d:**42019
4. A. Calogero, *Wavelets on general lattices*, ERA Amer. Math. Soc. **5** (1999), 1–10. MR **99i:**42042
5. A. Calogero, *A characterization of wavelets on general lattices*, J. Geom. Anal. **10** (2000), 597–622. MR **2002b:**42051

6. C. K. Chui, W. Czaja, M. Maggioni and G. Weiss, *Characterization of general tight wavelet frames with matrix dilations and tightness preserving oversampling*, J. Fourier Anal. Appl., To appear.
7. C. K. Chui and X. L. Shi, *Bessel sequences and affine frames*, Appl. Comp. Harmonic Anal. **1** (1993), 29–49. MR **95b:**42028
8. C. K. Chui and X. L. Shi, *$n\times$oversampling preserves any tight affine frame for odd n*, Proc. Amer. Math. Soc. **121** (1994), 511–517. MR **94h:**42052
9. C. K. Chui and X. L. Shi, *Inequalities on matrix-dilated Littlewood-Paley energy functions and oversampled affine frames*, SIAM J. Math. Anal. **28** (1997), 213–232. MR **98b:**42046
10. C. K. Chui and X. L. Shi, *Orthonormal wavelets and tight frames with arbitrary real dilations*, Appl. Comp. Harmonic Anal. **9** (2000), 243–264. MR **2002a:**42025
11. R. G. Catalán, *Oversampling and preservation of tightness in affine frames*, Proc. Amer. Math. Soc. **130** (2002), 1031–1034.
12. E. Hernandez and G. Weiss, *A First Course on Wavelets*, CRC Press, Boca Raton, FL, 1996. MR **97i:**42015
13. J. Stoeckler, "*Multivariate Affine Frames*", Habil.-Schr. Thesis, Duisburg Univ., 1996.
14. E. Weber, *On the translation invariance of wavelet subspaces*, J. Fourier Anal. Appl. **6** (2000), 551–558. MR **2001h:**42057

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