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# GEVREY VECTORS OF MULTI-QUASI-ELLIPTIC SYSTEMS

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ABSTRACT. We show that the multi-quasi-ellipticity is a necessary and sufficient condition for the property of elliptic iterates to hold for multi-quasi-homogenous differential operators.

## 1. INTRODUCTION

Let $P_j(x,D) = \sum_\alpha a_{j\alpha}(x) D^\alpha, j = 1,\dots,N$, henceforth denoted $(P_j)_{j=1}^N$, be linear differential operators with $C^\infty$ coefficients in an open subset $\Omega$ of $\mathbb{R}^n$.

The aim of this work is to prove the property of elliptic iterates for multi-quasi-elliptic systems of differential operators in generalized Gevrey spaces $G^{\mathcal{F},s}(\Omega)$, where $\mathcal{F}$ denotes Newton's polyhedron of the system $(P_j)_{j=1}^N$. The property of elliptic iterates for the system $(P_j)_{j=1}^N$ in the generalized Gevrey classes $G^{\mathcal{F},s}(\Omega)$ means the following inclusion:

$$G^s\left(\Omega,(P_j)_{j=1}^N\right) \subset G^{\mathcal{F},s}(\Omega).$$

**Definition 1.** Newton's polyhedron of the system $(P_j)_{j=1}^N$ at the point $x_0 \in \Omega$, denoted $\mathcal{F}(x_0)$, is the convex hull of the set $\{\alpha \in \mathbb{N}^n, \exists j \in \{1,\dots,N\}\,;\, a_{j\alpha}(x_0) \neq 0\}$. A Newton's polyhedron $\mathcal{F}$ is said to be regular if there exists a finite set $Q(\mathcal{F}) \subset (\mathbb{R}_+^*)^n$ such that

$$\mathcal{F}=\bigcap_{q\in Q(\mathcal{F})}\left\{\alpha \in \mathbb{R}_+^n, \langle \alpha, q\rangle \leq 1\right\}.$$

Set

$$\begin{aligned} k(\alpha,\mathcal{F}) &= \inf\left\{t>0,\ t^{-1}\alpha \in \mathcal{F}\right\},\ \alpha \in \mathbb{R}_+^n, \\ \mu(\mathcal{F}) &= \max_{1\leq j\leq n} \mu_j(\mathcal{F}), \\ \mu_j(\mathcal{F}) &= \max_{q\in Q(\mathcal{F})} q_j^{-1},\ j=1,\dots,n, \\ \theta(\mathcal{F}) &= \left(\frac{\mu(\mathcal{F})}{\mu_1(\mathcal{F})},\dots,\frac{\mu(\mathcal{F})}{\mu_n(\mathcal{F})}\right). \end{aligned}$$

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**Definition 2.** Let $\mathcal{F}$ be a regular Newton's polyhedron and $s \in \mathbb{R}_+$. We define the generalized Gevrey space $G^{\mathcal{F},s}(\Omega)$ by the space of $u \in C^\infty(\Omega)$ such that $\forall H$ *compact of* $\Omega, \exists C > 0, \forall \alpha \in \mathbb{N}^n$,

$$\sup_H |D^\alpha u| \leq C^{|\alpha|+1} \left[\Gamma\left(\mu(\mathcal{F}) k(\alpha, \mathcal{F}) + 1\right)\right]^s, \tag{1.1}$$

where $\Gamma$ is the gamma function.

*Remark* 1. One can take $\sup_H |D^\alpha u|$ or $\|D^\alpha u\|_{L^2(H)}$ in the definition, according to Sobolev imbedding theorems.

