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# ON THE SPECTRAL PICTURE OF AN IRREDUCIBLE SUBNORMAL OPERATOR II

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ABSTRACT. In this paper we show that the spectral picture of an irreducible subnormal operator may be arbitrarily prescribed subject only to certain natural necessary conditions. This completes work begun by the second author.

## 1. INTRODUCTION

In this paper $\mathcal{H}$ will denote a separable infinite dimensional complex Hilbert space, and $\mathcal{B}(\mathcal{H})$ will denote the algebra of all bounded linear operators on $\mathcal{H}$. An operator $S \in \mathcal{B}(\mathcal{H})$ is subnormal if it has a normal extension, and $S$ is irreducible if it has no reducing subspaces. The spectral picture of an operator $S$ consists of the spectrum, $\sigma(S)$, the essential spectrum, $\sigma_e(S)$, and the values of the Fredholm index function on the components of $\sigma(S) \setminus \sigma_e(S)$. If $K$ is a compact set in the complex plane $\mathbb{C}$, then $R(K)$ will denote the uniform closure of the rational functions with poles off $K$, and $C(K)$ denotes the algebra of all continuous functions on $K$. In 1972, Clancey and Putnam [1] characterized which compact sets can arise as the spectrum of a pure subnormal operator; namely, a compact set $K$ is the spectrum of a pure subnormal operator if and only if for every open disk $D$ that intersects $K$, we have $R(\mathrm{cl}D \cap K) \neq C(\mathrm{cl}D \cap K)$, equivalently, the nonpeak points for $R(K)$ are dense in $K$. Still another formulation is that if $\{G_k\}$ denotes the collection of nontrivial Gleason parts for $R(K)$, then $\bigcup_k G_k$ must be dense in $K$. In 1980, Olin and Thomson [9] showed that a compact set $K$ can be the spectrum of an irreducible subnormal operator if and only if $R(K)$ has only one non-trivial Gleason part $G$ and $G$ is dense in $K$. In 1988, McGuire [7] constructed irreducible subnormal operators with a prescribed spectrum, essential spectrum and Fredholm index, however the Fredholm index could only be prescribed to have values $\leq -2$.

In this paper we will completely characterize the "generalized spectral picture" of an irreducible subnormal operator, including the spectrum, approximate point spectrum, $\sigma_{ap}(S)$, essential spectrum, and the (semi) Fredholm index function on $\sigma(S) \setminus \sigma_{ap}(S)$. More precisely, if $K$ is a compact set in $\mathbb{C}$ such that $R(K)$ has only one non-trivial gleason part which is dense in $K$, and if $K_a$ and $K_e$ are two compact sets satisfying $\partial K \subseteq \partial K_e \subseteq K_a \subseteq K_e \subseteq K$, and if for each component $V_n$ of $\sigma(S) \setminus \sigma_e(S)$ we choose an integer $a_n$ satisfying $a_n \leq -1$, then there exists an

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irreducible subnormal operator $S$ such that $\sigma(S) = K$, $\sigma_{ap}(S) = K_a$, $\sigma_e(S) = K_e$, $ind(S - \lambda) = -\infty$ for $\lambda \in \sigma_e(S) \setminus \sigma_{ap}(S)$, and $ind(S - \lambda) = a_n$ for $\lambda \in V_n$. Furthermore, each of the above conditions is necessary.

The main new idea in this paper involves using the dual of a subnormal operator to create some new irreducible subnormal operators with certain spectral properties. It should be noted that it is easy to construct a subnormal operator with a prescribed spectral picture; it is finding one that is *irreducible* that is difficult.

The spectral pictures of pure subnormal operators were essentially characterized in 1972 by Clancey and Putnam [1].

**Theorem 1.1** (Spectral pictures for pure subnormal operators)**.** *If $K, K_a, K_e$ are compact sets in $\mathbb{C}$ and if for each bounded component $V_n$ of $\mathbb{C} \setminus K_e$ an integer $a_n$ has been chosen, then there is a pure subnormal operator $S$ such that*

$$\sigma(S) = K, \sigma_{ap}(S) = K_a, \sigma_e(S) = K_e, \text{ and } ind(S - \lambda) = a_n \text{ for all } \lambda \in V_n$$

*if and only if $\partial K \subseteq \partial K_e \subseteq K_a \subseteq K_e \subseteq K$, $K$ satisfies the Clancey-Putnam condition, and $a_n \leq -1$ for all $n$.*

Thomson's Theorem (see [11] or [4]) gives a very nice description of the spectral pictures for cyclic subnormal operators.

