

## ENDPOINT ESTIMATES FOR THE CIRCULAR MAXIMAL FUNCTION

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ABSTRACT. We consider the problem of endpoint estimates for the circular maximal function defined by

$$Mf(x) = \sup_{1 < t < 2} \left| \int_{S^1} f(x - ty) d\sigma(y) \right|$$

where  $d\sigma$  is the normalized surface area measure on  $S^1$ . Let  $\Delta$  be the closed triangle with vertices  $(0, 0)$ ,  $(1/2, 1/2)$ ,  $(2/5, 1/5)$ . We prove that for  $(1/p, 1/q) \in \Delta \setminus \{(1/2, 1/2), (2/5, 1/5)\}$ , there is a constant  $C$  such that  $\|Mf\|_{L^q(\mathbb{R}^2)} \leq C\|f\|_{L^p(\mathbb{R}^2)}$ . Furthermore,  $\|Mf\|_{L^{5,\infty}(\mathbb{R}^2)} \leq C\|f\|_{L^{5/2,1}(\mathbb{R}^2)}$ .

### 1. INTRODUCTION AND STATEMENT OF RESULTS

Bourgain [B1] showed that the circular maximal function defined by

$$\sup_{t>0} \left| \int_{S^1} f(x - ty) d\sigma(y) \right|$$

is bounded on  $L^p(\mathbb{R}^2)$  if  $p > 2$ . Mockenhaupt, Seeger and Sogge [MSS] later found a new proof of this result based on their local smoothing estimates. Their result actually implies that if one modifies the definition so that the supremum is taken over  $1 < t < 2$ , then the resulting maximal operator  $M$  (see below) is bounded from  $L^p(\mathbb{R}^2)$  to  $L^q(\mathbb{R}^2)$  for some  $q > p$ . Here  $M$  is defined by

$$(1.1) \quad Mf(x) = \sup_{1 < t < 2} \left| \int_{S^1} f(x - ty) d\sigma(y) \right|.$$

Let  $\Delta$  be the closed triangle with vertices  $P = (2/5, 1/5)$ ,  $Q = (1/2, 1/2)$ ,  $R = (0, 0)$ . Schlag [S] showed  $M$  is bounded from  $L^p(\mathbb{R}^2) \rightarrow L^q(\mathbb{R}^2)$  if  $(1/p, 1/q)$  lies in the interior of  $\Delta$ . His result was obtained using a combinatorial method. A different proof was later obtained by Schlag and Sogge [SS] which was based on some local smoothing estimates. It can easily be shown that  $M$  cannot be bounded from  $L^p(\mathbb{R}^2)$  to  $L^q(\mathbb{R}^2)$  if  $(1/p, 1/q) \in ([0, 1] \times [0, 1] \setminus \Delta) \cup \{(1/2, 1/2)\}$  (see [S], [SS]). Thus, when  $(1/p, 1/q) \in (R, P] \cup [P, Q)$ , the  $L^p - L^q$  estimates for  $M$  are still open. In this note these remaining endpoint estimates are considered. The following is

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our main result which gives the complete type set for  $M$  except for the  $L^{5/2} - L^5$  estimate.

**Theorem 1.1.** *Let  $M$  be defined by (1.1). Then for  $(1/p, 1/q) \in \Delta \setminus \{P, Q\}$ , there is a constant  $C$  such that*

$$\|Mf\|_{L^q(\mathbb{R}^2)} \leq C\|f\|_{L^p(\mathbb{R}^2)}.$$

Furthermore  $\|Mf\|_{L^{5,\infty}(\mathbb{R}^2)} \leq C\|f\|_{L^{5/2,1}(\mathbb{R}^2)}$ .

Theorem 1.1 will be proven using some sharp Carleson-Sjölin type estimate for the 2-dimensional wave equation. Let us define

$$U_t f(x) = \int_{\mathbb{R}^2} e^{i\langle x, \xi \rangle + it|\xi|} \widehat{f}(\xi) d\xi.$$

In Section 2, we will show the following  $L^p - L^q$  local smoothing estimates.

**Proposition 1.2.** *If  $\text{supp } \widehat{f} \subset \{\xi \in \mathbb{R}^2 : |\xi| \sim N\}$ , then for  $1/p + 3/q = 1$ ,  $14/3 < q \leq \infty$ ,*

$$(1.2) \quad \left( \int_{\mathbb{R}^2} \int_1^2 |U_t f(x)|^q dt dx \right)^{1/q} \leq CN^{3/2-6/q} \|f\|_{L^p(\mathbb{R}^2)}.$$

This is a slight improvement of the results obtained by Schlag and Sogge [SS], and Tao and Vargas [TV2, section 4]. In particular, the  $\epsilon$ -loss on regularity is removed. For the proof of (1.2) we use the bilinear cone restriction estimate of Wolff [W] and Tao [T] together with a modification of an argument in [TV2, section 4]. Let  $\Gamma = \{(\xi, \tau) \in \mathbb{R}^2 \times \mathbb{R} : |\xi| = \tau, \ 1 \leq \tau \leq 2\}$  and let  $\Omega_1, \Omega_2$  be closed subsets of  $S^1$  with  $\text{dist}(\Omega_1, \Omega_2)$  comparable to 1. Now set  $\Gamma_i = \{(\xi, \tau) \in \Gamma : \xi/\tau \in \Omega_i\}$  for  $i = 1, 2$ . The following is the bilinear cone restriction estimate in  $\mathbb{R}^3$  due to Wolff [W] (for  $r > 5/3$ ) and Tao [T] (for  $r = 5/3$ ):

If  $\text{supp } f \subset \Gamma_1$  and  $\text{supp } g \subset \Gamma_2$ , then for  $r \geq 5/3$

$$(1.3) \quad \|\widehat{f d\mu g d\mu}\|_r \leq C\|f\|_2 \|g\|_2$$

where  $d\mu$  is the surface measure of  $\Gamma$ .