**Definition 3.** The system $(P_j)_{j=1}^N$ is said to be multi-quasi-elliptic in $\Omega$ if
1) The $\mathcal{F}(x)$ do not depend on $x \in \Omega$, i.e. $\forall x, \mathcal{F}(x) = \mathcal{F}$.
2) $\mathcal{F}$ is regular.
3) $\forall x \in \Omega, \exists C > 0, \exists R \geq 0, \forall \xi \in \mathbb{R}^n, |\xi| \geq R$,

$$\sum_{j=1}^N |P_j(x,\xi)| \geq C \sum_{\alpha \in \mathbb{Z}_+^n \cap \mathcal{F}} |\xi^\alpha|.$$

**Definition 4.** Let $(P_j)_{j=1}^N$ be a system of linear differential operators satisfying conditions 1) and 2) of Definition 3 and $s \in \mathbb{R}_+$, the space of Gevrey vectors of the system $(P_j)_{j=1}^N$, denoted $G^s\left(\Omega, (P_j)_{j=1}^N\right)$, is the space of $u \in C^\infty(\Omega)$ such that $\forall H$ *compact of* $\Omega, \exists C > 0, \forall l \in \mathbb{N}, 1 \leq i_l \leq N$,

$$\|P_{i_1} \dots P_{i_l} u\|_{L^2(H)} \leq C^{l+1} (l!)^{s\mu(\mathcal{F})}. \tag{1.2}$$

The aim of this work is to show the following theorem.

**Theorem 1.** *Let $\Omega$ be an open subset of $\mathbb{R}^n, \sigma > s \geq 1$ and $(P_j)_{j=1}^N$ be a system of linear differential operators with $G^{\theta(\mathcal{F}),\sigma}(\Omega)$ coefficients. Then*

$$(P_j)_{j=1}^N \text{ is multi-quasi-elliptic in } \Omega \Longleftrightarrow G^s\left(\Omega, (P_j)_{j=1}^N\right) \subset G^{\mathcal{F},s}(\Omega).$$

Some consequences of this theorem are given in section 4. For differential operators with constant coefficients we have shown in [3] a more general result.

## 2. Sufficient condition

The proof of the sufficient condition follows essentially the work of Zanghirati [6], so we refer for details to this paper.

Instead of $Q(\mathcal{F}), k(\mathcal{F},\alpha), \mu(\mathcal{F}), \theta(\mathcal{F})$ we write, respectively, $Q, k(\alpha), \mu, \theta$. Denote $\mathcal{K} = \{k = k(\alpha) : \alpha \in \mathbb{N}^n\}$. If $\omega$ is an open subset of $\mathbb{R}^n$, $u \in C^\infty(\omega)$ and $k \in \mathcal{K}$, define $|u|_{k,\omega} = \sum\limits_{k(\alpha)=k} \|D^\alpha u\|_{L^2(\omega)}$. When $u \in C_0^\infty(\mathbb{R}^n)$ we write $|u|_k$.

Let $(P_j)_{j=1}^N$ be a system of linear differential operators with coefficients defined in an open neighborhood $\Omega$ of the origin satisfying the following conditions:
(i) The system $(P_j)_{j=1}^N$ is multi-quasi-elliptic in $\Omega$.
(ii) The coefficients $a_{j\alpha} \in G^{\theta,s}(\Omega), \forall \alpha \in \mathcal{F}, \forall j \in \{1,\dots,N\}$.

For $\rho > 0$, we denote $B_\rho = \{x \in \mathbb{R}^n, \sum\limits_{j=1}^n x_j^{2\mu_j/\mu} < \rho^2\}$. We define for $h \in \mathbb{N}$,

$$P_j^h(x,D) = \underbrace{P_j(x,D) \circ \dots \circ P_j(x,D)}_{h \text{ times}}, \ j = 1,\dots,N.$$

From the multi-quasi-ellipticity of the system $(P_j)_{j=1}^N$ and following the proof of Lemma 3.4 of [6], we obtain

**Lemma 1.** *There exist* $\rho_0 > 0$ *and* $C_1 > 0, \forall \varepsilon \in ]0, \frac{1}{v(n)}[$ *(*$v(n)$ *denote the number of elements of* $\mathcal{K} \cap [0, n[$*),* $\exists C_2(\varepsilon) > 0, \forall \delta \in ]0,1[, \forall \rho > 0, B_{\rho+\delta} \subset B_{\rho_0}, \forall u \in C^\infty(B_{\rho_0}), \forall p \geq n$,