**Theorem 1.2** (Spectral pictures for pure cyclic subnormal operators)**.** *If $S$ is a pure cyclic subnormal operator, then there exists a bounded simply connected open set $G \subseteq \mathbb{C}$ such that*

$$\sigma(S) = clG, \sigma_e(S) = \sigma_{ap}(S) = \partial G, \text{ and } ind(S - \lambda) = -1 \text{ for all } \lambda \in G. \tag{1}$$

*Conversely, if $G$ is any bounded, simply connected open set in $\mathbb{C}$, then there exists a pure cyclic subnormal operator $S$ such that* (1) *holds. Furthermore, $S$ is irreducible if and only if $G$ is connected.*

## 2. Preliminaries

Our main tool for constructing irreducible subnormal operators is a theorem due to McGuire [8, Theorem 4.1] (also see McGuire [7]) which allows one to "twist" the direct sum of a sequence of operators to create an irreducible operator. The original idea of the construction is similar to a construction in R. F. Olin and J. E. Thomson [10, p. 154] which was then extended by J. Conway [3, p. 276] and subsequently put into a more general setting by McGuire [8, Theorem 4.1].

We need a more general result than what appears in [8, Theorem 4.1], so our statement of the Conway-McGuire Theorem is slightly different than that in [8]; however, the same proof in [8] applies to this more general theorem.

**Theorem 2.1** (Conway-McGuire Theorem)**.** *Suppose that $1 \leq n \leq \infty$ and that $\{\mu_j\}_{j=0}^n$ is a collection of pairwise singular, positive regular Borel measures which are all supported on a common compact subset of $\mathbb{C}$. Suppose also that the following hold:*

1. *For each $j \geq 0$, $\mathcal{H}_j$ is an invariant subspace for $N_j = M_z$ on $L^2(\mu_j)$, and if $S_j = N_j|\mathcal{H}_j$, then $S_j$ is an irreducible subnormal operator with minimal normal extension $N_j$.*
2. *For $j \geq 1$, there exists a bounded linear operator $A_j : \mathcal{H}_0 \to \mathcal{H}_j$ such that $A_j S_0 = S_j A_j$.*

(3) *For $j \geq 1$, there exists a $\mu_j$-measurable set $\Delta_j \subseteq \sigma(S_j)$ such that the range of $P_j A_j$ is not contained in $\mathcal{H}_j$ where $P_j = M_{\chi_{\Delta_j}}$ is multiplication by the characteristic function of $\Delta_j$ on $L^2(\mu_j)$.*

*Then there exists an irreducible subnormal operator $S$ similar to $\bigoplus_{j=0}^{n} S_j$.*

**Example 2.2.** Suppose that for $j \geq 0$, $S_j = M_z$ on $H^2(D_j)$ where $D_j$ is an open disk. If $D_j \subseteq D_0$ for all $j \geq 1$, and if $D_j \neq D_k$ for $j \neq k$, then there exists an irreducible subnormal operator $S$ similar to $\bigoplus_{j=0}^{\infty} S_j$.

*Proof.* For $j \geq 1$, let $A_j : H^2(D_0) \to H^2(D_j)$ be given by $A_j f = f|D_j$. Clearly, $A_j$ intertwines $S_0$ with $S_j$. Each of the operators $S_j$ is irreducible and their spectral measures (arc length measure on $\partial D_j$) are mutually singular (this is where we use $D_j \neq D_k$). Finally, if $\Delta_j$ is any subset of $\partial D_j$ with positive Lebesgue measure, but not full measure, then the range of $P_j A_j$ will consist of functions that vanish off of $\Delta_j$, hence the range of $P_j A_j$ cannot be contained in $H^2(D_j)$. Thus the Conway-McGuire Theorem applies. □

If $K$ is a compact set in $\mathbb{C}$, then two points $a, b \in K$ belong to the same Gleason part of $R(K)$ if a Harnack type inequality holds, that is, if there exists a constant $c > 0$ such that $\frac{1}{c} Ref(a) \leq Ref(b) \leq cRef(a)$ for all $f \in R(K)$ with $Ref > 0$ (where $Ref$ denotes the real part of $f$). A Gleason part of $R(K)$ is said to be non-trivial if it contains more than one point. It is known that if $a, b \in K$ belong to the same Gleason part, then they have mutually absolutely continuous representing measures supported on $\partial K$. However, if $a, b$ belong to distinct Gleason parts, then every representing measure for $a$ is mutually singular with respect to every representing measure for $b$. This last condition is useful in showing that two points belong to the same Gleason part. For instance consider the following example.