Once Proposition 1.2 has been established, the proof of Theorem 1.1 is straightforward.

*Proof of Theorem 1.1.* It is well-known that

$$\widehat{d\sigma}(\xi) = e^{i|\xi|} a_+(\xi) + e^{-i|\xi|} a_-(\xi)$$

where  $a_{\pm}$  are smooth functions satisfying  $|\partial^\alpha a_{\pm}(\xi)| \leq C_\alpha (1 + |\xi|)^{-1/2-|\alpha|}$ . Therefore, it is sufficient to show that the maximal operator  $\mathcal{M}$  defined by

$$\mathcal{M}f(x) = \sup_{1 < t < 2} \left| \int e^{i\langle x, \xi \rangle + it|\xi|} a_+(t\xi) \widehat{f}(\xi) d\xi \right|$$

satisfies the estimates in Theorem 1.1 in place of  $M$ . Let  $\beta \in C_0^\infty(1/2, 2)$  satisfying  $\sum \beta(\cdot/2^j) = 1$  and let  $\beta_j(\xi) = \beta(|\xi|/2^j)$ . Let  $f_j, f_0$  be defined by  $\widehat{f_j} = \beta_j \widehat{f}$ ,  $f_0 = \sum_{j \leq 0} f_j$ , respectively. Set  $\mathcal{M}_j f = \mathcal{M}f_j$ . Trivially, we have

$$\mathcal{M}f(x) \leq \mathcal{M}f_0(x) + \sum_{j \geq 1} \mathcal{M}_j f(x).$$

It is easy to see that  $\|\mathcal{M}f_0\|_q \leq C\|f\|_p$  for  $1 \leq p \leq q$ , so we only need to consider  $\sum_{j \geq 1} \mathcal{M}_j f$ . Now we need the following well-known lemma.

**Lemma 1.3.** *Let  $I$  be an interval and let  $F$  be a smooth function defined on  $\mathbb{R}^n \times I$ . Then, for  $1 < p \leq \infty$ ,*

$$\left\| \sup_{t \in I} |F(x, t)| \right\|_{L^p(\mathbb{R}^n)} \leq C(\|F\|_{L^p(\mathbb{R}^n \times I)} + \|F\|_{L^p(\mathbb{R}^n \times I)}^{(p-1)/p} \|\partial_t F\|_{L^p(\mathbb{R}^n \times I)}^{1/p}).$$

By Lemma 1.3 and Plancherel's theorem it is easy to see that for  $j \geq 1$ ,

$$(1.4) \quad \|\mathcal{M}_j f\|_2 \leq C\|f\|_2.$$

Let  $I$  be the interval  $[1, 2]$ . Using Lemma 1.3 and Hölder's inequality, we have

$$\|\mathcal{M}_j f\|_q \leq C2^{-j/2} \|U_t f_j^1\|_{L^q(\mathbb{R}^2 \times I)} + C2^{-j/2+j/q} (\|U_t f_j^2\|_{L^q(\mathbb{R}^2 \times I)} + \|U_t f_j^3\|_{L^q(\mathbb{R}^2 \times I)})$$

where the supports of the Fourier transforms of  $f_j^1$ ,  $f_j^2$  and  $f_j^3$  are contained in the set  $\{\xi \in \mathbb{R}^2 : |\xi| \sim 2^j\}$ , and  $\|f_j^1\|_p$ ,  $\|f_j^2\|_p$  and  $\|f_j^3\|_p$  are bounded by  $C\|f\|_p$ . Applying Proposition 1.2 to the last inequality, it is easy to see that for  $1/p + 3/q = 1$  and  $q > 14/3$ ,

$$\|\mathcal{M}_j f\|_q \leq C2^{j(1-5/q)} \|f\|_p.$$

A complex interpolation between this and (1.4) shows that if  $(1/p, 1/q)$  is contained in the closed triangle with vertices  $(1, 0)$ ,  $(5/14, 3/14)$ ,  $(1/2, 1/2)$  but is not on the closed line segment  $[(1/2, 1/2), (5/14, 3/14)]$ , then

$$(1.5) \quad \|\mathcal{M}_j f\|_q \leq C2^{j(3/p-1/q-1)/2} \|f\|_p.$$

Using (1.5) and Lemma 2.6 (in Section 2) with  $n = 1$ , we have for  $(1/p, 1/q) \in [P, Q]$ ,

$$(1.6) \quad \|\mathcal{M} f\|_{q,\infty} \leq C\|f\|_{p,1}.$$

Since  $\mathcal{M}$  is a local operator, an interpolation (real interpolation) between these estimates and the trivial  $L^\infty - L^\infty$  estimate completes the proof of Theorem 1.1.  $\square$

We point out that similar endpoint estimates hold for the spherical maximal function in  $\mathbb{R}^n$ ,  $n \geq 3$ . Set

$$M^n f(x) = \sup_{1 < t < 2} \left| \int_{S^{n-1}} f(x - ty) d\sigma^n(y) \right|$$

where  $d\sigma^n$  is the normalized surface area measure on  $S^{n-1}$ . Set  $P_1 = (0, 0)$ ,  $P_2 = ((n-1)/n, (n-1)/n)$ ,  $P_3 = ((n-1)/n, 1/n)$ ,  $P_4 = ((n^2-n)/(n^2+1), (n-1)/(n^2+1))$ . Let  $\mathcal{Q}$  be the closed quadrangle with vertices  $P_1, P_2, P_3, P_4$ . It was shown in [SS] that  $\|M^n f\|_q \leq C\|f\|_p$  if  $(1/p, 1/q)$  is contained in the interior of  $\mathcal{Q}$  and that these maximal inequalities can never hold if  $(1/p, 1/q)$  is outside of  $\mathcal{Q}$ . Using an argument similar to the one used for the circular maximal function, we can show the following.