(2.1)

$$|u|_{p+1,B_\rho} \leq C_1 \left( \sum_{j=1}^N \left| P_j^n(x,D)u \right|_{p-n+1,B_{\rho+\delta}} + \varepsilon \left| u \right|_{p+1,B_{\rho+\delta}} + (\varepsilon\delta)^{-n\mu} \left| u \right|_{p-n+1,B_{\rho+\delta}} + \sum_{h=0}^p \left( \frac{(p+1)!}{h!} \right)^{s\mu} C_2(\varepsilon)^{p+1-h} \left| u \right|_{h,B_{\rho+\delta}} \right),$$

*and for* $p \leq n$*, we have*

(2.2)

$$|u|_{p+1,B_\rho} \leq C_1 \left( \sum_{j=1}^N \left| P_j^n(x,D)u \right|_{p-n+1,B_{\rho+\delta}} + \varepsilon \left| u \right|_{p+1,B_{\rho+\delta}} + (\varepsilon\delta)^{-(p+1)\mu} \left| u \right|_{0,B_{\rho+\delta}} \right).$$

Let $\lambda > 0$ and $R > 0$. For $p \in \mathbb{N}$, we set

$$\sigma_p(u,\lambda) = (p!)^{-s\mu} \lambda^{-p} \sup_{R/2 \leq \rho < R} (R-\rho)^{p\mu} \left| u \right|_{p,B_\rho}.$$

**Lemma 2.** *Let* $\rho_0$ *be as in the Lemma* 1 *and let* $0 < R < 1$ *such that* $\overline{B}_R \subset B_{\rho_0}$*. Then there exists* $\lambda_0 > 0$ $\left(\lambda_0 \textit{ depends only on } R \textit{ and } (P_j)_{j=1}^N\right)$, $\forall u \in C^\infty(B_{\rho_0})$, $\forall \lambda \geq \lambda_0, \forall p \geq n$,

(2.3)

$$\sigma_{p+1}(u,\lambda) \leq \left[ (p-n+2)\dots(p+1) \right]^{-s\mu} \sum_{j=1}^N \sigma_{p-n+1}\left(P_j^n u, \lambda\right) + \sum_{h=0}^p \sigma_h(u,\lambda),$$

*and for* $p \leq n-1$,

$$\sigma_{p+1}(u,\lambda) \leq (p+1)!^{-s\mu} \sum_{j=1}^N \sigma_0\left(P_j^{p+1} u, \lambda\right) + \sigma_0(u,\lambda). \tag{2.4}$$

*Proof.* Let $p \geq n$, multiply both sides of (2.1) by $(p+1)!^{-s\mu} \lambda^{-p-1} (R-\rho)^{p\mu}$, put $\delta = \frac{R-\rho}{p-n+2}$ and then taking the sup over $\rho \in [R/2, R[$, we obtain

$$\sigma_{p+1}(u,\lambda) \leq C_1 \left( I_1 + \varepsilon I_2 + \varepsilon^{-n\mu} I_3 + I_4 \right),$$

where $I_1, I_2, I_3$ and $I_4$ are such that

$$\begin{aligned}
I_1 &\leq \sum_{j=1}^N \left( \frac{(p-n+1)!}{(p+1)!} \right)^{s\mu} \frac{e^\mu}{\lambda^n} \sigma_{p-n+1}\left(P_j^n u, \lambda\right), \\
I_2 &\leq (2^n e)^\mu \sigma_{p+1}(u,\lambda), \\
I_3 &\leq \frac{e^\mu}{\lambda^n} \sigma_{p-n+1}(u,\lambda), \\
I_4 &\leq \frac{e^\mu C_2(\varepsilon)}{\lambda} \sum_{h=0}^p \left( \frac{C_2(\varepsilon)}{\lambda} \right)^{p-h} \sigma_h(u,\lambda).
\end{aligned}$$