**Example 2.3.** Suppose that $G_1$ and $G_2$ are disjoint simply connected regions bounded by rectifiable Jordan curves and $K = cl(G_1 \cup G_2)$. If $\partial G_1 \cap \partial G_2$ has positive arc length measure, yet contains no subarc with positive length, then $R(K)$ has only one non-trivial Gleason part which is equal to $G_1 \cup G_2$, yet the interior of $K$ has two components $G_1$ and $G_2$.

*Sketch of the proof.* Choose points $a_i \in G_i$, $i = 1, 2$, and let $\omega_i$ be harmonic measure for $G_i$ evaluated at $a_i$. Then $\omega_i$ is a representating measure for $R(K)$ at the point $a_i$. Since $\partial G_i$ is a rectifiable Jordan curve, $\omega_i$ is mutually absolutely continuous with respect to arc length measure. Since $\partial G_1 \cap \partial G_2$ intersects in a set of positive length, it follows that $\omega_1$ and $\omega_2$ are not mutually singular measures. Hence $a_1$ and $a_2$ must belong to the same Gleason part. Since $a_1$ and $a_2$ are arbitrary points in $G_1$ and $G_2$, respectively, it follows that $G_1$ and $G_2$ belong to the same Gleason part. □

It what follows, $R^2(K, \mu)$ will denote the $L^2(\mu)$ closure of $R(K)$ and $\|f\|_\mu$ denotes the $L^2(\mu)$ norm of a function $f$.

If $S$ is a pure subnormal operator on $\mathcal{H}$ with minimal normal extension $N$ acting on $\mathcal{K}$, then the dual of $S$, $dual(S)$, is the subnormal operator $N^*|\mathcal{H}^\perp$ where $\mathcal{H}^\perp = \mathcal{K} \ominus \mathcal{H}$. For example, if $S_\mu = M_z$ on $R^2(K, \mu)$, then $dual(S_\mu) = M_{\overline{z}}$ on $R^2(K, \mu)^\perp$. In [2], Conway established several fundamental properties of $dual(S)$ which we summarize in the following theorem. In what follows if $K$ is a compact set, then $K^* = \{z : \overline{z} \in K\}$.

**Theorem 2.4** (Properties of the dual)**.** *If $S$ is a pure subnormal operator with minimal normal extension $N$ and $T = dual(S)$, then the following hold:*

(1) *$T$ is a pure subnormal operator.*
(2) *$T$ is irreducible if and only if $S$ is irreducible.*
(3) *$T$ is essentially normal if and only if $S$ is essentially normal.*
(4) $\sigma(T) = \sigma(S)^*$.
(5) *If $S$ is essentially normal, then $\sigma_e(N) = \sigma_e(S) \cup \sigma_e(T)^*$.*
(6) *If $S$ is essentially normal and $\lambda \notin \sigma_e(N)$, then $ind(S - \lambda) = ind(T - \overline{\lambda})$.*

*Proof.* Properties (1), (2), and (4) were established in Conway [2]. The other properties follow easily from the following observation. Write $N = \left[\begin{smallmatrix} S & A \\ 0 & T^* \end{smallmatrix}\right]$ with respect to $\mathcal{H} \oplus \mathcal{H}^\perp$. It follows that $[S^*, S] = AA^*$ and $[T^*, T] = A^*A$. Thus, $S$ has a compact self-commutator if and only if $A$ is compact, which happens if and only if $T$ has a compact self-commutator. Thus (3) follows. To see (5) and (6), simply notice that if $S$ is essentially normal, then $A$ is compact, hence $N$ is a compact perturbation of $S \oplus T^*$. □

For further background on subnormal operators and rational approximation, see Conway [4].

## 3. Main results

We begin by stating the well known necessary spectral conditions for an irreducible subnormal operator.

**Proposition 3.1.** *If $S$ is an irreducible subnormal operator, then the following hold:*

(1) $\partial\sigma(S) \subseteq \partial\sigma_e(S) \subseteq \sigma_{ap}(S) \subseteq \sigma_e(S) \subseteq \sigma(S)$.
(2) *$R(\sigma(S))$ has only one non-trivial Gleason part $G$ and $clG = \sigma(S)$.*
(3) *$ind(S - \lambda) \leq -1$ for all $\lambda \in \sigma(S) \setminus \sigma_e(S)$ and $ind(S - \lambda)$ is constant on the components of $\sigma(S) \setminus \sigma_e(S)$.*
(4) *$ind(S - \lambda) = -\infty$ for all $\lambda \in \sigma_e(S) \setminus \sigma_{ap}(S)$.*

We now formulate our main theorem which shows that the above necessary conditions are also sufficient for describing the spectral picture of an irreducible subnormal operator.