**Theorem 1.4.** *Suppose  $M^n$  is defined as in the above for  $n \geq 3$ . Then there is a constant  $C$  such that*

$$\|M^n f\|_{L^q(\mathbb{R}^n)} \leq C\|f\|_{L^p(\mathbb{R}^n)}$$

*if  $(1/p, 1/q)$  is contained in  $\mathcal{Q} \setminus \{P_2, P_3, P_4\}$ . Furthermore,*

$$\|M^n f\|_{L^{q,\infty}(\mathbb{R}^n)} \leq C\|f\|_{L^{p,1}(\mathbb{R}^n)}$$

*if  $(1/p, 1/q) = P_2, P_3, P_4$ .*

*Proof of Theorem 1.4.* Now recall  $\widehat{d\sigma^n}(\xi) = e^{i|\xi|}a_+(\xi) + e^{-i|\xi|}a_-(\xi)$  with  $|\partial_\xi^\alpha a_\pm(\xi)| \leq C_\alpha(1 + |\xi|)^{-(n-1)/2 - |\alpha|}$ . Define

$$\mathcal{M}f(x) = \sup_{1 < t < 2} \left| \int_{\mathbb{R}^n} e^{i\langle x, \xi \rangle + it|\xi|} \frac{\widehat{f}(\xi)}{(1 + |\xi|)^{(n-1)/2}} d\xi \right|.$$

It is sufficient to consider  $\mathcal{M}$  instead of  $M^n$ . As before, by Littlewood-Paley decomposition,  $f = f_0 + \sum_{j \geq 1} f_j$ . Set  $\mathcal{M}_j f = \mathcal{M}(f_j)$ . It can be easily seen that  $\|\mathcal{M}_0 f\|_q \leq C\|f\|_p$  for  $1 \leq p \leq q$ . Theorem 1.4 follows from Lemma 2.6 and the following estimates. There is a constant  $C$  such that for  $j \geq 1$ ,

$$(1.7) \quad \|\mathcal{M}_j f\|_1 \leq C2^j \|f\|_1,$$

$$(1.8) \quad \|\mathcal{M}_j f\|_\infty \leq C2^j \|f\|_1,$$

$$(1.9) \quad \|\mathcal{M}_j f\|_2 \leq C2^{-j \frac{n-2}{2}} \|f\|_2,$$

$$(1.10) \quad \|\mathcal{M}_j f\|_{2(n+1)/(n-1)} \leq C2^{-j \frac{n^2-2n-1}{2n+2}} \|f\|_2.$$

Interpolations (by Lemma 2.6) between (1.7) and (1.9), (1.8) and (1.9), (1.8) and (1.10) give the restricted weak types  $(n/(n-1), n/(n-1))$ ,  $(n/(n-1), n)$ ,  $((n^2+1)/(n^2-n), (n^2+1)/(n-1))$ , respectively. Therefore, we only need to show (1.7), (1.8), (1.9), (1.10).

It is easy to see that (1.8) and (1.9) follow from the fact that if  $\beta \in C_0^\infty(\mathbb{R}^n \setminus 0)$ ,

$$\left| \int_{\mathbb{R}^n} e^{i\langle x, \xi \rangle + it|\xi|} \frac{\beta(\xi/2^j)}{(1 + |\xi|)^{(n-1)/2}} d\xi \right| \leq C \min(2^j, 2^{j(n+1)/2} (1 + 2^j|x| - t)^{-N})$$

for every  $N$ . Using Lemma 1.3 and Plancherel's theorem, we have (1.9). To see (1.10), let  $U_t f(x) = \int_{\mathbb{R}^n} e^{i\langle x, \xi \rangle + it|\xi|} \widehat{f}(\xi) d\xi$  and use Lemma 1.3 to get

$$(1.11) \quad \begin{aligned} \|\mathcal{M}_j f\|_{L^{\frac{2n+2}{n-1}}(\mathbb{R}^n)} &\leq C2^{-j \frac{n-1}{2}} \|U_t f_j^1\|_{L^{\frac{2n+2}{n-1}}(\mathbb{R}^n \times I)} \\ &\quad + C2^{-j \frac{n-1}{2}} 2^{j \frac{n-1}{2n+2}} (\|U_t f_j^2\|_{L^{\frac{2n+2}{n-1}}(\mathbb{R}^n \times I)} + \|U_t f_j^3\|_{L^{\frac{2n+2}{n-1}}(\mathbb{R}^n \times I)}) \end{aligned}$$

where the supports of the Fourier transforms of  $f_j^1, f_j^2$  and  $f_j^3$  are contained in the set  $\{\xi \in \mathbb{R}^n : |\xi| \sim 2^j\}$ , and  $\|f_j^1\|_p, \|f_j^2\|_p$  and  $\|f_j^3\|_p$  are bounded by  $C\|f\|_p$ . By the Strichartz estimates and re-scaling we can see that for  $l = 1, 2, 3$ ,

$$\|U_t f_j^l\|_{L^{\frac{2n+2}{n-1}}(\mathbb{R}^n \times I)} \leq C2^{j/2} \|f\|_2.$$

Putting this into (1.11) yields (1.10).  $\square$

Finally, we mention that an analogue of Theorem 1.4 holds for the maximal operators associated with smoothly varying hypersurfaces, which were considered in [SS] (Theorem 4.1). This can be shown in the same way.

## 2. PROOF OF PROPOSITION 1.2

This section is devoted to the proof of the sharp local smoothing estimate (1.2). It will be deduced from (1.3) and the following proposition. We only need to set  $r_0 = 5/3$  in Proposition 2.1.