By a suitable choice of $\varepsilon$, we find

$$\sigma_{p+1}(u,\lambda) \leq \left(\frac{(p-n+1)!}{(p+1)!}\right)^{s\mu} \frac{\widetilde{C}_1}{\lambda^n}\sum_{j=1}^{N}\sigma_{p-n+1}\left(P_j^n u,\lambda\right)+\frac{\widetilde{C}_2}{\lambda^n}\sigma_{p-n+1}(u,\lambda) + \frac{\widetilde{C}_3}{\lambda}\sum_{h=0}^{p}\left(\frac{\widetilde{C}_4}{\lambda}\right)^{p-h}\sigma_h(u,\lambda).$$

It suffices to take $\lambda_0=\widetilde{C}_1+\widetilde{C}_2+\widetilde{C}_3+\widetilde{C}_4$ to get $(2.3)$. For the inequality (2.4) we multiply both sides of inequality (2.2) by $\frac{(R-\rho)^{(p+1)\mu}}{(p+1)!^{s\mu}\lambda^{p+1}}$, take $\delta=\frac{R-\rho}{2}$ and then we follow the same procedure for obtaining (2.3). $\square$

**Lemma 3.** *Let $\rho_0, R$ and $\lambda_0$ be as in Lemma 2. Then for any $u\in C^\infty(B_{\rho_0})$, $\forall\lambda\geq\lambda_0$, $\forall p\in\mathbb{N}$, we have*

(2.5)

$$\sigma_{p+1}(u,\lambda)\leq 2^{p+1}\sigma_0(u,\lambda)+\sum_{l=1}^{p+1}2^{p+1-l}C_{p+1}^{l}\frac{1}{(l!)^{s\mu}}\sum_{1\leq i_1,\ldots,i_l\leq N}\sigma_0\left(P_{i_1}\ldots P_{i_l}u,\lambda\right).$$

*Proof.* It is obtained by recurrence over $p$. $\square$

Our first result is the following theorem, wich generalizes the results of [6], [7] and [8] to systems.

**Theorem 2.** *Let $\Omega$ be an open subset of $\mathbb{R}^n$, $s\geq 1$ and $(P_j(x,D))_{j=1}^N$ be a system of linear differential operators with $G^{\theta,s}(\Omega)$ coefficients. Then*

$$(P_j)_{j=1}^N \text{ is multi-quasi-elliptic in } \Omega \Rightarrow G^s\left(\Omega,(P_j)_{j=1}^N\right)\subset G^{\mathcal{F},s}(\Omega).$$

*Proof.* It is sufficient to check (1.1) in a neighborhood of every point $x$ of $\Omega$. Let us assume $x$ is the origin. Then there exist $\rho_0,\lambda_0$ and $R$ such that the precedent lemmas hold. Let $u\in G^s(\Omega,(P_j)_{j=1}^N)$. Then there is $C_1>0$ such that

$$\sigma_0\left(P_{i_1}\ldots P_{i_l}u,\lambda_0\right)\leq C_1^{l+1}(l!)^{s\mu},\ \forall l\in\mathbb{N},$$

hence from $(2.5)$, we obtain

$$\sigma_{p+1}(u,\lambda_0)\leq C_1(2+NC_1)^{p+1},\ \forall p\in\mathbb{N},$$

which gives

$$|u|_{p+1,B_{R/2}}\leq (p+1)!^{s\mu}C_2^{(p+1)\mu+1},\forall p\in\mathbb{N}. \tag{2.6}$$

Following the same steps as in [6] we obtain

$$|u|_{k,B_{R/2}}\leq\widetilde{C}^{k\mu+1}\left(\Gamma(k+1)\right)^{s\mu}. \tag{2.7}$$

Consequently from (2.7) it is easy, as in [6], to obtain the estimate $(1.1)$. $\square$

## 3. Necessary condition

In this section we prove the converse of Theorem 2. For this aim we need a characterization of the multi-quasi-ellipticity of the system $(P_j(x,D))_{j=1}^N$, known in the case of a scalar operator; see [4].