**Theorem 3.2.** *If $K, K_a, K_e$ are compact sets in $\mathbb{C}$ such that:*

(1) $\partial K \subseteq \partial K_e \subseteq K_a \subseteq K_e \subseteq K$,
(2) *$R(K)$ has only one non-trivial Gleason part $G$ and $clG = K$,*
(3) *for each component $V_n$ of $K \setminus K_e$ an integer $a_n \leq -1$ has been chosen,*

*then there exists an irreducible subnormal operator $S$ for which:*

(a) $\sigma(S) = K$, $\sigma_{ap}(S) = K_a$, $\sigma_e(S) = K_e$,
(b) *$ind(S - \lambda) = a_n$ for all $\lambda \in V_n$,*
(c) *$ind(S - \lambda) = -\infty$ for all $\lambda \in \sigma_e(S) \setminus \sigma_{ap}(S)$.*

We will present the proof in steps, as a series of propositions. This first proposition allows us to prescribe the spectral picture with Fredholm index function $-1$.

**Proposition 3.3.** *If $K$ and $K_a$ are compact sets in $\mathbb{C}$ such that $\partial K \subseteq K_a \subseteq K$ and $R(K)$ has only one non-trivial Gleason part $G$ such that $clG = K$, then there exists a measure $\nu$ and an irreducible essentially normal subnormal operator $S$ of* *the form* $S = M_z$ *on* $\mathcal{H} \subseteq L^2(\nu)$*, such that* $\sigma(S) = K$*,* $\sigma_{ap}(S) = \sigma_e(S) = K_a$*, and* $ind(S - \lambda) = -1$ *for all* $\lambda \in \sigma(S) \setminus \sigma_e(S)$*.*

*Proof.* If $K_a = \partial K$, then McGuire [7, Theorem 5] constructs a measure $\mu$ supported on $\partial K \cup \mathrm{cl}(G \setminus \mathrm{int}K)$ such that if $S_\mu$ denotes multiplication by $z$ on $R^2(K, \mu)$, then $S_\mu$ is irreducible, $\sigma(S_\mu) = K$ and $\sigma_{ap}(S_\mu) = \sigma_e(S_\mu) = \partial K$. It follows, since $S_\mu$ is rationally cyclic, that $\mathrm{ind}(S_\mu - \lambda) = -1$ for all $\lambda \in \sigma(S_\mu) \setminus \sigma_e(S_\mu)$. It also follows that $\mu$ is supported on $\partial K$ because $\mathrm{cl}(G \setminus \mathrm{int}K) \subseteq \partial K$. So, in this case we are done.

Now suppose that $K_a \neq \partial K$; in this case we must have that $\mathrm{int}K \neq \emptyset$ and that $K_a \cap \mathrm{int}K \neq \emptyset$. Let $\mu$ be the measure supported on $\partial K$ that is constructed above, so that $S_\mu$ is irreducible and has the above-mentioned spectral properties. We will use $\mu$ to create a new measure $\nu$.

Let $\{z_j\}$ be a countable subset of $\mathrm{int}K$ which clusters at each point of $K_a \cap \mathrm{int}K$, and whose only other cluster points lie in $\partial K$. For instance, we could choose $\{z_j\}$ to be a countable dense subset of $K_a \cap \mathrm{int}K$, however, if $K_a \cap \mathrm{int}K$ has some isolated points, we will need to choose some more $z_j$'s in $\mathrm{int}K$ that converge to these isolated points. We want to define a measure $\nu = \mu + \sum_{j=1}^{\infty} c_j \delta_j$ where $c_j > 0$ will be chosen and $\delta_j$ denotes the point mass at $z_j$.