**Proposition 2.1.** *Let  $I$  be the interval  $[1, 2]$ . Suppose that (1.3) holds for  $r \geq r_0$ ,  $r_0 < 2$ . Then for every  $f$  with  $\text{supp } \widehat{f} \subset \{\xi \in \mathbb{R}^2 : |\xi| \sim N\}$ ,*

$$(2.1) \quad \|U_t f\|_{L^q(\mathbb{R}^2 \times I)} \leq C N^{3/2-6/q} \|f\|_{L^p(\mathbb{R}^2)}$$

*provided  $(3 + r_0) < q \leq \infty$  and  $3/q + 1/p = 1$ .*

Let  $f$  be a function with  $\text{supp } \widehat{f} \subset \{\xi \in \mathbb{R}^2 : |\xi| \sim N\}$ . Without loss of generality we may assume the support of  $\widehat{f}$  is contained in a small conic neighborhood of the direction  $(1, 0)$ .

To exploit the bilinear estimates, we use a decomposition technique which was used in [TV2, section 4]. For each  $j \geq 1$ , let us dyadically divide the circle  $S^1$  into  $2^j$  arcs  $I_k^j$  of length  $\pi 2^{1-j}$ . We will write  $I_k^j \sim I_{k'}^j$  to mean that  $I_k^j$  and  $I_{k'}^j$  are not adjacent but have adjacent parent arcs of length  $\pi 2^{2-j}$ . Then, by a Whitney decomposition of  $S^1 \times S^1$  away from the diagonal  $D$  of  $S^1 \times S^1$  (see [TVV]), we have

$$S^1 \times S^1 \setminus D = \bigcup_{j \geq 1} \bigcup_{(k, k') : I_k^j \sim I_{k'}^j} I_k^j \times I_{k'}^j.$$

Let  $\psi$  be a smooth function in  $C_0^\infty([1/2, 4])$  satisfying  $\psi(x) = 1$  if  $1 \leq x \leq 2$ . Let  $f_k^j$  be given in polar coordinates by

$$\widehat{f_k^j}(r, \theta) = \widehat{f}(r, \theta) \chi_{I_k^j}(\theta) \psi(r/N).$$

Since the support of  $\widehat{f}$  is contained in a small conic neighborhood of the direction  $(1, 0)$ , we may assume that all the  $I_k^j$  are contained in a small neighborhood of  $(1, 0)$ . Trivially, it follows that

$$U_t f(x) \cdot U_t f(x) = \sum_{j \geq 1} \sum_{(k, k') : I_k^j \sim I_{k'}^j} U_t f_k^j(x) \cdot U_t f_{k'}^j(x).$$

Thus, it is more convenient to consider a bilinear operator than a linear one. For each  $j \geq 1$ , define a bilinear operator  $B_j^N$  by

$$B_j^N(f, g)(x) = \sum_{I_k^j \sim I_{k'}^j} U_t f_k^j(x) \cdot U_t g_{k'}^j(x).$$

We want to compute the operator norm of  $B_j^N$  from  $L^p(\mathbb{R}^2) \times L^p(\mathbb{R}^2)$  to  $L^{q/2}(\mathbb{R}^2 \times I)$ .

**Lemma 2.2.** *Suppose for some  $p, q$  satisfying  $4 \leq q$ ,  $2 \leq p$  and  $1/2 \leq 1/q + 1/p$ , there is a constant  $B$  such that if  $I_k^j \sim I_{k'}^j$ , then*

$$(2.2) \quad \|U_t f_k^j \cdot U_t g_{k'}^j\|_{L^{\frac{q}{2}}(\mathbb{R}^2 \times I)} \leq B \|f_k^j\|_{L^p(\mathbb{R}^2)} \|g_{k'}^j\|_{L^p(\mathbb{R}^2)}.$$

*Then there is a constant  $C$ , independent of  $j$  and  $N$ , such that*

$$(2.3) \quad \|B_j^N(f, g)\|_{L^{\frac{q}{2}}(\mathbb{R}^2 \times I)} \leq C B \|f\|_{L^p(\mathbb{R}^2)} \|g\|_{L^p(\mathbb{R}^2)}.$$

*Proof.* Since the  $I_k^j$  are contained in a small neighborhood of  $(1, 0)$ , it is easy to see that if  $I_k^j \sim I_{k'}^j$ , then  $\text{supp } \widehat{f_k^j} + \text{supp } \widehat{g_{k'}^j}$  is contained in the set  $\{\xi \in \mathbb{R}^2 : \text{dist}(\xi/|\xi|, I_k^j) \leq 2^{4-j}\}$ . From this we can see that the Fourier supports of  $\{U_t f_k^j \cdot U_t g_{k'}^j\}_{(k, k') : I_k^j \sim I_{k'}^j}$  are contained in essentially disjoint rectangles and the overlap among these rectangles is uniformly bounded in  $j$ . Now we use a lemma in [TV2], which can be proven using Plancherel's theorem and a standard argument.

**Lemma 2.3** ([TV2], Lemma 7.1). *Let  $\{R_k\}$  be a collection of rectangles in frequency space such that the dilates  $\{2R_k\}$  are essentially disjoint, and suppose that  $\{F_k\}$  are a collection of functions whose Fourier supports are contained in  $R_k$ . Then for  $1 \leq p \leq \infty$  we have*

$$(\sum_k \|F_k\|_p^{p^*})^{1/p^*} \lesssim \|\sum_k F_k\|_p \lesssim (\sum_k \|F_k\|_p^{p^*})^{1/p^*}$$

where  $p_* = \min(p, p')$  and  $p^* = \max(p, p')$ .

