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KODAIRA DIMENSION OF SYMMETRIC POWERS

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ABSTRACT. We compute the plurigenera and the Kodaira dimension of the dth symmetric power S^dX of a smooth projective variety X. As an application we obtain genus estimates for the curves lying on X.

We work over the complex numbers. When X is a smooth projective curve of genus g, elementary arguments show that the dth symmetric power S^dX is uniruled as soon as $d > g$, and therefore that the plurigenera vanish. When the dimension of X is greater than one, the situation is quite different. Using ideas of Roitman [Ro1, Ro2] and Reid [Re], we prove:

Theorem 1. *Let X be smooth projective variety with $n = dimX > 1$. Let Σ_d be a desingularization of S^dX. Then there are isomorphisms*

$$S^dH^0(X, \omega_X^{\otimes m}) \cong H^0(\Sigma_d, \omega_{\Sigma_d}^{\otimes m})$$

whenever mn is even.

Corollary 1. *With the previous assumptions, the mth plurigenus*

$$P_m(\Sigma_d) = \left(\begin{array}{c} d + P_m(X) - 1 \\ d \end{array} \right)$$

whenever mn is even. The Kodaira dimension $\kappa(\Sigma_d) = d\kappa(X)$.

Proof. The first formula is an immediate consequence of the theorem. It implies that

$$P_m(\Sigma_d) = O(P_m(X)^d) = O(m^{d\kappa(X)})$$

which yields the second formula. □

(D. Huybrechts pointed out to us that these statements were already known in the case $dimX = 2$ and were used to compute the Kodaira dimension of the Hilbert scheme of d points on X; see [HL], theorem 11.1.2.)

Recall that a projective variety Z is uniruled provided there exists a variety Z' and dominant rational map $Z' \times \mathbb{P}^1 \dashrightarrow Z$ which is nontrivial on the second factor. The reference [K] is more than adequate for standard properties of uniruled varieties.

Corollary 2. *If X has nonnegative Kodaira dimension, then S^dX is not uniruled for any d.*

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Proof. Since uniruledness is a birational property, it is enough to observe that Σ_d is not uniruled because it has nonnegative Kodaira dimension. □

The most interesting corollaries involve genus estimates for curves lying on X. The phrase "d general points of X lie on an irreducible curve with genus g normalization" will mean that there is an irreducible quasiprojective family $\mathcal{C} \to T$ of smooth projective genus g curves and a morphism $\mathcal{C} \to X$ which is a generically one-to-one on the fibers $\mathcal{C}_t$ and such that the morphism from the relative symmetric power

$$\mathcal{S}^d\mathcal{C} := \mathcal{C} \times_T \mathcal{C} \times_T \ldots \mathcal{C}/S_d$$

to S^dX is dominant.

Corollary 3. *Suppose that the Kodaira dimension of X is nonnegative and that d general points lie on an irreducible curve with genus g normalization. Then $g \geq d$.*

Proof. Assume the contrary that $g < d$, and let $\mathcal{C} \to T$ be the corresponding family. Then each fiber $\mathcal{S}^d\mathcal{C}_t$ is a projective space bundle over the Jacobian $J(\mathcal{C}_t)$ by Abel-Jacobi; in particular, it is uniruled. Therefore $\mathcal{S}^d\mathcal{C}$ and hence S^dX are uniruled, but this contradicts the previous corollary. □

Corollary 4. *Suppose that X has general type and that d general points lie on an irreducible curve with genus g normalization. Then $g > d$.*

Proof. We assume that $g \leq d$ for some family $\mathcal{C} \to T$. By the previous corollary, we may suppose that $g = d$. Denote the maps $\mathcal{S}^d\mathcal{C} \to S^dX$ and $\mathcal{S}^d\mathcal{C} \to T$ by p and π, respectively. The map p is dominant and generically injective on the fibers of π. If a general fiber $p^{-1}(Z)$ has positive dimension, then an irreducible hyperplane section $H \subset T$ meets $\pi(p^{-1}(Z))$. Therefore $\mathcal{S}^d\mathcal{C} \times_T H \to S^dX$ is still dominant, and we may replace T by H and $\mathcal{S}^d\mathcal{C}$ by the fiber product. By continuing in this way, we can assume that p is generically finite. Choose a desingularization $\overline{T}$ of a compactification of T, and a nonsingular compactification of S of $\mathcal{S}^d\mathcal{C} \times_T \overline{T}$ such that p extends to a morphism of S to a desingularization Σ_d of S^dX. We then have $\kappa(\Sigma_d) \leq \kappa(S)$ which implies that S has general type. On the other hand the general fiber of $S \to \overline{T}$ is $\mathcal{S}^d\mathcal{C}_t$ is birational to an Abelian variety by the theorems of Abel and Jacobi. This implies that

$$\kappa(S) \leq dim\overline{T} + \kappa(\mathcal{S}^d\mathcal{C}_t) = dim\overline{T} < dimS$$

by [Mo, 2.3], but this is impossible since S has general type. □

1. Proof of the main theorem

Recall [Re] that a variety Y has canonical singularities provided that

(1) Y is normal.
(2) $\omega_Y^{[r]} := (\omega_Y^{\otimes r})^{**}$ is locally free for some $r > 0$, where $\omega_Y = (\Omega_Y^{dimY})^{**}$.
(3) If $f : Y' \to Y$ is a resolution of singularities, then $f_*\omega_{Y'}^{\otimes r} = \omega_Y^{[r]}$.