**Proposition 1.** *A system $(P_j)_{j=1}^N$, satisfying 1) and 2) of Definition 2, is multi-quasi-elliptic in $\Omega$ if and only if for any $x \in \Omega$, $\forall q \in Q$,*

$$\sum_{j=1}^{N} |P_{jq}(x,\xi)| \neq 0, \quad \forall \xi \in \mathbb{R}^n, \xi_1 \dots \xi_n \neq 0\,,$$

*where $P_{jq}$ is the $q$-quasi-homogenous part of $P_j$, i.e.*

$$P_{jq}(x,\xi) = \sum_{\langle \alpha, q\rangle = 1} a_{j\alpha}(x)\,\xi^{\alpha}.$$

**Theorem 3.** *Let $\Omega$ be an open subset of $\mathbb{R}^n$ and $P_j(x,D)$, $j = 1,\dots,N$, be differential operators with $G^{\theta,\sigma}(\Omega)$ coefficients. If $s > \sigma \geq 1$, then*

$$G^{s}\left(\Omega, (P_j)_{j=1}^N\right) \subset G^{\mathcal{F},s}(\Omega) \Rightarrow (P_j)_{j=1}^N \text{ is multi-quasi-elliptic in } \Omega.$$

*Proof.* Assume that the system $(P_j)_{j=1}^N$ is not multi-quasi-elliptic. Then there exist $x_0 \in \Omega, q \in Q$ and $\xi_0 \in S^{n-1}, \xi_{0,1} \dots \xi_{0,n} \neq 0$, such that

$$P_{jq}(x_0,\xi_0) = 0, \quad \forall j = 1,\dots,N. \tag{3.1}$$

We construct a function $u \in G^s(\Omega, (P_j)_{j=1}^N)$ such that $u \notin G^{\mathcal{F},s}(\Omega)$, which contradicts the hypothesis. Put $\eta = \frac{1-\varepsilon/\mu}{\mu s}$, and choose $\varepsilon$ satisfying

$$0 < \varepsilon \leq \frac{\mu(s-\sigma)}{2\mu s - \sigma} < \frac{1}{2} \text{ and } \varepsilon < \min_{\langle\beta,q\rangle<1} \mu\left(1 - \langle\beta,q\rangle\right).$$

Let $\delta > 0$ such that the ball $B_0 = B(x_0, 2\delta)$ is relatively compact in $\Omega$ and $\varphi \in G^{q,\sigma\mu}(\mathbb{R}^n)$ with compact support in $B(0,2\delta)$ and $\varphi(x) \equiv 1$ in $B(0,\delta)$. The desired function is defined by

$$u(x) = \int_1^{+\infty} \varphi\left[r^{\varepsilon q}(x-x_0)\right] e^{-r^{\eta}} e^{i\langle x-x_0, r^q\xi_0\rangle} dr,$$

where $r^q x = (r^{q_1}x_1, r^{q_2}x_2, \dots, r^{q_n}x_n)$.

Following [5] and [8] it is easy to show that $u \notin G^{\mathcal{F},s}(U)$ for any neighborhood $U$ of $x_0$.

Let us verify that $u \in G^s(\Omega, (P_j)_{j=1}^N)$. Since the coefficients of the operators $P_j$ are in $G^{\theta,\sigma}(\Omega) \subset G^{q,\sigma\mu}(\Omega)$, then $\exists M > 0$, $\forall \alpha \in \mathbb{Z}_+^n$, $\forall \beta \in \mathbb{Z}_+^n$, $\forall x \in B_0$, $\forall r \geq 1$, $\forall j = 1,\dots,N$, such that

$$\left|\left(D_x^{\beta} P_j^{(\alpha)}\right)(x, r^q\xi_0)\right| \leq M^{|\beta|+1}\left[\Gamma(\langle\beta,q\rangle+1)\right]^{\sigma\mu} r^{1-\langle\alpha,q\rangle}. \tag{3.2}$$