Clearly, if $f \in R(K)$, then $\|f\|_\mu \leq \|f\|_\nu$. We want to choose the constants $c_j > 0$ such that there exists a constant $M > 0$ with $\|f\|_\nu \leq M\|f\|_\mu$. To see how to do this, notice that each point in the interior of $K$ is an analytic bounded point evaluation for $R^2(K, \mu)$ (because $\sigma_{ap}(S_\mu) = \partial K$). So let $K_{z_j}$ denote the reproducing kernel for $R^2(K, \mu)$ at the point $z_j$. Choose $c_j$ such that $\sum_j c_j \|K_{z_j}\|_\mu^2 < \infty$. Then if we choose $M^2 = 1 + \sum_j c_j \|K_{z_j}\|_\mu^2$, we get the above inequality. Thus $\mu$ and $\nu$ give equivalent norms on $R(K)$. It follows that $S_\nu = M_z$ on $R^2(K, \nu)$ is similar to $S_\mu$. Hence $S_\nu$ has the same spectral properties as $S_\mu$. It also follows that $S_\nu$ is irreducible, because if $S_\nu$ commutes with a projection, then $S_\mu$ will commute with an idempotent. However, since the commutant of $S_\mu$ consists of multiplication operators, one easily sees that any idempotent that commutes with $S_\mu$ is actually a projection, and since $S_\mu$ is irreducible, this cannot happen.

Now, $S_\nu$ is not the operator we are looking for, because $\sigma_{ap}(S_\nu) = \sigma_{ap}(S_\mu) = \partial K$, but if we let $T = dual(S_\nu)$, then this is (almost) the required operator. It follows by Theorem 2.4 that $T$ is an irreducible essentially normal subnormal operator, $\sigma(T) = K^*$, $\sigma_e(T) = \sigma_{ap}(T)$ (because $T$ is essentially normal) and that $\sigma_e(S_\nu) \cup \sigma_e(T)^* = \sigma_e(N_\nu) = K_a$. Since $\sigma_e(S_\nu) = \partial K$, we have $\partial K \cup \sigma_e(T)^* = K_a$. However, $\partial K = \partial\sigma(T)^* \subseteq \sigma_e(T)^*$. Thus, $\sigma_e(T)^* = K_a$. Also, by Theorem 2.4, $\mathrm{ind}(T - \lambda) = -1$ for all $\lambda \in \sigma(T) \setminus \sigma_e(T)$.

In order to get the required operator $S$, simply let $\mathcal{H}$ be the complex conjugates of all functions in $R^2(K, \nu)^\perp$ and $S = M_z$ on $\mathcal{H}$. Then $S$ has the required properties. $\square$

Suppose that $G$ is a bounded region in $\mathbb{C}$ and $\rho : G \to [0, \infty)$ is a non-negative continuous function. Let $\mathcal{F}_\rho$ be the collection of all analytic functions $f$ on $G$ such that there exists a constant $C_f$ such that $|f(z)| \leq C_f \rho(z)$ for all $z \in G$. Also, if $\mu$ is a measure on $G$, then $L_a^2(G, \mu)$ will denote the set of all analytic functions in $G$ that belong to $L^2(\mu)$. Note that $L_a^2(G, \mu)$ may not be a closed subspace of $L^2(\mu)$.

The following lemma is well known.

**Lemma 3.4.** *If $G$ is a bounded open set in $\mathbb{C}$, and $K$ and $L$ are compact subsets of $G$, then there exists a constant $M > 0$ such that $|f(z)|^2 \leq M \int_{G-L} |f|^2 dA$ for all $z \in K$.*

**Proposition 3.5.** *If $G$ is a bounded region in $\mathbb{C}$ and $\rho : G \to [0, \infty)$ is continuous, then there exists a sequence $\{\mu_j\}_{j=1}^{\infty}$ of positive finite regular Borel measures on $G$ that are pairwise mutually singular and such that for each $j \geq 1$, the following hold:*

1. *$\mathcal{F}_\rho \subseteq L^2_a(G, \mu_j)$,*
2. *$L^2_a(G, \mu_j)$ is a closed subspace of $L^2(\mu_j)$,*
3. *the point evalution functionals $e_\lambda : L^2_a(G, \mu_j) \to \mathbb{C}$ given by $e_\lambda(f) = f(\lambda)$ are continuous for each $\lambda \in G$, and*
4. *$S_j = M_z$ on $L^2_a(G, \mu_j)$ is an irreducible subnormal operator.*

