PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 131, Number 5, Pages 1515–1525
S 0002-9939(02)06748-5
Article electronically published on October 1, 2002

FINITE RANK OPERATORS IN CLOSED MAXIMAL TRIANGULAR ALGEBRAS II

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(Communicated by David R. Larson)

ABSTRACT. In this paper, we discuss finite rank operators in a closed maximal triangular algebra $\mathcal{S}$. Based on the following result that each finite rank operator of $\mathcal{S}$ can be written as a finite sum of rank one operators each belonging to $\mathcal{S}$, we proved that $(\mathcal{S} \cap \mathcal{F}(\mathcal{H}))^{w^*} = \{T \in \mathcal{B}(\mathcal{H}) : TN \subseteq N_\sim, \forall N \in \mathcal{N}\}$, where $N_\sim = N$, if $dim N \ominus N_- \leq 1$; and $N_\sim = N_-$, if $dim N \ominus N_- = \infty$. We also proved that the Erdos Density Theorem holds in $\mathcal{S}$ if and only if $\mathcal{S}$ is strongly reducible.

1. Introduction

Finite rank operators and rank one operators are important to the theory of nest algebras. In a nest algebra, each finite rank operator can be written as a finite sum of rank one operators which belong to itself (This result is in [6], but belongs to Ringrose); the w^*-closure of all finite rank operators is the whole of the nest algebra ([6], it is known as the famous Erdos Density Theorem). Naturally, we may ask what happens in the case of maximal triangular algebras?

We have proved in [4] that each finite rank operator of a closed maximal triangular algebra $\mathcal{S}$ can be represented as a finite sum of rank one operators in $\mathcal{S}$. This is first appeared in [4], but for completeness and reader-friendly reasons, we state it in Section 2. In Section 3, using the decomposability of finite rank operators in $\mathcal{S}$ and the technique of annihilators, we calculate the w^*-closure of all finite rank operators in $\mathcal{S}$. In the last section, we give some remarks on Rosenthal's famous note [16], and obtain a sufficient and necessary condition for which the Erdos Density Theorem holds in $\mathcal{S}$.

Now we give some notation and terminology. Let $\mathcal{H}$ be a complex separable infinite–dimensional Hilbert space, $\mathcal{B}(\mathcal{H})$ the set of all bounded operators on $\mathcal{H}$ and $\mathcal{F}(\mathcal{H})$ the set of all finite rank operators in $\mathcal{B}(\mathcal{H})$. A nest $\mathcal{N}$ is a chain of closed subspaces of Hilbert space $\mathcal{H}$ containing (0) and $\mathcal{H}$ which is closed under intersection and closed span. For $N \in \mathcal{N}$, define

$$N_- = \bigvee\{N^{'} \in \mathcal{N} : N^{'} < N\}.$$

Received by the editors December 12, 2000 and, in revised form, December 16, 2001.
2000 *Mathematics Subject Classification.* Primary 47L75.
Key words and phrases. Closed maximal triangular algebra, finite rank operator, w^*-closure.

If $N \neq N_-$, the subspace $N \ominus N_-$ is called an atom of $\mathcal{N}$. If $\dim N \ominus N_- \leq 1$ for any $N \in \mathcal{N}$, $\mathcal{N}$ is called a maximal nest. If $\mathcal{N}$ is a nest, the nest algebra $\mathcal{T}(\mathcal{N})$ is the set of all operators T such that $TN \subseteq N$ for every element N in $\mathcal{N}$.

Let $\mathcal{S}$ be a subalgebra of $\mathcal{B}(\mathcal{H})$, and define $\mathcal{S}^* = \{A^* : A \in \mathcal{S}\}$. Following Kadison and Singer [8], we shall say that $\mathcal{S}$ is a triangular algebra if $\mathcal{D} = \mathcal{S} \cap \mathcal{S}^*$ is a maximal abelian subalgebra of $\mathcal{B}(\mathcal{H})$. The maximal abelian *–algebra $\mathcal{D}$ is called the diagonal of $\mathcal{S}$. A maximal triangular algebra is a triangular algebra which is not properly contained in any other such algebra. Applying Zorn's Lemma, we conclude that any triangular algebra is contained in a maximal triangular algebra with the same diagonal.

Let $\mathcal{S}$ be a maximal triangular algebra over $\mathcal{H}$. It is shown in [8], Lemma 2.3.3, that $Lat\mathcal{S}$ is totally ordered by inclusion. Hence it forms a nest $\mathcal{N}$, we shall call $\mathcal{N}$ the hull nest of $\mathcal{S}$ and $\mathcal{T}(\mathcal{N})$ the hull nest algebra of $\mathcal{S}$. In general, the hull nest $\mathcal{N}$ is quasi–maximal, that is the subspace $N \ominus N_-$ has dimension 0, 1 or infinity for any $N \in \mathcal{N}$ (see [5], Theorem 1). Following [8], we shall say that $\mathcal{S}$ is irreducible if the hull nest $\mathcal{N} = \{(0), \mathcal{H}\}$, and that $\mathcal{S}$ is strongly reducible if $\mathcal{N}$ is maximal. It is shown in [11] and [12] that not all maximal triangular algebras are norm closed. However, one feels that non-norm-closed maximal triangular algebras are rather pathological and that the proper objects for study should at least be complete. If a triangular algebra is norm-closed, we shall simply say it is closed.

Suppose that $\mathcal{S}$ is a subspace of $\mathcal{B}(\mathcal{H})$, if $\mathcal{S} \cap \mathcal{F}(\mathcal{H})$ is weakly dense in $\mathcal{S}$, we say that the Erdos Density Theorem holds in $\mathcal{S}$.