On the other hand in view of (3.1) it is easy to obtain $\forall \delta > 0$, $\exists C_1 > 0$, $\forall r \geq 1$, $\forall x \in \Omega, |x - x_0| < 2\delta r^{-\varepsilon/\mu}$, $\forall j = 1,\dots,N$,

$$|P_j(x, r^q\xi_0)| \leq C_1 r^{1-\varepsilon/\mu}. \tag{3.3}$$

Now we need a convenient form of $P_{i_k} \dots P_{i_1} u$, for any integer $k \geq 1$. The generalized Leibniz formula $P_j(x,D)(uv) = \sum\limits_{\alpha} \frac{1}{\alpha!} P_j^{(\alpha)} u D^{\alpha} v$ gives

$$P_{i_k} \dots P_{i_0} u(x) = \int_1^{+\infty} A_{i_k \dots i_0}(x,r)\, e^{-r^{\eta}} e^{i\langle x-x_0, r^q\xi_0\rangle} dr,$$

where $1 \leq i_l \leq N$, for any integer $l \leq k$, $P_{i_0}$ designs the identity operator, and

$$\left\{ \begin{array}{l} A_{i_0}(x,r) = \varphi\left[r^{\varepsilon q}(x-x_0)\right], \\ A_{i_{k+1},i_k\ldots i_0}(x,r) = \sum\limits_{\langle\alpha,q\rangle\leq 1} \frac{1}{\alpha!} P_{i_{k+1}}^{(\alpha)}(x, r^q\xi_0)\, D_x^\alpha A_{i_k\ldots i_0}(x,r). \end{array} \right. \tag{3.4}$$

To complete the proof we need the following

**Lemma 4.** $\exists L > 0, \exists L_0 > 0, \exists C_0 > 0, \forall k \in \mathbb{Z}_+$, $\forall \gamma \in \mathbb{Z}_+^n$, $\forall x \in B_0$, $\forall r \geq 1$,

$$\begin{aligned} \left|D_x^\gamma A_{i_k\ldots i_0}(x,r)\right| \leq C_0 \left(L_0 r^\varepsilon\right)^{\langle\gamma,q\rangle} L^k \Big( & r^{(1-\varepsilon/\mu)k} \left[\Gamma(\langle\gamma,q\rangle+1)\right]^{\sigma\mu} \\ & + \left[\Gamma(\langle\gamma,q\rangle+k+1)\right]^{\sigma\mu} r^{k\varepsilon(2-1/\mu)} \Big). \end{aligned} \tag{3.5}$$

*Proof.* It is obtained by recurrence over $k$. In fact for $k = 0$, the estimate $(3.5)$ means $\varphi \in G_0^{q,\sigma\mu}(\mathbb{R}^n)$. So suppose that the estimate $(3.5)$ holds up to the order $k$ and let us check it at the order $k+1$. Set $\lambda = r^{1-\varepsilon/\mu}$ and $\tau = r^{\varepsilon(2-1/\mu)}$. Then the estimate $(3.5)$ is written as

$$\left|D_x^\gamma A_{i_k\ldots i_0}(x,r)\right| \leq C_0 \left(L_0 r^\varepsilon\right)^{\langle\gamma,q\rangle} L^k S(k,\gamma),$$

where

$$S(k,\beta) = \lambda^k \left[\Gamma(\langle\beta,q\rangle+1)\right]^{\sigma\mu} + \left[\Gamma(\langle\beta,q\rangle+k+1)\right]^{\sigma\mu} \tau^k.$$

Let $\omega = \min\limits_{1\leq j\leq n} q_j$. Then we have

$$\lambda^{1-\langle\alpha,q\rangle} \tau^{\langle\alpha,q\rangle} S(k,\beta+\alpha) \leq 2^{\frac{\sigma\mu}{\omega}+1} S(k+1,\beta), \quad \langle\alpha,q\rangle \leq 1. \tag{3.6}$$

From $(3.4)$, we have

$$\left|D_x^\gamma A_{i_{k+1}\ldots i_0}(x,r)\right| \leq I_1 + I_2 + I_3,$$

where

$$\begin{aligned} I_1 &= \left|P_{i_{k+1}}(x, r^q\xi_0)\right| \left|D_x^\gamma A_{i_k\ldots i_0}(x,r)\right|, \\ I_2 &= \sum_{\beta<\gamma} \binom{\gamma}{\beta} \left|D_x^{\gamma-\beta} P_{i_{k+1}}(x, r^q\xi_0)\right| \left|D_x^\beta A_{i_k\ldots i_0}(x,r)\right|, \\ I_3 &= \sum_{0<\langle\alpha,q\rangle\leq 1} \sum_{\beta\leq\gamma} \frac{1}{\alpha!} \binom{\gamma}{\beta} \left|D_x^{\gamma-\beta} P_{i_{k+1}}^{(\alpha)}(x, r^q\xi_0)\right| \left|D_x^{\alpha+\beta} A_{i_k\ldots i_0}(x,r)\right|. \end{aligned}$$