*Proof.* We first construct a single measure $\mu$ of a certain form that satisfies these conditions. Suppose that $G = \bigcup_n K_n$ where $K_n$ is compact and $K_n \subseteq \operatorname{int} K_{n+1}$. Let $A_n = \operatorname{int} K_{n+1} \setminus K_n$. Choose constants $c_n > 0$ satisfying

$$c_n \leq \frac{1}{2^n \left[1 + \int_{A_n} \rho(z)^2 dA\right]}.$$

Next form the measure $\mu = \sum_{n=1}^{\infty} c_n dA|A_n$, where $dA$ denotes area measure. One easily checks that $\mathcal{F}_\rho \subseteq L^2_a(G, \mu)$. For if $f \in \mathcal{F}_\rho$, then $|f(z)| \leq C_f \rho(z)$ for all $z \in G$. Thus

$$\int |f(z)|^2 d\mu \leq C_f^2 \int \rho(z)^2 d\mu = C_f^2 \sum_{n=1}^{\infty} c_n \int_{A_n} \rho(z)^2 dA \leq C_f^2 \sum_{n=1}^{\infty} \frac{1}{2^n} < \infty.$$

Point evaluations are also continuous on $L^2_a(G, \mu)$ because if $\lambda \in K_n$ for some $n \geq 1$ and $f \in L^2_a(G, \mu)$, then $f \in L^2_a(\operatorname{int} K_{n+1}, dArea)$ and so by Lemma 3.4 there is a constant $M$ such that

$$|f(\lambda)|^2 \leq M \int_{A_{n+1}} |f|^2 dA = \frac{M}{c_{n+1}} \int_{A_{n+1}} |f|^2 c_{n+1} dA \leq \frac{M}{c_{n+1}} \int |f|^2 d\mu.$$

It follows from this that $L^2_a(G, \mu)$ is a closed subspace of $L^2(\mu)$; because any Cauchy sequence from $L^2_a(G, \mu)$ will be bounded in $L^2(\mu)$ norm, hence by the above inequality it will be uniformly bounded on compact subsets of $G$, and thus is a normal family.

Clearly, $S = M_z$ on $L^2_a(G, \mu)$ is a subnormal operator and since its commutant is the algebra of multiplication operators with symbols in $H^\infty(G)$, it follows that $S$ is irreducible provided $G$ is connected.

Now to get a sequence of measures $\{\mu_j\}$ satisfying the above conditions, simply write the natural numbers as a countably infinite disjoint union of countably infinite sets, say $\mathbb{N} = \bigcup_{j=1}^{\infty} E_j$. Then let $\mu_j = \sum_{n \in E_j} c_n dA|A_n$. Since the measures $\mu_j$ have the same form as the measure $\mu$, one easily sees that the four properties above are satisfied and the $\mu_j$'s are pairwise singular; thus these are the required measures. $\square$

A general result due to Grauert [6] (see also Cowen and Douglas [5, p. 194]) allows us to choose eigenvectors for $S^*$ in a co-analytic manner whever the Fredholm index function for $S$ is $-1$. This is the content of the next proposition.

**Proposition 3.6.** *If $S \in \mathcal{B}(\mathcal{H})$ and $ind(S - \lambda) = -1$ for all $\lambda \in G := \sigma(S) \setminus \sigma_{ap}(S)$, then there exists a co-analytic function $h : G \to \mathcal{H}$ that is not identically zero on* *any component of* $G$ *such that* $h(\lambda) \in ker(S^* - \overline{\lambda})$*. In particular, for every* $x \in \mathcal{H}$*, the function* $\lambda \mapsto \langle x, h(\lambda) \rangle$ *is analytic on* $G$*.*

*Proof of Theorem 3.2.* Suppose we are given compact sets $K$, $K_a$, and $K_e$ satisfying the prescribed conditions. In order to apply the Conway-McGuire Theorem we need to construct irreducible operators $S_j$, intertwining maps $A_j$, and some measurable sets $\Delta_j$ such that $\bigoplus_j S_j$ has the required spectral properties.

By Proposition 3.3, there exists an irreducible subnormal operator $S_0$ of the form $S_0 = M_z$ on a subspace $\mathcal{H}_0 \subseteq L^2(\mu_0)$ where $\sigma(S_0) = K$, $\sigma_{ap}(S_0) = \sigma_e(S_0) = K_a$ and $\operatorname{ind}(S_0 - \lambda) = -1$ for all $\lambda \in \sigma(S_0) \setminus \sigma_e(S_0)$. Furthermore, the measure $\mu_0$ restricted to $\sigma(S_0) \setminus \sigma_{ap}(S_0)$ is a purely discrete measure.