2. Finite rank operators

Definition 2.1. Let $\mathcal{A}$ be a subalgebra of $\mathcal{B}(\mathcal{H})$, and let n be a positive integer. $\mathcal{A}$ is *n-fold* transitive if for any choice of elements $x_1, \ldots, x_n, y_1, \ldots, y_n \in \mathcal{H}$ with $\{x_i\}_{i=1}^n$ linearly independent, there exists a sequence $\{A_k\} \subseteq \mathcal{A}$ such that

$$\lim_k A_k x_i = y_i, \qquad \forall 1 \leq i \leq n.$$

Thus $\mathcal{A}$ is 1-*fold* transitive if and only if $Lat\mathcal{A} = \{(0), \mathcal{H}\}$.

Lemma 2.2. *Let $\mathcal{S}$ be a closed irreducible triangular algebra, then $\mathcal{S}$ is n-fold transitive, $\forall n \geq 1$.*

Proof. Since the Hilbert space $\mathcal{H}$ is separable infinite–dimensional, then the diagonal $\mathcal{D} = \mathcal{S} \cap \mathcal{S}^*$ is a countably decomposable maximal abelian *–subalgebra of $\mathcal{B}(\mathcal{H})$, and since $\mathcal{S}$ is irreducible, so by [1], Theorem 3.3, $\mathcal{S}$ is strongly dense in $\mathcal{B}(\mathcal{H})$.

Suppose that $x_1, \ldots, x_n, y_1, \ldots, y_n \in \mathcal{H}$ with $\{x_i\}_{i=1}^n$ linearly independent. By the Hahn-Banach Theorem, we can choose bounded operators $F_1, \ldots, F_n$ such that $F_i(x_j) = \delta_{ij}$. Set

$$Tx = \sum_{i=1}^n F_i(x) y_i.$$

Then $T \in \mathcal{B}(\mathcal{H})$ and $Tx_i = y_i$. Since $\mathcal{S}$ is strongly dense in $\mathcal{B}(\mathcal{H})$, we can find, for each $k \geq 1$, an $A_k \in \mathcal{S}$ such that

$$\| A_k x_i - Tx_i \| \leq 1/k, \qquad i = 1, 2, \ldots, n.$$

Hence $\lim_k A_k x_i = Tx_i = y_i$, proving that $\mathcal{S}$ is *n-fold* transitive. $\square$

If x, y are nonzero vectors in $\mathcal{H}$, we define the rank one operator $x \otimes y$ by

$$(x \otimes y)(z) = (z, y)x, \qquad \forall z \in \mathcal{H}.$$

Lemma 2.3 (F.Y. Lu [10]). *Let $\mathcal{S}$ be a norm-closed subalgebra of $\mathcal{B}(\mathcal{H})$ that satisfies the following conditions:*
(1) $I \in \mathcal{S}$;
(2) $Lat\mathcal{S} = \{(0), \mathcal{H}\}$;
(3) $\mathcal{S} \cap \mathcal{S}^*$ abelian.
Then $\mathcal{S}$ contains no rank one operators.

Proof. Suppose that there is a nonzero rank one operator $x \otimes y \in \mathcal{S}$. Since $Lat\mathcal{S} = \{(0), \mathcal{H}\}$ and $I \in \mathcal{S}$, it follows that $[\mathcal{S}x] = \mathcal{H}$. Hence for any $z \in \mathcal{H}$, there exists $\{S_\alpha\} \subseteq \mathcal{S}$ such that $\lim_\alpha S_\alpha x = z$. Since $\mathcal{S}$ is norm-closed, it follows that

$$z \otimes y = (\lim_\alpha S_\alpha x) \otimes y = \lim_\alpha S_\alpha (x \otimes y) \in \mathcal{S}.$$

Since $Lat\mathcal{S}^*$ is also trivial, similarly, for any $w \in \mathcal{H}$ there exists $\{S_\beta\} \subseteq \mathcal{S}$ such that $\lim_\beta S_\beta^* y = w$. Hence,

$$z \otimes w = \lim_\beta z \otimes (S_\beta^* y) = \lim_\beta (z \otimes y) S_\beta \in \mathcal{S}.$$

Thus $\mathcal{S}$ contains all rank one operators in $\mathcal{B}(\mathcal{H})$.

Now suppose that u, v are linearly independent vectors in $\mathcal{H}$ and $(u, v) \neq 0$. Then the self-adjoint rank one operators $u \otimes u$ and $v \otimes v$ belong to $\mathcal{S} \cap \mathcal{S}^*$. However,

$$(u \otimes u)(v \otimes v) = (v, u) u \otimes v \neq (u, v) v \otimes u = (v \otimes v)(u \otimes u);$$

this contradicts condition (3). $\square$

Proposition 2.4. *Let $\mathcal{S}$ be a closed irreducible triangular algebra, then $\mathcal{S}$ contains no nonzero finite rank operators.*

Proof. Suppose that there exists a rank n operator F in $\mathcal{S}$. Set

$$F = \sum_{i=1}^{n} x_i \otimes z_i,$$

where $\{x_i\}_{i=1}^n$ and $\{z_i\}_{i=1}^n$ are both linearly independent.