The smallest such r is called the index. If Y is canonical, then the third condition holds for all $r \geq 1$ [Re, 1.3]. Thus the index is the smallest r for which $\omega_Y^{[r]}$ is locally free. It is enough to test the last condition for a particular resolution of singularities. This condition is equivalent to a more widely used condition involving pullbacks of canonical divisors.

Lemma 1. *Let Z be a smooth variety on which a finite group G acts. Let $Y = Z/G$. If Y has canonical singularities of index dividing r, then*

$$H^0(Y, \omega_Y^{[r]}) \subseteq H^0(Z, \omega_Z^{\otimes r})^G.$$

If the fix point locus has codimension greater than one, equality holds.

Proof. Construct a commutative diagram

$$\begin{tikzcd} Z' \arrow[r] \arrow[d, "g'"'] & Z \arrow[d, "g"'] \\ Y' \arrow[r, "f"] & Y \end{tikzcd}$$

where $Y' \to Y$ is a desingularization, and Z' is a G-equivariant desingularization of the fiber product. Then there are inclusions

$$\omega_Y^{[r]} = f_* \omega_{Y'}^{\otimes r} \subseteq (f \circ g')_* (\omega_{Z'}^{\otimes r})^G = g_*(\omega_Z^{\otimes r})^G.$$

This implies the first part of the lemma. The inclusion $\omega_Y^{[r]} \subseteq g_*(\omega_Z^{\otimes r})^G$ is an equality on the complement of the fixed point locus. Since $\omega_Y^{[r]}$ is locally free (hence reflexive) and $g_*(\omega_Z^{\otimes r})^G$ is torsion free, the second statement follows. □

Proposition 1. *Let X be a smooth variety of dimension $n > 1$. Then $S^d X$ has canonical singularities of index 1 if n is even, and canonical singularities of index at most 2 if n is odd.*

Proof. As the result is local analytic for X, we may replace it by $\mathbb{C}^n$. Consider the action of the symmetric group S_d on $\mathbb{C}^{nd} = \mathbb{C}^n \times \ldots \times \mathbb{C}^n$ by permutation of factors, and let $h : S_d \to GL_{nd}(\mathbb{C})$ be the corresponding homomorphism. This action is equivalent to a direct sum of n copies of the standard representation $\mathbb{C}^d$ where S_d acts via permutation matrices. Therefore $h(S_d)$ does not contain any quasi-reflections (because $n > 1$) and $det(h(\sigma)) = sign(\sigma)^n$. When n is even, $h(S_d) \subset SL_{nd}(\mathbb{C})$. This implies that $S^d X = \mathbb{C}^{nd}/S_d$ is Gorenstein by [W], and therefore canonical of index one [Re, 1.8].

The case when n is odd is more laborious. For any element $\sigma \in S_d$ of order r, define $S(\sigma)$ as follows: choose a primitive rth root of unity ϵ and express the eigenvalues of $h(\sigma)$ as $\lambda_i = \epsilon^{a_i}$ where $0 \le a_i < r$, set $S(\sigma) = \sum a_i$. By [Re, 3.1], to prove that $S^d X$ is canonical it will suffice to verify that $S(\sigma) \ge r$ for every element σ of order r. Let $\mathbb{C}^d$ be the permutation representation of S_d. If $e_1, \ldots, e_d$ is the standard basis then $\sigma \cdot e_i = e_{\sigma(i)}$. If ϵ is a primitive rth root of unity, then it is easy to see that the eigenvectors of the cycle $\sigma = (12\ldots r)$ acting on $\mathbb{C}^d$ are $e_1 + \epsilon^i e_2 + \ldots + \epsilon^{(r-1)i} e_r$ and $e_{r+1}, \ldots, e_d$. Therefore the nonunit eigenvalues are $\epsilon, \ldots, \epsilon^{r-1}$ and these occur with multiplicity one. Hence $S(\sigma) = nr(r-1)/2 \ge r$ as required. The general case is similar. Let σ be a permutation of order r and ϵ as before. Write σ as a product of disjoint cycles of length r_i. Therefore r is the least common multiple of the r_i, and let $r_i' = r/r_i$. A list (with possible repetitions) of the nonunit eigenvalues of σ acting on $\mathbb{C}^d$ is

$$\epsilon^{r_1'}, \ldots, \epsilon^{r_1'(r_1-1)}, \epsilon^{r_2'}, \ldots, \epsilon^{r_2'(r_2-1)}, \ldots.$$

Therefore

$$S(\sigma) = \frac{n}{2}[r_1' r_1 (r_1 - 1) + r_2' r_2 (r_2 - 1) + \ldots] \ge r$$

and this proves that S^dX is canonical. It remains to check that the index is at most 2. For this it suffices to observe that if x_i are coordinates on $\mathbb{C}^{nd}$, then

$$(dx_1 \wedge \ldots \wedge dx_{nd})^{\otimes 2}$$

is S_d invariant. This determines a generator of $\omega^{[2]}_{S^dX}(X)$, which shows that this module is free. $\square$

Proof of the main theorem. Let m be an integer such that mn is even (hence a multiple of the index of S^dX). Then

$$H^0(\omega^{\otimes m}_{\Sigma_d}) = H^0(\omega^{[m]}_{S^dX}) = H^0(\omega^{\otimes m}_{X^d})^{S_d}.$$

By Künneth's formula, this equals

$$[H^0(\omega^{\otimes m}_X) \otimes \ldots \otimes H^0(\omega^{\otimes m}_X)]^{S_d} = S^dH^0(\omega^{\otimes m}_X).$$

$\square$

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