Using this and the observation made above, we have for  $q/2 \geq 2$ ,

$$\|\sum_{I_k^j \sim I_{k'}^j} U_t f_k^j \cdot U_t g_{k'}^j\|_{L^{\frac{q}{2}}(\mathbb{R}^2)} \leq C(\sum_{I_k^j \sim I_{k'}^j} \|U_t f_k^j \cdot U_t g_{k'}^j\|_{L^{\frac{q}{2}}(\mathbb{R}^2)}^{(\frac{q}{2})'})^{1-\frac{2}{q}}$$

with  $C$  independent of  $j$  and  $t$ . After raising both sides to the power  $q/2$ , we integrate on the interval  $I$  to get

$$\|\sum_{I_k^j \sim I_{k'}^j} U_t f_k^j \cdot U_t g_{k'}^j\|_{L^{\frac{q}{2}}(\mathbb{R}^2 \times I)}^{\frac{q}{2}} \leq C \int_I (\sum_{I_k^j \sim I_{k'}^j} \|U_t f_k^j \cdot U_t g_{k'}^j\|_{L^{\frac{q}{2}}(\mathbb{R}^2)}^{(\frac{q}{2})'})^{\frac{q}{2}-1} dt.$$

Since  $q/2 \geq 2$ , by Minkowski's inequality

$$\|B_j^N(f, g)\|_{L^{\frac{q}{2}}(\mathbb{R}^2 \times I)} \leq C(\sum_{I_k^j \sim I_{k'}^j} \|U_t f_k^j \cdot U_t g_{k'}^j\|_{L^{\frac{q}{2}}(\mathbb{R}^2 \times I)}^{(\frac{q}{2})'})^{1-\frac{2}{q}}.$$

From (2.2) we have  $\|B_j^N(f, g)\|_{L^{\frac{q}{2}}(\mathbb{R}^2 \times I)} \leq CB(\sum_{I_k^j \sim I_{k'}^j} \|f_k^j\|_{L^p}^{(\frac{q}{2})'} \|g_{k'}^j\|_{L^p}^{(\frac{q}{2})'})^{1-\frac{2}{q}}$ .

Since there are at most four values of  $k$  associated to each  $k'$ , it follows from Schwarz's inequality and the condition  $1/2 - 1/q \leq 1/p$  that

$$\|B_j^N(f, g)\|_{L^{\frac{q}{2}}(\mathbb{R}^2 \times I)} \leq CB(\sum_k \|f_k^j\|_{L^p}^p)^{1/p} (\sum_k \|g_k^j\|_{L^p}^p)^{1/p}.$$

Now it is sufficient to show  $(\sum_k \|f_k^j\|_{L^p}^p)^{1/p} \leq C\|f\|_{L^p}$  for  $p \geq 2$ . But this follows from Lemma 2.3, because the Fourier supports of  $f_k^j$  are contained in essentially disjoint rectangles.  $\square$

In view of Lemma 2.2, to compute the norm of  $B_j^N$  it is sufficient to consider  $U_t f_k^j \cdot U_t g_{k'}^j$  when  $I_k^j \sim I_{k'}^j$ . This will be done by treating the cases  $2^j > N^{\frac{1}{2}}$  and  $2^j \leq N^{\frac{1}{2}}$  separately. By rotation, we may assume that  $\widehat{f}_k^j$  and  $\widehat{g}_{k'}^j$  are supported on the sets

$$\{\xi \in \mathbb{R}^2 : \xi_1 \sim N, |\xi_2| \sim N2^{-j}\}, \quad \{\xi \in \mathbb{R}^2 : \eta_1 \sim N, |\xi_2| \ll N2^{-j}\},$$

respectively.

First we claim that if  $2^j > N^{\frac{1}{2}}$ , then for  $p \leq q$ ,  $1/p + 3/q = 1$ ,

$$(2.4) \quad \|U_t f_k^j \cdot U_t g_{k'}^j\|_{L^{q/2}(\mathbb{R}^2 \times I)} \leq CN^{(3-12/q)} 2^{-j(1-4/q)} \|f_k^j\|_p \|g_{k'}^j\|_p.$$

To see this, note that  $\widehat{f}_k^j$  is supported in a rectangle of size  $N \times N2^{-j}$ . By dilation  $(\xi_1, \xi_2) \rightarrow (N\xi_1, N2^{-j}\xi_2)$

$$U_t f_k^j(x) = N2^{2j} \int_{\mathbb{R}^2} e^{i\langle x, (N\xi_1, N2^{-j}\xi_2) \rangle + it|(N\xi_1, N2^{-j}\xi_2)|} \beta(\xi_1, \xi_2) \widehat{f}_k^j(N\xi_1, N2^{-j}\xi_2) d\xi$$

where  $\beta \in C_0^\infty([1/2, 4] \times [-2, 2])$  and  $\beta \equiv 1$  on the set  $[1, 2] \times [-1, 1]$ . For  $\xi \in [1/2, 4] \times [-2, 2]$ , we have

$$t|(N\xi_1, N2^{-j}\xi_2)| = tN\xi_1 \sqrt{1 + (2^{-j} \frac{\xi_2}{\xi_1})^2} = tN\xi_1 + O(t2^{-2j}N).$$

Since  $2^j > N^{\frac{1}{2}}$ ,  $e^{iO(t2^{-2j}N)}\beta$  is uniformly contained in  $C_0^\infty([1/2, 4] \times [-2, 2])$ , so it can be expanded in a Fourier series so that  $e^{O(t2^{-2j}N)}\beta = \sum_{l \in \mathbb{Z}^2} C_l(t) e^{i\langle \xi, l \rangle}$  with  $\sum_{l \in \mathbb{Z}^2} |C_l(t)| < M$  uniformly in  $t$ . By re-scaling,  $(\xi_1, \xi_2) \rightarrow (\xi_1/N, 2^j \xi_2/N)$

$$U_t f_k^j(x) = \sum_{l \in \mathbb{Z}^2} C_l(t) \int_{\mathbb{R}^2} e^{i\langle x, \xi \rangle + it\xi_1} e^{i\langle (\xi_1/N, 2^j \xi_2/N), l \rangle} \widehat{f}_k^j(\xi) d\xi.$$

This is essentially a sum of Fourier transforms. We need the following elementary lemma known as Bernstein's inequality.