Following Lemma 2.2, $\mathcal{S}$ is *n-fold* transitive. So there exists a sequence $\{A_k\} \subseteq \mathcal{S}$ such that

$$\lim_k A_k x_1 = x_1 \quad \text{and} \quad \lim_k A_k x_i = 0, \quad 1 < i \leq n.$$

Since $\mathcal{S}$ is norm-closed, then

$$x_1 \otimes z_1 = \lim_k A_k (\sum_{i=1}^{n} x_i \otimes z_i) = \lim_k A_k F \in \mathcal{S}.$$

This is a contradiction to Lemma 2.3. Hence $\mathcal{S}$ does not contain nonzero finite rank operators. $\square$

Lemma 2.5. *Let $\mathcal{S}$ be a maximal triangular algebra with hull nest $\mathcal{N}$. If $N \in \mathcal{N}$ and $\dim(N \ominus N_-) \leq 1$, then $P(N) T P(N_-)^\perp \in \mathcal{S}$, $\forall T \in \mathcal{B}(\mathcal{H})$.*

Proof. Following the proof of [15], Lemma 5.2. $\square$

For the purpose of this paper, we give another form of [10], Theorem 5.2.3.

Lemma 2.6. *Suppose that $\mathcal{S}$ is a closed maximal triangular algebra. Then a rank one operator $x \otimes y \in \mathcal{S}$ if and only if there exists an element N in $\mathcal{N}$ such that:*

(1) if $dim N \ominus N_- \leq 1$, $x \in N$ and $y \in N_-^{\perp}$;

(2) if $dim N \ominus N_- = \infty$, $x \in N, y \in N^{\perp}$; or $x \in N_-, y \in N_-^{\perp}$.

Proof. Sufficiency. It follows from Lemma 2.5 and [8], Lemma 2.3.2.

Necessity. Since $x \otimes y \in \mathcal{S} \subseteq \mathcal{T}(\mathcal{N})$, there exists an element $N \in \mathcal{N}$ such that $x \in N$ and $y \in N_-^{\perp}$. Write

$$\begin{aligned} x &= x_1 + x_2 \in N_- \oplus (N \ominus N_-), \\ y &= y_1 + y_2 \in (N \ominus N_-) \oplus N^{\perp}, \end{aligned}$$

then

$$x \otimes y = x_1 \otimes y + x_2 \otimes y_2 + x_2 \otimes y_1.$$

It follows from [8], Lemma 2.3.2 that $x_1 \otimes y$ and $x_2 \otimes y_2$ belong to $\mathcal{S}$; thus, $x_2 \otimes y_1$ also belongs to $\mathcal{S}$.

If $dim N \ominus N_- = \infty$, following the proof of [5], Theorem 1, $P(N \ominus N_-)\mathcal{S}P(N \ominus N_-)$ is a closed irreducible triangular algebra in $\mathcal{B}(N \ominus N_-)$. Thus by Proposition 2.4,

$$x_2 \otimes y_1 = P(N \ominus N_-)(x_2 \otimes y_1)P(N \ominus N_-) = 0.$$

Then $x_2 = 0$ or $y_1 = 0$. If $x_2 = 0$, $x \in N_-$ and $y \in N_-^{\perp}$; if $y_1 = 0$, $x \in N$ and $y \in N^{\perp}$.

If $dim N \ominus N_- \leq 1$, $x \in N$ and $y \in N_-^{\perp}$. □

Theorem 2.7. *Suppose that $\mathcal{S}$ is a closed maximal triangular algebra, and F is a finite rank operator in $\mathcal{S}$, then F can be written as a finite sum of rank one operators each belonging to $\mathcal{S}$, and the number of rank one operators necessary to form F is bounded above 3 times the rank of F.*

Proof. Set $\mathcal{N}$ to be the hull nest of $\mathcal{S}$, and let F be a rank n operator in $\mathcal{S}$. Since $F \in \mathcal{S} \subseteq \mathcal{T}(\mathcal{N})$, then by [6], Theorem 1, there exist $\{N_i\}_{i=1}^n \subseteq \mathcal{N}$ and $\{x_i\}_{i=1}^n, \{y_i\}_{i=1}^n$ with $x_i \in N_i, y_i \in N_{i-}^{\perp}, i = 1, 2, \ldots, n$ such that

$$F = x_1 \otimes y_1 + x_2 \otimes y_2 + \cdots + x_n \otimes y_n.$$

Write

$$\begin{aligned} x_i &= x_i^1 + x_i^2 \in N_{i-} \oplus (N_i \ominus N_{i-}), \\ y_i &= y_i^1 + y_i^2 \in (N_i \ominus N_{i-}) \oplus N_i^{\perp}; \end{aligned}$$

then

$$F = \sum_{i=1}^{n} (x_i^1 \otimes y_i^1 + x_i \otimes y_i^2 + x_i^2 \otimes y_i^1) = F_1 + F_2$$

with $F_1 = \sum\limits_{i=1}^{n} (x_i^1 \otimes y_i^1 + x_i \otimes y_i^2), F_2 = \sum\limits_{i=1}^{n} (x_i^2 \otimes y_i^1)$. Following [8], Lemma 2.3.2, the rank one operators $x_i^1 \otimes y_i^1$ and $x_i \otimes y_i^2 (i = 1, 2, \ldots, n)$ belong to $\mathcal{S}$. Hence $F_1 \in \mathcal{S}$, so $F_2 \in \mathcal{S}$. In the following, we shall prove that $x_i^2 \otimes y_i^1 \in \mathcal{S}, i = 1, 2, \ldots, n$.

Without loss of generality, let

$$N_1 \leq N_2 \leq \cdots \leq N_n.$$

If $N_i = N_{i-}$, then $x_i^2 = y_i^1 = 0$. So we can suppose that $N_i \neq N_{i-}, \forall 1 \leq i \leq n$. For a fixed i, suppose that

$$N_{i-q-1} < N_{i-q} = \cdots = N_i = \cdots = N_{i+p} < N_{i+p+1}.$$

Since $P(N) \in \mathcal{S} \cap \mathcal{S}^*$ for any $N \in \mathcal{N}$, then

$$(P(N_i) - P(N_{i-}))F_2(P(N_i) - P(N_{i-})) = \sum_{j=i-q}^{i+p} x_j^2 \otimes y_j^1 \in \mathcal{S}.$$

Now we distinguish two cases.