Since $A_{i_k\ldots i_0}$ are functions of compact supports in $B\left(x_0, 2\delta r^{-\varepsilon/\mu}\right)$, and according to $(3.3)$ and $(3.6)$, we have

$$I_1 \leq 2^{\frac{\sigma\mu}{\omega}+1} C_1 C_0 \left(L_0 r^\varepsilon\right)^{\langle\gamma,q\rangle} S(k+1,\gamma) L^k. \tag{3.7}$$

The estimates $(3.2)$ and $(3.6)$ give

$$\begin{aligned} I_2 \leq & \sum_{\beta<\gamma} \binom{\gamma}{\beta} \left[\Gamma(\langle\gamma-\beta,q\rangle+1)\right]^{\sigma\mu} M^{|\gamma-\beta|+1} r^\varepsilon C_0 \left(L_0 r^\varepsilon\right)^{\langle\beta,q\rangle} \\ & \cdot 2^{\frac{\sigma\mu}{\omega}+1} S(k+1,\beta) L^k. \end{aligned}$$

On the other hand, using properties of the gamma function, we have

$$\binom{\gamma}{\beta} \left[\Gamma(\langle\gamma-\beta,q\rangle+1)\right]^{\sigma\mu} S(k,\beta) \leq C_2^{\sigma\mu\langle\gamma-\beta,q\rangle} S(k,\gamma). \tag{3.8}$$

Thus we obtain

$$I_2 \leq \frac{nMC_2^{\sigma\mu}}{L_0 r^{\varepsilon}} \sum_{\beta \geq 0} \left( \frac{MC_2^{\sigma\mu}}{L_0 r^{\varepsilon}} \right)^{\langle \beta, q \rangle} r^{\varepsilon} 2^{\frac{\sigma\mu}{\omega}+1} MC_0 \left( L_0 r^{\varepsilon} \right)^{\langle \gamma, q \rangle} S\left(k+1, \gamma\right) L^k.$$

Set $C_3 = \sum_{\alpha \geq 0} \left(\frac{1}{2}\right)^{\langle \alpha, q \rangle}$, take $L_0 \geq 2MC_2^{\sigma\mu}$ and $r \geq 1$, and then

$$I_2 \leq \frac{nMC_2^{\sigma\mu}}{L_0} C_3 2^{\frac{\sigma\mu}{\omega}+1} MC_0 \left( L_0 r^{\varepsilon} \right)^{\langle \gamma, q \rangle} S\left(k+1, \gamma\right) L^k. \tag{3.9}$$

Finally in view of (3.2)

$$\begin{aligned} I_3 \quad \leq \quad & \sum_{0 < \langle \alpha, q \rangle \leq 1} \sum_{\beta \leq \gamma} \binom{\gamma}{\beta} \left[ \Gamma(\langle \gamma - \beta, q \rangle + 1) \right]^{\sigma\mu} M^{|\gamma - \beta| + 1} r^{1 - \langle \alpha, q \rangle} \\ & \times C_0 \left( L_0 r^{\varepsilon} \right)^{|\beta + \alpha|} S\left(k, \beta + \alpha\right) L^k. \end{aligned}$$

For any $\alpha \in Z_+^n$, $0 < \langle \alpha, q \rangle \leq 1$, we have $r^{1 - \langle \alpha, q \rangle + \varepsilon \langle \alpha, q \rangle} \leq \lambda^{1 - \langle \alpha, q \rangle} \tau^{\langle \alpha, q \rangle}$, which gives, with (3.6) and (3.8),

$$I_3 \leq \sum_{0 < \langle \alpha, q \rangle \leq 1} \sum_{\beta \leq \gamma} \left( \frac{MC_2^{\sigma\mu}}{L_0 r^{\varepsilon}} \right)^{|\gamma|} 2^{\frac{\sigma\mu}{\omega}+1} MC_0 L_0^{|\alpha|} \left( L_0 r^{\varepsilon} \right)^{|\gamma|} S\left(k+1, \gamma\right) L^k.$$