**Lemma 2.4.** *If  $\widehat{f}$  is supported on a rectangle  $Q$ , then for  $1 \leq p \leq q \leq \infty$*

$$\|f\|_q \leq |Q|^{\frac{1}{p} - \frac{1}{q}} \|f\|_p.$$

Using this, we can easily see that for  $1 \leq p \leq q$ ,

$$\|U_t f_k^j\|_{L^q(\mathbb{R}^2 \times I)} \leq CN^{2(1/p-1/q)} 2^{-j(1/p-1/q)} \|f_k^j\|_p$$

because  $\widehat{f}_k^j$  is supported in a rectangle of size  $N \times N2^{-j}$ . In particular, if one sets  $1/p = 1 - 3/q$ , then

$$\|U_t f_k^j\|_{L^q(\mathbb{R}^2 \times I)} \leq CN^{(3/2-6/q)} 2^{-j(1/2-2/q)} \|f_k^j\|_p$$

because  $2^j \geq N^{1/2}$  and  $q \geq 4$ . The same estimates also hold for  $g_{k'}^j$ . Thus, by Hölder's equality, we can see that (2.4) follows. From Lemma 2.2 and (2.4), it follows that if  $2^j > N^{1/2}$ , then for  $1/p + 3/q = 1$ ,  $p \geq 2$  and  $q \geq 4$ ,

$$(2.5) \quad \|B_j^N(f, g)\|_{L^{q/2}(\mathbb{R}^2 \times I)} \leq CN^{(3-12/q)} 2^{-(1-4/q)j} \|f\|_p \|g\|_p.$$

Now we turn to the case  $2^j \leq N^{\frac{1}{2}}$ . To begin with, we want to show that if  $2^j \leq N^{\frac{1}{2}}$ , then

$$(2.6) \quad \|U_t f_k^j \cdot U_t g_{k'}^j\|_{L^\infty(\mathbb{R}^2 \times I)} \leq CN 2^{-2j} \|f_k^j\|_\infty \|g_{k'}^j\|_\infty.$$

Dividing the arc  $I_k^j$  into  $\sqrt{N}2^{-j}$  sub-arcs  $I_{k,n}^j$ , we decompose  $f_k^j$  into  $\sqrt{N}2^{-j}$  functions  $\{f_{k,n}^j\}_n$  such that the  $\widehat{f}_{k,n}^j$  are supported on rectangles of size  $N \times N^{1/2}$ , whose major directions are  $(1, \theta_n)$ . Then using the same method as in the previous case, for these decomposed functions we linearize the phase  $t|\xi|$  to obtain

$$(2.7) \quad U_t f_k^j(x) = \sum_n \sum_{l \in \mathbb{Z}^2} C_{l,n}(t) \int_{\mathbb{R}^2} e^{i\langle x, \xi \rangle} e^{i(t\xi_1 + t\xi_2 \theta_n)(1 + \theta_n^2)^{-1/2}} e^{i\langle \xi/N, l \rangle} \widehat{f}_{k,n}^j(\xi) d\xi$$

with  $\sum_{l \in \mathbb{Z}^2} |C_{l,n}(t)| < M$  uniformly in  $t, n$ . Indeed, after the dilation  $\xi \rightarrow N\xi$  on the support of  $\widehat{f}_{k,n}^j(\cdot N)$ , we have  $t|(N\xi_1, N\xi_2)| = (tN\xi_1 + tN\xi_2 \theta_n)/\sqrt{1 + \theta_n^2} + N\xi_1 O((\xi_2/\xi_1 - \theta_n)^2)$ . Since the diameter of the angular support of  $\widehat{f}_{k,n}^j(\cdot N)$  is about  $N^{-1/2}$ , we have  $N\xi_1 O((\xi_2/\xi_1 - \theta_n)^2) = O(1)$ . Introducing a cut-off function  $\beta$ , we expand  $\beta e^{iN\xi_1 O((\xi_2/\xi_1 - \theta_n)^2)}$  in a Fourier series  $\sum_{l \in \mathbb{Z}^2} C_{l,n}(t) e^{i\langle \xi, l \rangle}$  and re-scale by  $\xi \rightarrow N\xi$  to get (2.7). Since the number of  $n$  is about  $\sqrt{N}2^{-j}$ , an application of Lemma 2.4 to each  $f_{k,n}^j$  in (2.7) gives

$$\|U_t f_k^j\|_{L^\infty(\mathbb{R}^2 \times I)} \leq C\sqrt{N}2^{-j} \|f_k^j\|_\infty.$$

Trivially, the same estimate holds for  $g_{k'}^j$ . Therefore, (2.6) follows.

Now we want to show that if  $I_k^j \sim I_{k'}^j$ , then

$$(2.8) \quad \|U_t f_k^j \cdot U_t g_{k'}^j\|_{L^r(\mathbb{R}^2 \times I)} \leq CN^{2-3/r} 2^{j(3/r-1)} \|f_k^j\|_2 \|g_{k'}^j\|_2.$$

By re-scaling it suffices to show the following.