Case 1. $dimN_i \ominus N_{i-} = \infty$. Following the proof of [5], Theorem 1, we have that $P(N_i \ominus N_{i-})\mathcal{S}P(N_i \ominus N_{i-})$ is a closed irreducible triangular algebra in $\mathcal{B}(N_i \ominus N_{i-})$. Thus by Proposition 2.4, $P(N_i \ominus N_{i-})\mathcal{S}P(N_i \ominus N_{i-})$ does not contain any nonzero finite rank operators. Hence, if $\sum_{j=i-q}^{i+p} x_j^2 \otimes y_j^1 \neq 0$, we have

$$\sum_{j=i-q}^{i+p} x_j^2 \otimes y_j^1 \notin \mathcal{S}.$$

This is a contradiction, so $\sum_{j=i-q}^{i+p} x_j^2 \otimes y_j^1 = 0$.

Case 2. $\dim N_i \ominus N_{i-} = 1$. Following Lemma 2.5, we have

$$x_j^2 \otimes y_j^1 \in \mathcal{S}, \qquad j = i-q, \ldots, i+p.$$

Since the hull nest is quasi–maximal, the two cases are jointly exhaustive. Since i is arbitrary, we obtain that F_2 is also a finite sum of rank one operators in $\mathcal{S}$. So any rank n operator can be written as a finite sum of rank one operators each belonging to $\mathcal{S}$. □

3. The w^*-closure of finite rank operators

In this section, we will describe the w^*-closure of finite rank operators in $\mathcal{S}$. Set

$$\mathcal{W} = \{X \in \mathcal{B}(\mathcal{H}) : XN \subseteq \widetilde{N}, \forall N \in \mathcal{N}\},$$

where $\widetilde{N} = N_-$ if $dimN \ominus N_- \leq 1$; and $\widetilde{N} = N$ if $dimN \ominus N_- = \infty$.

Lemma 3.1. *$\mathcal{W}$ is a weakly closed $\mathcal{T}(\mathcal{N})$–ideal determined by the order homomorphism $N \to \widetilde{N}$ of $\mathcal{N}$ into itself; and a rank one operator $x \otimes y \in \mathcal{W}$ if and only if there exists an element N in $\mathcal{N}$ such that $x \in N, y \in N_\sim^\perp$, where $N_\sim = N$, if $dimN \ominus N_- \leq 1$; $N_\sim = N_-$, if $dimN \ominus N_- = \infty$.*

Proof. The fact that $\mathcal{W}$ is a weakly closed $\mathcal{T}(\mathcal{N})$–ideal is obvious from the definition of $\mathcal{W}$.

By virtue of [7], Lemma 1.1, a rank one operator $x \otimes y \in \mathcal{W}$ if and only if there exists an element $N \in \mathcal{N}$ such that $x \in N, y \in N_\sim^\perp$. In the following, we will compute $N_\sim$. For any $N \in \mathcal{N}$, we consider separately three cases. Recall that $N_\sim = \bigvee\{N' : \widetilde{N'} < N\}$ defined in [7].

Case 1. $dimN \ominus N_- = 1$. In this case, $\widetilde{N} = N_- < N$. If $N' > N$, $N'_- \geq N$. Thus $\widetilde{N'} \geq N$. So $N_\sim = N$.

Case 2. $dimN \ominus N_- = \infty$. In this case, $\widetilde{N} = N$. Since $\widetilde{N_-} \leq N_- < N$, $N_\sim = N_-$.

Case 3. $dimN \ominus N_- = 0$. Thus, $\widetilde{N} = N_- = N$. In this case, we can prove that

$$\{N' \in \mathcal{N} : N' < N\} = \{N' \in \mathcal{N} : \widetilde{N'} < N\}.$$

Indeed, since $\widetilde{N'} \leq N'$, we have that $\{N' \in \mathcal{N} : N' < N\} \subseteq \{N' \in \mathcal{N} : \widetilde{N'} < N\}$. Conversely, if $N' \notin \{N' \in \mathcal{N} : N' < N\}$, that is $N' \geq N$ and $\widetilde{N'} \geq \widetilde{N} = N$. So $N' \notin \{N' \in \mathcal{N} : \widetilde{N'} < N\}$. Hence $\{N' \in \mathcal{N} : N' < N\} \supseteq \{N' \in \mathcal{N} : \widetilde{N'} < N\}$. Therefore,

$$N_{\sim} = \bigvee\{N' \in \mathcal{N} : \widetilde{N'} < N\} = \bigvee\{N' \in \mathcal{N} : N' < N\} = N_{-} = N.$$

Since the hull nest $\mathcal{N}$ is quasi-maximal, the three cases are jointly exhaustive. This completes the proof. $\square$

Set $\mathcal{C}_1(\mathcal{H})$ as the ideal of all trace class operators in $\mathcal{B}(\mathcal{H})$.

Theorem 3.2. *Suppose that $\mathcal{S}$ is a closed maximal triangular algebra with hull nest $\mathcal{N}$, then $\rho \in \mathcal{B}(\mathcal{H})_*$ annihilates $\mathcal{S} \cap \mathcal{F}(\mathcal{H})$ if and only if ρ is of the form*

$$\rho(\cdot) = tr(X\cdot),$$

where X is a trace class operator in $\mathcal{W}$.