Put $C_4 = \sum_{0 < \langle \alpha, q \rangle \leq 1} L_0^{\langle \alpha, q \rangle}$. Then we obtain

$$I_3 \leq 2^{\frac{\sigma\mu}{\omega}+1} MC_4 C_3 C_0 \left( L_0 r^{\varepsilon} \right)^{\langle \gamma, q \rangle} S\left(k+1, \gamma\right) L^k. \tag{3.10}$$

If we choose

$$L \geq 2^{\frac{\sigma\mu}{\omega}+1} \left( C_1 + \frac{nM^2 C_2^{\sigma\mu}}{L_0} C_3 + MC_3 C_4 \right),$$

we get, from (3.7), (3.9) and (3.10),

$$I_1 + I_2 + I_3 \leq C_0 \left( L_0 r^{\varepsilon} \right)^{\langle \gamma, q \rangle} S\left(k+1, \gamma\right) L^{k+1},$$

which means that (3.5) holds at the order $k+1$. $\square$

*End of the Proof of Theorem* 3. Applying the last lemma for $\gamma = 0$, we find

$$\left| A_{i_k \dots i_0}\left(x, r\right) \right| \leq C_0' L^k \left( r^{(1 - \varepsilon/\mu)k} + (k!)^{\sigma\mu} r^{k\varepsilon(2 - 1/\mu)} \right). \tag{3.11}$$

Thus we obtain

$$\left| A_{i_k \dots i_0}\left(x, r\right) \right| \leq C_0' L^k \left(2s\mu\right)^{k\mu s} (k!)^{s\mu} \left[ \exp\left( \frac{r^{\eta}}{2} \right) + \exp\left( \frac{r^{\eta'}}{2} \right) \right],$$

where $\eta' = \frac{\varepsilon(2 - 1/\mu)}{\mu(s - \sigma)} \leq \eta = \frac{1 - \varepsilon/\mu}{\mu s}$, since $\varepsilon \leq \dfrac{\mu\left(s - \sigma\right)}{2\mu s - \sigma}$.

Therefore

$$\begin{aligned} \left| P_{i_k \dots} P_{i_0} u\left(x\right) \right| \quad & \leq \quad 2C_0' L'^k (k!)^{s\mu} \int_1^{+\infty} \exp\left( -\frac{r^{\eta}}{2} \right) dr \\ & \leq \quad C^{k+1} \left(k\right)!^{s\mu}, \end{aligned}$$

which means that $u \in G^s \left( \Omega, \left(P_j\right)_{j=1}^N \right)$. $\square$

## 4. Consequences

A first consequence of Theorem 2 is a result on Gevrey-hypoellipticity for multi-quasi-elliptic systems.

**Corollary 1.** *Under the assumptions of Theorem* 2*, the following propositions are equivalent:*
(i) $u \in \mathcal{D}'(\Omega)$, $P_j u \in G^{\mathcal{F},s}(\Omega)$, $\forall j = 1, \ldots, N$.
(ii) $u \in G^{\mathcal{F},s}(\Omega)$.

The theorems of this work unify the results of Bolley-Camus [1] and Métivier [5] in the homogenous case, the results of Zanghirati [7] and [8] in the scalar quasi-homogenous case and generalize them to quasi-homogenous systems.

**Corollary 2.** *Let $\Omega$ be an open subset of $\mathbb{R}^n$ and $\sigma > s \geq 1$, and let $(P_j)_{j=1}^N$ be a system of linear differential operators with coefficients in $G^{q,\sigma}(\Omega)$. Then*

$$(P_j)_{j=1}^N \text{ is } q\text{-quasi-elliptic in } \Omega \Longleftrightarrow G^s\left(\Omega, (P_j)_{j=1}^N\right) \subset G^{q,s}(\Omega).$$

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