**Lemma 2.5.** *Suppose (1.3) holds. If  $\widehat{f}$  and  $\widehat{g}$  are supported on the sets  $\{\xi \in \mathbb{R}^2 : \xi_1 \sim 1, |\xi_2| \sim 2^{-j}\}$  and  $\{\xi \in \mathbb{R}^2 : \xi_1 \sim 1, |\xi_2| \ll 2^{-j}\}$ , respectively, then*

$$\|U_t f \cdot U_t g\|_{L^r(\mathbb{R}^2 \times \mathbb{R})} \leq C 2^{j(3/r-1)} \|f\|_2 \|g\|_2.$$

*Proof.* By Plancherel's theorem, it is sufficient to show

$$\|\widehat{f d\mu g d\mu}\|_r \leq C 2^{-j} 2^{3j/r} \|f\|_2 \|g\|_2$$

whenever  $f$  and  $g$  are supported on the sets  $\{(\xi, \tau) \in \Gamma : \xi_1 \sim 1, |\xi_2| \sim 2^{-j}\}$  and  $\{(\xi, \tau) \in \Gamma : \xi_1 \sim 1, |\xi_2| \ll 2^{-j}\}$ , respectively. To obtain this from the bilinear cone restriction estimate, observe that the cone  $\Gamma$  can be rotated without affecting the estimate (1.3). Now make a coordinate transform given by

$$\eta_1 = \frac{\tau + \xi_1}{\sqrt{2}}, \quad \eta_2 = \frac{\tau - \xi_1}{\sqrt{2}}, \quad \rho = \xi_2.$$

This rotation will move the forward light cone  $\Gamma$  to the cone  $\widetilde{\Gamma}$  which is tangent to the  $\eta_2$ -axis. Note that the defining equation for  $\Gamma$  (i.e.  $\tau^2 = \xi_1^2 + \xi_2^2$ ,  $\tau \sim 1$ ) is transformed to  $2\eta_1\eta_2 = \rho^2$  with  $\eta_1 \sim 1$ . Therefore,  $\widehat{f d\mu}$  is essentially the same as

$$Tf(x, t) = \int_{\widetilde{\Gamma}} e^{i(x, (\eta_1, \rho)) + it\rho^2/2\eta_1} f(\eta, \rho) d\widetilde{\mu}(\eta, \rho)$$

where  $d\widetilde{\mu}$  is the surface measure on  $\widetilde{\Gamma}$ . It is sufficient to consider  $Tf \cdot Tg$  instead of  $\widehat{f d\mu g d\mu}$ . In  $(\eta, \rho)$ -coordinates the conditions imposed on the supports of  $f, g$  should be read as: the supports of  $f, g$  are contained in  $\{(\eta, \rho) \in \widetilde{\Gamma} : \eta_1 \sim 1, \rho \sim 2^{-j}\}$ ,  $\{(\eta, \rho) \in \widetilde{\Gamma} : \eta_1 \sim 1, |\rho| \ll 2^{-j}\}$ , respectively. Now make the change of variables  $\rho \rightarrow 2^{-j}\rho$  to get  $Tf(x, t) = 2^{-j}T(f(\cdot, 2^{-j}\cdot))(x_1, 2^{-j}x_2, 2^{-2j}t)$ . For the bilinear operator

$$Tf(x, t) \cdot Tg(x, t) = 2^{-2j}Tf(\cdot, 2^{-j}\cdot)(x_1, 2^{-j}x_2, 2^{-2j}t) \cdot Tg(\cdot, 2^{-j}\cdot)(x_1, 2^{-j}x_2, 2^{-2j}t)$$

the functions  $f(\cdot, 2^{-j}\cdot)$  and  $g(\cdot, 2^{-j}\cdot)$  have disjoint angular supports with distance comparable to 1, contained in the cone  $\widetilde{\Gamma}$ . Therefore, by the bilinear cone restriction estimate (1.3) we see that  $\|Tf(\cdot, 2^{-j}\cdot) \cdot Tg(\cdot, 2^{-j}\cdot)\|_r \leq C 2^j \|f\|_2 \|g\|_2$ . Now, by re-scaling it follows that

$$\|Tf \cdot Tg\|_r \leq C 2^{-j} 2^{3j/r} \|f\|_2 \|g\|_2.$$

This completes the proof.  $\square$

Since for some fixed  $I_k^j$  and  $I_{k'}^j$ , the expression  $U_t f_k^j \cdot U_t g_{k'}^j$  can be considered as a bilinear operator, an interpolation between (2.6) and (2.8) shows that if  $I_k^j \sim I_{k'}^j$ , then for  $p, q$  satisfying  $q \geq 2r$  and  $1/p = r/q$ ,

$$\|U_t f_k^j U_t g_{k'}^j\|_{q/2} \leq C N^{-6/q+2/p+1} 2^{j(6/q+2/p-2)} \|f_k^j\|_p \|g_{k'}^j\|_p$$

with  $C$  independent of  $I_k^j, I_{k'}^j$ . Therefore, from Lemma 2.2 and the above it follows that if  $2^j \leq N^{1/2}$ , then for  $p, q$  satisfying  $q \geq 2r$ ,  $1/p = r/q$ , and  $1/p + 1/q \geq 1/2$ ,  $p \geq 2$ ,  $q \geq 4$ ,

$$(2.9) \quad \|B_j^N(f, g)\|_{L^{q/2}} \leq C N^{-6/q+2/p+1} 2^{j(6/q+2/p-2)} \|f\|_p \|g\|_p.$$



To sum up the last estimates, we use the following lemma which is a multilinear extension of a result implicit in [B3]. An explicit statement can also be found in [CSW]. We denote by  $L^{p,r}$  the Lorentz spaces.