Proof. Necessity. If $\rho \in \mathcal{B}(\mathcal{H})_* \cong \mathcal{C}_1(\mathcal{H})$, there exists an operator $X \in \mathcal{C}_1(\mathcal{H})$ such that $\rho(\cdot) = tr(X\cdot)$ and ρ annihilates $\mathcal{S} \cap \mathcal{F}(\mathcal{H})$. For any $Y \in \mathcal{F}(\mathcal{H})$ and $N \in \mathcal{N}$, by [8], Lemma 2.3.2 and Lemma 2.5, the operator $P(N)YP(\widetilde{N})^{\perp} \in \mathcal{S} \cap \mathcal{F}(\mathcal{H})$. Thus

$$tr(P(\widetilde{N})^{\perp}XP(N)Y) = tr(XP(N)YP(\widetilde{N})^{\perp}) = 0, \quad \forall Y \in \mathcal{F}(\mathcal{H}).$$

From $\overline{\mathcal{F}(\mathcal{H})}^{w^*} = \mathcal{B}(\mathcal{H})$ and the w^*-continuity of the map $tr(P(\widetilde{N})^{\perp}XP(N)\cdot)$ it follows that

$$tr(P(\widetilde{N})^{\perp}XP(N)Y) = 0, \quad \forall Y \in \mathcal{B}(\mathcal{H}).$$

Then

$$P(\widetilde{N})^{\perp}XP(N) = 0, \qquad \forall N \in \mathcal{N}.$$

So

$$X \in \mathcal{W} \cap \mathcal{C}_1(\mathcal{H}).$$

Sufficiency. If $X \in \mathcal{W} \cap \mathcal{C}_1(\mathcal{H})$, let $x \otimes y$ be any rank one operator of $\mathcal{S}$. Then, by Lemma 2.6, there exists an element $N \in \mathcal{N}$ such that:

(1) if $dimN \ominus N_{-} \leq 1$, then $x \in N$ and $y \in N_{-}^{\perp}$. Since $\widetilde{N} = N_{-}$, we have that

$$\begin{aligned} tr(X(x \otimes y)) &= tr(XP(N)(x \otimes y)P(N_{-})^{\perp}) \\ &= tr(P(N_{-})^{\perp}XP(N)(x \otimes y)) = 0. \end{aligned}$$

(2) if $dimN \ominus N_{-} = \infty$, we distinguish two cases.

Case 1. $x \in N, y \in N^{\perp}$. Since $\widetilde{N} = N$,

$$tr(X(x \otimes y)) = tr(P(N)^{\perp}XP(N)(x \otimes y)) = 0.$$

Case 2. $x \in N_{-}, y \in N_{-}^{\perp}$. Since $X \in \mathcal{W}$, $XN_{-} \subseteq \widetilde{N_{-}} \subseteq N_{-}$. Thus,

$$tr(X(x \otimes y)) = tr(P(N_{-})^{\perp}XP(N_{-})(x \otimes y)) = 0.$$

Therefore the map $tr(X\cdot)$ annihilates any rank one operators in $\mathcal{S}$. Since the map $tr(X\cdot)$ is linear, it follows from Theorem 2.7 that

$$tr(XF) = 0, \qquad \forall F \in \mathcal{S} \cap \mathcal{F}(\mathcal{H}).$$

So $\rho(\cdot) = tr(X\cdot)$ annihilates $\mathcal{S} \cap \mathcal{F}(\mathcal{H})$. $\square$

Theorem 3.2 tells us that $(\mathcal{S}\cap\mathcal{F}(\mathcal{H}))_\perp = \mathcal{W}\cap\mathcal{C}_1(\mathcal{H})$. Since $(\mathcal{S}\cap\mathcal{F}(\mathcal{H}))\cap\mathcal{K}(\mathcal{H}) = \mathcal{S}\cap\mathcal{F}(\mathcal{H})$, Theorem 3.2 also shows that $(\mathcal{S}\cap\mathcal{F}(\mathcal{H}))^\perp = \mathcal{W}\cap\mathcal{C}_1(\mathcal{H})$.

In order to calculate the annihilator of $\mathcal{W}\cap\mathcal{C}_1(\mathcal{H})$, we need some results about weakly closed $\mathcal{T}(\mathcal{N})$–modules. These results have their own interest. Note that the symbol "$\sim$" in the following results 3.3–3.5 is not the same as that defined in the beginning of Section 3.

Lemma 3.3. *Suppose that $E, \widetilde{E}$ are comparable projections in $\mathcal{B}(\mathcal{H})$. If $A \in \mathcal{C}_1(\mathcal{H})$ and $(I-\widetilde{E})AE = 0$, then A can be decomposed as $A = A_1 + A_2$ such that*

1) $(I-\widetilde{E})A_1 = 0, \quad A_2E = 0$;

2) $\| A \|_1 = \| A_1 \|_1 + \| A_2 \|_1$.

Proof. We consider separately two cases.

Case 1. $\widetilde{E} \le E$. We decompose $\mathcal{H}$ as $\widetilde{E}\oplus(E\ominus\widetilde{E})\oplus E^\perp$. Since $(I-\widetilde{E})AE = 0$, corresponding to the decomposition of $\mathcal{H}$, the trace class operator A has the matrix form

$$A = \begin{pmatrix} B_{11} & B_{12} & B_{13} \\ 0 & 0 & B_{23} \\ 0 & 0 & B_{33} \end{pmatrix}.$$

Thus, following [9], Lemma 3.3, A can be written as

$$A = \begin{pmatrix} B_{11} & B_{12} & C \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & D \\ 0 & 0 & B_{23} \\ 0 & 0 & B_{33} \end{pmatrix} = A_1 + A_2$$

and $\| A \|_1 = \| A_1 \|_1 + \| A_2 \|_1$. It follows from the matrix form of A_1, A_2 that $(I-\widetilde{E})A_1 = 0$ and $A_2E = 0$.