Let $V = K \setminus K_a = \sigma(S_0) \setminus \sigma_e(S_0)$. By Proposition 3.6, there exists a co-analytic function $h : V \to \mathcal{H}_0$ that is not identically zero on any component of $V$ such that $h(z) \in \ker(S_0^* - \overline{z})$ for all $z \in V$. It follows that for each $f \in \mathcal{H}_0$, the function $\hat{f} : V \to \mathbb{C}$ given by $\hat{f}(z) = \langle f, h(z) \rangle$ is an analytic function on $V$. Furthermore, $|\hat{f}(z)| \le \|f\|_{\mu_0} \|h(z)\|_{\mu_0}$. In particular, if we set $\rho(z) = \|h(z)\|_{\mu_0}$, then $\rho : V \to [0, \infty)$ is continuous and for any $f \in \mathcal{H}_0$, the function $\hat{f} \in \mathcal{F}_\rho$. Also, notice that

$$\widehat{S_0 f}(z) = \langle S_0 f, h(z) \rangle = \langle f, S_0^* h(z) \rangle = \langle f, \overline{z} h(z) \rangle = z \langle f, h(z) \rangle = z \hat{f}(z).$$

Thus the "hat map" from $\mathcal{H}_0$ to the space of analytic functions on $V$ intertwines $S_0$ with multiplication by $z$.

*Prescribing Finite Values of the Index.* Let $\{V_n\}$ be an enumeration of the components of $K \setminus K_e$; for each component $V_n$ we have a negative integer $a_n$. Let $\mathcal{I} \subseteq \mathbb{N}$ be those values of $n$ where $a_n \le -2$. For each $n \in \mathcal{I}$, use Proposition 3.5 together with the function $\rho$ given above to find irreducible subnormal operators $S_{n,j}$, $j = 1, \ldots, (|a_n| - 1)$, of the form $S_{n,j} = M_z$ on $L^2_a(V_n, \mu_{n,j})$, where $\mathcal{F}_\rho \subseteq L^2_a(V_n, \mu_{n,j})$ and the measures $\mu_{n,i}$ and $\mu_{n,j}$ are singular whenever $i \ne j$.

For each $n \in \mathcal{I}$ and $j = 1, \ldots (|a_n| - 1)$, let $A_{n,j} : \mathcal{H}_0 \to L^2_a(V_n, \mu_{n,j})$ be given by $A_{n,j} f = \hat{f}$. Since $\mathcal{F}_\rho \subseteq L^2_a(V_n, \mu_{n,j})$, it follows that $A_{n,j}$ maps into $L^2_a(V_n, \mu_{n,j})$. Also, as in the proof of Proposition 3.5 we get that

$$\int |\hat{f}(z)|^2 d\mu_{n,j} \le C_{\hat{f}}^2 \sum_{n=1}^{\infty} \frac{1}{2^n} \le C_{\hat{f}}^2.$$

However, as shown above, $C_{\hat{f}} = \|f\|_{\mu_0}$. Thus, we have that $A_{n,j}$ is a bounded linear operator. Notice that $A_{n,j}$ is not the zero operator, since $h : V_n \to \mathcal{H}_0$ is not identically zero. As shown above $A_{n,j}$ intertwines $S_0$ with $S_{n,j}$. Also, choose $\Delta_{n,j}$ to be any compact subset of $V_n$ with positive $\mu_{n,j}$ measure. If $P_{n,j}$ is given as multiplication by the characterisitic function of $\Delta_{n,j}$ on $L^2(\mu_{n,j})$, then clearly $P_{n,j} A_{n,j}$ does not have its range entirely contained in $L^2_a(V_n, \mu_{n,j})$ because functions in the range of $P_{n,j} A_{n,j}$ are zero on sets of positive area. Finally notice that $\mu_{n,j}$ is singular with respect to $\mu_{m,k}$ when $m \ne n$ since they are carried by disjoint open sets. Also, $\mu_{n,j}$ is singular with respect to $\mu_0$ because $\mu_0 | V_n$ is purely discrete where $\mu_{n,j}$ is absolutely continuous with respect to area measure. Notice that the operator $S_0 \oplus \bigoplus_{j=1}^{(|a_n|-1)} S_{n,j}$ has index $a_n$ on the component $V_n$.

*Prescribing Infinite Values of the Index.* This part is almost identical to that above and will not be repeated in the same detail. But for each component $U_n$ of $K_e \setminus K_a$ and with the function $\rho$ as given above, use Proposition 3.5 to choose an infinite sequence (compared to the finite sequence obtained above) of pairwise singular measures $\{\nu_{n,j}\}_{j=1}^{\infty}$ and irreducible subnormal operators $S'_{n,j} = M_z$ on $L^2_a(U_n, \nu_{n,j})$. As above we have intertwining maps $B_{n,j} : \mathcal{H}_0 \to L^2_a(U_n, \nu_{n,j})$ and measurable sets $\Delta'_{n,j}$. Similar to that above, the operator $S_0 \oplus \bigoplus_{j=1}^{\infty} S'_{n,j}$ has index $-\infty$ on the component $U_n$.