**Lemma 2.6** (An interpolation lemma). *Let  $\varepsilon_1, \varepsilon_2 > 0$ . Suppose that  $\{T_j\}$  is a sequence of  $n$ -linear (or sublinear) operators such that for some  $1 \leq p_1^i, p_2^i < \infty$ ,  $i = 1, \dots, n$  and  $1 \leq q_1, q_2 < \infty$ ,*

$$\|T_j(f^1, \dots, f^n)\|_{q_1} \leq M_1 2^{\varepsilon_1 j} \prod \|f^i\|_{p_1^i}, \quad \|T_j(f^1, \dots, f^n)\|_{q_2} \leq M_2 2^{-\varepsilon_2 j} \prod \|f^i\|_{p_2^i}.$$

*Then  $T = \sum T_j$  is bounded from  $L^{p_1^1,1} \times \dots \times L^{p_n^1,1}$  to  $L^{q,\infty}$  with*

$$\|T(f^1, \dots, f^n)\|_{L^{q,\infty}} \leq C M_1^\theta M_2^{1-\theta} \prod \|f^i\|_{L^{p_i^1,1}}$$

*where  $\theta = \varepsilon_2/(\varepsilon_1 + \varepsilon_2)$ ,  $1/q = \theta/q_1 + (1-\theta)/q_2$ ,  $1/p^i = \theta/p_1^i + (1-\theta)/p_2^i$ .*

*Proof of Lemma 2.6.* Let  $N \in \mathbb{Z}$ , which will be chosen later. Let  $E_1, \dots, E_n$  be measurable sets and let  $\lambda > 0$ . Set  $T_N = \sum_{-\infty}^N T_j$  and  $T^N = \sum_{N+1}^\infty T_j$ . Note

$$\|T_N\|_{L^{p_1^1} \times \dots \times L^{p_n^1} \rightarrow L^{q_1}} \leq C M_1 2^{N\varepsilon_1}, \quad \|T_N\|_{L^{p_2^1} \times \dots \times L^{p_n^2} \rightarrow L^{q_2}} \leq C M_2 2^{-N\varepsilon_2}$$

and  $|\{x : |T(\chi_{E_1}, \dots, \chi_{E_n})(x)| > \lambda\}| \leq |\{x : |T_N(\chi_{E_1}, \dots, \chi_{E_n})(x)| > \frac{1}{2}\lambda\}| + |\{x : |T^N(\chi_{E_1}, \dots, \chi_{E_n})(x)| > \frac{1}{2}\lambda\}|$ . By Tchebyshev's inequality, the measure of the set  $\{x : |T(\chi_{E_1}, \dots, \chi_{E_n})(x)| > \lambda\}$  is bounded above by

$$C(M_1^{q_1} 2^{\varepsilon_1 N q_1} \prod |E_i|^{q_1/p_1^i} \lambda^{-q_1} + M_2^{q_2} 2^{-\varepsilon_2 N q_2} \prod |E_i|^{q_2/p_2^i} \lambda^{-q_2}).$$

Now choosing  $N$  which optimizes this yields

$$|\{x : |T(\chi_{E_1}, \dots, \chi_{E_n})(x)| > \lambda\}| \leq C(M_1^\theta M_2^{1-\theta} \prod |E_i|^{1/p^i} \lambda^{-1})^q.$$

This completes the proof.  $\square$

Using Lemma 2.6 and (2.9), we see that if  $1/p + 3/q = 1$  and  $q = r + 3$  (note that  $4 < q \leq 5$  and the conditions imposed on  $p, q$  in (2.9) are satisfied),

$$(2.10) \quad \left\| \sum_{2^j \leq N^{1/2}} B_j^N(f, g) \right\|_{L^{q/2, \infty}(\mathbb{R}^2 \times I)} \leq C N^{3-12/q} \|f\|_{p,1} \|g\|_{p,1}.$$

Indeed, observe that in (2.9) the exponent on  $2^j$  is negative if  $3/q + 1/p > 1$ , and positive if  $3/q + 1/p < 1$ . Use Lemma 2.6, and solve the conditions  $1/p + 3/q = 1$  and  $1/p = r/q$  to get (2.10) (which is an estimate at the point of intersection of these two lines). Since (1.3) holds for  $r_0 \leq r$ , it follows that for  $1/p + 3/q = 1$  and  $r_0 + 3 \leq q \leq 5$ ,

$$\left\| \sum_{2^j \leq N^{1/2}} B_j^N(f, g) \right\|_{L^{q/2, \infty}(\mathbb{R}^2 \times I)} \leq C N^{3-12/q} \|f\|_{p,1} \|g\|_{p,1}.$$

Since  $q/2 > 2$ , by real interpolation for bilinear operators between these estimates (note that  $r_0 + 3 < 5$ ), the  $L^{q/2, \infty}$ -norm in the left-hand side can be replaced by an  $L^{q/2}$ -norm (see [BL], Exercise 3.13.5). Therefore we have that for  $1/p + 3/q = 1$  and  $r_0 + 3 < q < 5$ ,

$$\left\| \sum_{2^j \leq N^{1/2}} B_j^N(f, g) \right\|_{L^{q/2}(\mathbb{R}^2 \times I)} \leq C N^{3-12/q} \|f\|_{p,1} \|g\|_{p,1}.$$

Recall that  $(U_t f)^2 = \sum_{j \geq 1} B_j^N(f, f)$ . Since  $q > 4$ , by the last inequality and (2.5) we have

$$\begin{aligned} \|U_t f\|_q^2 &\leq \left\| \sum_{2^j > N^{1/2}} B_j^N(f, f) \right\|_{q/2} + \left\| \sum_{2^j \leq N^{1/2}} B_j^N(f, f) \right\|_{q/2} \\ &\leq C N^{(3-12/q)} \left( \sum_{2^j \geq N^{1/2}} 2^{-(1-4/q)j} + 1 \right) \|f\|_{p,1}^2 \\ &\leq C N^{(3-12/q)} \|f\|_{p,1}^2 \end{aligned}$$

provided that  $1/p + 3/q = 1$  and  $r_0 + 3 < q < 5$ . This can be interpolated (via real interpolation) with the trivial estimate  $\|U_t f\|_{L^\infty(\mathbb{R} \times I)} \leq C N^{3/2} \|f\|_1$  to replace the  $L^{p,1}$ -norm by an  $L^p$ -norm. This completes the proof of Proposition 2.1.

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