Case 2. $E \le \widetilde{E}$. Decompose $\mathcal{H}$ as $E\oplus(\widetilde{E}\ominus E)\oplus\widetilde{E}^\perp$. In this case A has the matrix form

$$A = \begin{pmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ 0 & B_{32} & B_{33} \end{pmatrix}.$$

Similarly, by [9], Lemma 3.3, we have

$$A = \begin{pmatrix} B_{11} & C_{12} & C_{13} \\ B_{21} & C_{22} & C_{23} \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & D_{12} & D_{13} \\ 0 & D_{22} & D_{23} \\ 0 & B_{32} & B_{33} \end{pmatrix} = A_1 + A_2$$

and $\| A \|_1 = \| A_1 \|_1 + \| A_2 \|_1$. Following the matrix form of A_1, A_2, we have that $(I-\widetilde{E})A_1 = 0$ and $A_2E = 0$. □

Lemma 3.4. *Let $\mathcal{U} = \{X \in \mathcal{B}(\mathcal{H}) : XN \subseteq \widetilde{N}, \forall N \in \mathcal{N}\}$, where the map $N \to \widetilde{N}$ is an order homomorphism of $\mathcal{N}$ into $\mathcal{N}$. Then $P(\widetilde{N})TP(N)^\perp \in \mathcal{U}$, for any $N \in \mathcal{N}, T \in \mathcal{B}(\mathcal{H})$.*

Proof. The proof is routine. □

Proposition 3.5. *Suppose that $\mathcal{U}$ is a weakly closed $\mathcal{T}(\mathcal{N})$–module determined by the order homomorphism $N \to \widetilde{N}$, then each extreme point of the unit ball $b_1(\mathcal{U}\cap\mathcal{C}_1(\mathcal{H}))$ is a norm-one rank one operator in $\mathcal{U}\cap\mathcal{C}_1(\mathcal{H})$.*

Proof. Suppose that A is an extreme point of $b_1(\mathcal{U} \cap \mathcal{C}_1(\mathcal{H}))$. First, we shall prove that there exists an element $N_0 \in \mathcal{N}$ such that $A = P(N_{0*})AP(N_0)^{\perp}$. Recall that $N_{0*} = \bigwedge\{\widetilde{N} : N > N_0, \forall N \in \mathcal{N}\}$ defined in [7].

For $N \in \mathcal{N}$, suppose that $A \neq P(\widetilde{N})A$ and $AP(N) \neq 0$. Since $A \in \mathcal{U}$ and $N, \widetilde{N} \in \mathcal{N}$, we have that $(I - P(\widetilde{N}))AP(N) = 0$ and $P(N), P(\widetilde{N})$ are comparable projections. Thus, it follows from Lemma 3.3 that the trace class operator A can be decomposed as $A = A_1 + A_2$ and $P(\widetilde{N})^{\perp}A_1 = 0, A_2P(N) = 0$. So $A_1P(N) = AP(N), P(\widetilde{N})^{\perp}A_2 = P(\widetilde{N})^{\perp}A$. Owing to the hypothesis $A \neq P(\widetilde{N})A$ and $AP(N) \neq 0$, we have that $A_2 \neq 0$ and $A_1 \neq 0$. Now we shall prove that $A_1, A_2 \in \mathcal{U} \cap \mathcal{C}_1(\mathcal{H})$. Following Lemma 3.3 (2), we only need to prove $A_1, A_2 \in \mathcal{U}$. Since $P(\widetilde{N})^{\perp}A_1 = 0$,

$$\begin{aligned} A_1 = P(\widetilde{N})A_1 &= P(\widetilde{N})A_1P(N) + P(\widetilde{N})A_1P(N)^{\perp} \\ &= P(\widetilde{N})AP(N) + P(\widetilde{N})A_1P(N)^{\perp}. \end{aligned}$$

Since $A \in \mathcal{U}, P(\widetilde{N}), P(N) \in \mathcal{T}(\mathcal{N})$ and $\mathcal{U}$ is a weakly closed $\mathcal{T}(\mathcal{N})$–module, the operator $P(\widetilde{N})AP(N) \in \mathcal{U}$; and by virtue of Lemma 3.4, $P(\widetilde{N})A_1P(N)^{\perp} \in \mathcal{U}$. Hence $A_1 \in \mathcal{U}$. Similarly, we can prove $A_2 \in \mathcal{U}$. Thus, it follows from Lemma 3.3 (2) that A is not an extreme point of $b_1(\mathcal{U} \cap \mathcal{C}_1(\mathcal{H}))$. This is a contradiction. Therefore, for any $N \in \mathcal{N}$, either $A = P(\widetilde{N})A$ or $AP(N) = 0$. Set

$$N_0 = \bigvee\{N \in \mathcal{N} : AP(N) = 0\}.$$

Naturally, $AP(N_0) = 0$ and for any $N > N_0, A = P(\widetilde{N})A$. Thus we have

$$A = P(\widetilde{N})AP(N_0)^{\perp}, \quad \forall N > N_0.$$

Taking a limit, it follows from the definition of N_{0*} that

$$A = P(N_{0*})AP(N_0)^{\perp}.$$

In the following, we shall prove that A is a rank one operator. Since $A \in \mathcal{C}_1(\mathcal{H}) \subseteq \mathcal{K}(\mathcal{H})$, A can be written as

$$A = \sum_{k=1}^{+\infty} \lambda_k e_k \otimes f_k,$$

where $\sum$ is convergent according to the norm topology, $\{\lambda_k\}$ are s-numbers of A and $\| e_k \| = \| f_k \| = 1$. Thus,

$$A = P(N_{0*})AP(N_0)^{\perp} = \sum_{k=1}^{+\infty} \lambda_k P(N_{0*})(e_k \otimes f_k)P(N_0)^{\perp}.$$