Thus we have operators $S_0$, $S_{n,j}$ and $S'_{n,j}$, and their direct sum

$$T := S_0 \oplus \bigoplus_{n=1}^{\infty} \left( \bigoplus_{j=1}^{(|a_n|-1)} S_{n,j} \right) \oplus \bigoplus_{n=1}^{\infty} \left( \bigoplus_{j=1}^{\infty} S'_{n,j} \right)$$

has $\sigma(T) = \sigma(S_0) = K$, $\sigma_{ap}(T) = \sigma_{ap}(S_0) = K_a$, $\mathrm{ind}(T - \lambda) = a_n$ for $\lambda \in V_n$ and $\mathrm{ind}(T - \lambda) = -\infty$ for $\lambda \in U_n$ where $\{V_n\}$ are the components of $K \setminus K_e$ and $\{U_n\}$ are the components of $K_e \setminus K_a$.

We only need to verify that $\sigma_e(T) = K_e$. However, since $K_e = K_a \cup \bigcup_n U_n$, it follows immediately that $\sigma_e(T) = K_e$. We may now appeal to Theorem 2.1 to find an irreducible subnormal operator $S$ similar to $T$. $\square$

## 4. Final remarks

As mentioned in the Introduction, the spectral pictures are known for the class of pure subnormal operators and for the class of pure cyclic subnormal operators. For a pure rationally cyclic subnormal operator $S = M_z$ on $R^2(K, \mu)$ we must have $\sigma_e(S) = \sigma_{ap}(S)$ and $ind(S - \lambda) = -1$ for all $\lambda \in \sigma(S) \setminus \sigma_e(S)$. So describing the spectral picture of a rationally cyclic subnormal operator amounts to describing which pairs of compact sets $(K, K_e)$ can arise as the spectrum and essential spectrum of such an operator.

**Question 4.1.** Can we characterize the spectral pictures for the class of pure (or irreducible) rationally cyclic subnormal operators?

In this paper we have constructed irreducible subnormal operators with prescribed spectral pictures, however, the operators we have constructed are similar to reducible operators and thus commute with a nontrivial idempotent. An operator is said to be *strongly irreducible* if it does not commute with any nontrivial idempotents.

**Question 4.2.** Can we characterize the spectral pictures of the strongly irreducible subnormal operators?

## References

[1] K. Clancey & C.R. Putnam, *The local spectral behavior of completely subnormal operators*, Trans. Amer. Math. Soc. **163** (1972), 239–244. MR **45:**934

[2] J.B. Conway, *The dual of a subnormal operator* J. Operator Theory **5** (1981), no. 2, 195–211. MR **84j:**47037

[3] J.B. Conway, *Subnormal Operators*, Research Notes in Mathematics, Vol. 51, Pitman, London, 1981. MR **83i:**47030

[4] J.B. Conway, *The Theory of Subnormal Operators*, Amer. Math. Soc., Providence, RI, 1991. MR **92h:**47026

[5] M.J. Cowen & R.G. Douglas, *Complex Geometry and Operator Theory*, Acta. Math. **141** (1978), 187–261. MR **80f:**47012

[6] H. Grauert, *Analytische Faserungen uber holomorph vollstandigen Raumen* Math. Ann. **135** (1958), 263–273. MR **20:**4661

[7] P. McGuire, *On the spectral picture of an irreducible subnormal operator*, Proc. Amer. Math. Soc. **104** (1988), 801–808. MR **90d:**47026

[8] P. McGuire, *$C^*$–algebras generated by subnormal operators*, J. Funct. Anal. **79** (1988), 423–445. MR **90a:**47063

[9] R.F. Olin & J.E. Thomson, *Irreducible operators whose spectra are spectral sets*, Pacific J. Math. **91** (1980), 431–434. MR **82j:**47012

[10] R.F. Olin & J.E. Thomson, *Lifting the commutant of a subnormal operator*, Can. J. Math. **31** (1979), 148–156. MR **80b:**47034

[11] J.E. Thomson, *Approximation in the mean by polynomials*, Ann. of Math. (2) **133** (1991), no. 3, 477–507. MR **93g:**47026

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