Since $\| A \|_1 = \sum_{k=1}^{+\infty} \lambda_k \leq \sum_{k=1}^{+\infty} \lambda_k \| P(N_{0*})(e_k \otimes f_k)P(N_0)^{\perp} \|_1$, we have

$$\lambda_k = \lambda_k \| P(N_{0*})(e_k \otimes f_k)P(N_0)^{\perp} \|_1, \qquad \forall k = 1, 2, \cdots.$$

Therefore, if $\lambda_k \neq 0$, we have $\| P(N_{0*})e_k \| \cdot \| P(N_0)^{\perp} f_k \| = 1$. Hence

$$e_k \in N_{0*} \quad \text{and} \quad f_k \in N_0^{\perp}.$$

By [7], Lemma 1.1, for any $k \geq 1$, $e_k \otimes f_k \in \mathcal{U} \cap \mathcal{C}_1(\mathcal{H})$. Since A is an extreme point of $b_1(\mathcal{U} \cap \mathcal{C}_1(\mathcal{H}))$,

$$\lambda_2 = \lambda_3 = \cdots = 0.$$

Thus A is a norm-one rank one operator. $\square$

We come back to study finite rank operators in $\mathcal{S}$.

Lemma 3.6. *The unit ball $b_1(\mathcal{W} \cap \mathcal{C}_1(\mathcal{H}))$ is the norm-closed convex hull of its extreme points, where $\mathcal{W}$ is defined in the beginning of Section 3.*

Proof. Following Theorem 3.2 and $\mathcal{S} \cap \mathcal{F}(\mathcal{H}) = (\mathcal{S} \cap \mathcal{F}(\mathcal{H})) \cap \mathcal{K}(\mathcal{H})$, we obtain

$$(\mathcal{S} \cap \mathcal{F}(\mathcal{H})^{\|\cdot\|})^{\perp} = (\mathcal{S} \cap \mathcal{F}(\mathcal{H}))^{\perp} = \mathcal{W} \cap \mathcal{C}_1(\mathcal{H}).$$

Hence,

$$\mathcal{W} \cap \mathcal{C}_1(\mathcal{H}) \cong (\mathcal{K}(\mathcal{H})/\mathcal{S} \cap \mathcal{F}(\mathcal{H})^{\|\cdot\|})^{*}.$$

By virtue of the Krein-Milman Theorem, $b_1(\mathcal{W} \cap \mathcal{C}_1(\mathcal{H}))$ is the w^*- closed convex hull of its extreme points. It follows from [3], Corollary 16.4, that the boundary points of $b_1(\mathcal{W} \cap \mathcal{C}_1(\mathcal{H}))$ belong to the norm-closed convex hull of its extreme points. Therefore $b_1(\mathcal{W} \cap \mathcal{C}_1(\mathcal{H}))$ is the norm-closed convex hull of its extreme points. □

Now we are in the position to compute the $(\mathcal{W} \cap \mathcal{C}_1(\mathcal{H}))^{\perp}$. Set

$$\mathcal{V} = \{T \in \mathcal{B}(\mathcal{H}) : TN \subseteq N_{\sim}, \forall N \in \mathcal{N}\},$$

where $N_{\sim} = N$, if $dim N \ominus N_{-} \leq 1$; and $N_{\sim} = N_{-}$, if $dim N \ominus N_{-} = \infty$.

Theorem 3.7. $(\mathcal{W} \cap \mathcal{C}_1(\mathcal{H}))^{\perp} = \mathcal{V}$.

Proof. Suppose that $T \in (\mathcal{W} \cap \mathcal{C}_1(\mathcal{H}))^{\perp}$. For any $N \in \mathcal{N}$, nonzero vectors $x \in N$ and $y \in N_{\sim}^{\perp}$. By virtue of Lemma 3.1, the rank one operator $x \otimes y$ belongs to $\mathcal{W} \cap \mathcal{F}(\mathcal{H}) \subseteq \mathcal{W} \cap \mathcal{C}_1(\mathcal{H})$. Hence

$$0 = tr(Tx \otimes y) = (Tx, y), \quad \forall x \in N, y \in N_{\sim}^{\perp}.$$

So $TN \subseteq N_{\sim}$ for any $N \in \mathcal{N}$, and $T \in \mathcal{V}$. Thus, $(\mathcal{W} \cap \mathcal{C}_1(\mathcal{H}))^{\perp} \subseteq \mathcal{V}$.

Conversely, let $T \in \mathcal{V}$. For any $N \in \mathcal{N}, x \in N$ and $y \in N_{\sim}^{\perp}$, we have that

$$tr(Tx \otimes y) = (Tx, y) = (P(N_{\sim})^{\perp} T P(N) x, y) = 0.$$

Thus, T annihilates all rank-one operators in $\mathcal{W}$. It follows from Lemma 3.1 and Proposition 3.5 that T annihilates all extreme points of $b_1(\mathcal{W} \cap \mathcal{C}_1(\mathcal{H}))$. Thus by Lemma 3.6, T annihilates $b_1(\mathcal{W} \cap \mathcal{C}_1(\mathcal{H}))$ and $T \in (\mathcal{W} \cap \mathcal{C}_1(\mathcal{H}))^{\perp}$. Therefore $(\mathcal{W} \cap \mathcal{C}_1(\mathcal{H}))^{\perp} = \mathcal{V}$. □

Theorem 3.8. $(\mathcal{S} \cap \mathcal{F}(\mathcal{H}))^{w^*} = \mathcal{V} = \{T \in \mathcal{B}(\mathcal{H}) : TN \subseteq N_{\sim}, \forall N \in \mathcal{N}\}$, *where* $N_{\sim} = N$, *if* $dim N \ominus N_{-} \leq 1$; $N_{\sim} = N_{-}$, *if* $dim N \ominus N_{-} = \infty$.

Proof. It follows from Theorem 3.2 and Theorem 3.7 that

$$(\mathcal{S} \cap \mathcal{F}(\mathcal{H}))^{w^*} = [(\mathcal{S} \cap \mathcal{F}(\mathcal{H}))_{\perp}]^{\perp} = (\mathcal{W} \cap \mathcal{C}_1(\mathcal{H}))^{\perp} = \mathcal{V}.$$

□

Corollary 3.9. $(\mathcal{S} \cap \mathcal{F}(\mathcal{H}))^{w} = (\mathcal{S} \cap \mathcal{F}(\mathcal{H}))^{s} = (\mathcal{S} \cap \mathcal{F}(\mathcal{H}))^{w^*} = \mathcal{V}$.

Proof. Since $\mathcal{V}$ is weakly closed and $(\mathcal{S} \cap \mathcal{F}(\mathcal{H}))^{w^*} = \mathcal{V}$, we have $(\mathcal{S} \cap \mathcal{F}(\mathcal{H}))^{w} = \mathcal{V}$. Owing to the convexity of $\mathcal{S} \cap \mathcal{F}(\mathcal{H})$, $(\mathcal{S} \cap \mathcal{F}(\mathcal{H}))^{w} = (\mathcal{S} \cap \mathcal{F}(\mathcal{H}))^{s}$. □

4. The Erdos density theorem in $\mathcal{S}$

In this section, we will prove that the Erdos Density Theorem holds in $\mathcal{S}$ if and only if $\mathcal{S}$ is strongly reducible.

Proposition 4.1. *Suppose that $\mathcal{S}$ is a maximal triangular algebra, and that $\mathcal{N}$ is the hull nest of $\mathcal{S}$. Then $\mathcal{S}$ is weakly dense in $\mathcal{T}(\mathcal{N})$.*

Proof. Set $\mathcal{A} = \mathcal{S}^w$. It is easy to show that $Lat\mathcal{A} = Lat\mathcal{S} = \mathcal{N}$ and $\mathcal{A} \supseteq \mathcal{S} \cap \mathcal{S}^*$, so $\mathcal{A}$ is a weakly closed algebra which contains a m.a.s.a and $Lat\mathcal{A}$ is completely ordered. Following [13], Theorem 9.24, $\mathcal{A}$ is a reflexive algebra. Hence $\mathcal{A} = Alglat\mathcal{A} = Alg\mathcal{N} = \mathcal{T}(\mathcal{N})$, that is, $\mathcal{S}^w = \mathcal{T}(\mathcal{N})$. □

Note that in Proposition 4.1, $\mathcal{S}$ is not assumed to be closed. Following Proposition 4.1, we can obtain [16] Rosenthal's famous result: a weakly closed maximal triangular algebra is hyper-reducible, that is, $\mathcal{S} = \mathcal{T}(\mathcal{N})$. If a maximal triangular algebra $\mathcal{S}$ is not weakly closed, we have $\mathcal{S}^w \supset \mathcal{S}$. Owing to the maximality of $\mathcal{S}$, $\mathcal{S}^w$ is not a triangular algebra. Hence Rosenthal's result does not imply Proposition 4.1, which is more general. Now we give an application of Proposition 4.1.

Corollary 4.2. *Let $\mathcal{S}$ be a maximal triangular algebra, then $\mathcal{S}' = CI$.*

Proof. Following [2], Lemma 3.6, the commutant of a nest algebra is trivial. So

$$\mathcal{S}' = (\mathcal{S}^w)' = \mathcal{T}(\mathcal{N})' = CI.$$

□

Theorem 4.3. *Suppose that $\mathcal{S}$ is a closed maximal triangular algebra, then $\mathcal{S} \cap \mathcal{F}(\mathcal{H})$ is weakly dense in $\mathcal{S}$ if and only if $\mathcal{S}$ is strongly reducible.*

Proof. If $\mathcal{S}$ is strongly reducible, it follows from Corollary 3.9 that $(\mathcal{S} \cap \mathcal{F}(\mathcal{H}))^w = \mathcal{T}(\mathcal{N})$. Thus $\mathcal{S} \cap \mathcal{F}(\mathcal{H})$ is weakly dense in $\mathcal{S}$.

Suppose, on the contrary, that $\mathcal{S} \cap \mathcal{F}(\mathcal{H})$ is weakly dense in $\mathcal{S}$. It follows from Corollary 3.9 and Proposition 4.1 that

$$\mathcal{V} = (\mathcal{S} \cap \mathcal{F}(\mathcal{H}))^w = \mathcal{S}^w = \mathcal{T}(\mathcal{N}).$$

Thus $\mathcal{T}(\mathcal{N}) = \mathcal{V} = \{T \in \mathcal{B}(\mathcal{H}) : TN \subseteq N_\sim, \forall N \in \mathcal{N}\}$, where $N_\sim = N$, if $dimN \ominus N_- \leq 1$; and $N_\sim = N_-$, if $dimN \ominus N_- = \infty$. It is easy to prove that $dimN \ominus N_- \leq 1$ for any $N \in \mathcal{N}$. Indeed, suppose that there exists an element N in $\mathcal{N}$ such that $dimN \ominus N_- = \infty$. In this case, $N_\sim = N_-$. So the identity operator $I \in \mathcal{T}(\mathcal{N})$ and $I \notin \mathcal{V}$. This contradicts $\mathcal{T}(\mathcal{N}) = \mathcal{V}$. Hence for any $N \in \mathcal{N}$, $dimN \ominus N_- \leq 1$. Thus $\mathcal{S}$ is strongly reducible. □

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