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# $p$-RIDER SETS ARE $q$-SIDON SETS

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ABSTRACT. The aim of this paper is to prove that for every $p < \frac{4}{3}$, every $p$-Rider set is a $q$-Sidon set for all $q > \frac{p}{2-p}$. This gives some positive answers for the union problem of $p$-Sidon sets. We also obtain some results on the behavior of the Fourier coefficient of a measure with spectrum in a $p$-Rider set.

Let $G$ be an infinite metrizable compact abelian group, equipped with its normalized Haar measure $dx$, and $\Gamma$ its dual group (discrete and countable). For example, when $G$ is the unit circle of the complex plane, then $\Gamma$ will be identified with $\mathbb{Z}$ by $p \mapsto e_p$, where $e_p(x) = e^{2i\pi px}$. The space of complex regular Borel measures over $G$, equipped with the norm of total variation will be denoted by $M(G)$. If $\mu \in M(G)$, its Fourier transform at the point $\gamma$ is defined by $\hat{\mu}(\gamma) = \int_G \gamma(-x)d\mu(x)$. As usual $C(G)$ is the space of continuous functions on $G$ equipped with the supremum norm and $\mathcal{P}(G)$ is the space of trigonometric polynomials.

For $B \subset M(G)$ and $\Lambda \subset \Gamma$, set

$$B_\Lambda = \{f \in B |\ \forall \gamma \notin \Lambda, \quad \hat{f}(\gamma) = 0\}.$$

$B_\Lambda$ is the set of elements of $B$ whose spectrum is contained in $\Lambda$.

**Definition 0.1.** Let $1 \leq p < 2$ and $\Lambda$ a subset of $\Gamma$; $\Lambda$ is a $p$-Sidon set if there exists $C > 0$ such that for all $f \in \mathcal{P}_\Lambda(G) : (\sum_{\lambda\in\Lambda} |\hat{f}(\lambda)|^p)^{1/p} \leq C\|f\|_\infty$.

1-Sidon sets are simply called Sidon sets.

The best constant $C$ is called the $p$-Sidonicity constant of $\Lambda$ and is denoted by $S_p(\Lambda)$. Obviously, $\Lambda$ is a $p$-Sidon set implies $\Lambda$ is a $q$-Sidon set for $q > p$. If $\Lambda$ is a $p$-Sidon set and not a $q$-Sidon set for any $q < p$, $\Lambda$ is called a true $p$-Sidon set.

For more about these sets, see [B], [BP], [JW], [W] and more recently [L], [LQR].

**Definition 0.2.** A subset $A$ of $\Gamma$ is quasi-independent (resp. dissociated) if for every $(n_\gamma)_{\gamma\in A} \in \{-1; 0; 1\}^A$ (resp. $\forall (n_\gamma)_{\gamma\in A} \in \{-2; \ldots; 2\}^A$) with almost all $n_\gamma$ equal to zero:

$$\prod_{\gamma\in A} \gamma^{n_\gamma} = 1 \Rightarrow \forall \gamma \in A : \gamma^{n_\gamma} = 1.$$

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We recall that if $A$ is quasi-independent, then $A$ is a Sidon set, with $S_1(A) \leq 8$.

In the sequel, we shall denote by $[Q]$ the mesh spanned by a subset $Q$ of $\Gamma$:

$$[Q] = \Big\{ \prod_{\substack{\text{finite} \\ \gamma\in Q}} \gamma^{n_\gamma} \,|\, n_\gamma \in \{-1,0,1\} \Big\}.$$

We introduce the following notion. Lemma 1.2 below shows that quasi-independent sets are a particular case of it.

**Definition 0.3.** Let $\beta \geq 1$. A subset $A$ of $\Gamma$ shall be called an $R$-set of type $\beta$ if there is some constant $B > 0$ such that

$$\sum_{k\geq 0} r_k(1_\Gamma)\beta^{-k} \leq B \quad \text{and} \quad \forall\gamma\in\Gamma,\ \sum_{k\geq 1} r_{2k+1}(\gamma)\beta^{-(2k+1)} \leq B$$

where $r_k(\gamma)$ denotes the number of words equal to $\gamma$, built on $A$, with length $k$.

Note that $r_k(\gamma) = \text{card}\Big\{(\alpha_a)_{a\in A} \in \{-1,0,1\}^A |\ \gamma = \prod_{a\in A} a^{\alpha_a} \text{ and } \sum|\alpha_a| = k\Big\}$. Moreover, in order to be an $R$-set of type $\beta$, it is sufficient to verify that for every $\gamma \in \Gamma$ and every $k \geq 0$, $r_k(\gamma) \leq Bb^k$, for some $b < \beta$.

**Definition 0.4.** We shall denote by $C^{a\cdot s}(G)$ the completion of trigonometric polynomials for the norm

$$[\![f]\!] = \mathbb{E}\Big\|\sum_{\gamma\in\Gamma} \varepsilon_\gamma \hat{f}(\gamma)\gamma\Big\|_\infty$$

where $f \in \mathcal{P}$ and $(\varepsilon_\gamma)_{\gamma\in\Gamma}$ is a family of independent Bernoulli variables with values in $\{-1,1\}$.

An equivalent norm is obtained replacing $(\varepsilon_\gamma)_{\gamma\in\Gamma}$ by a family $(g_\gamma)_{\gamma\in\Gamma}$ of standard gaussian independent variables (see [P1]). This space is intensively studied in [MP] and [P1].

Following the result of Drury ("the union of two Sidon sets is a Sidon set"), many improvements were achieved in the 70s for Sidon sets. Rider [R], particularly, showed that they may be characterized by the inequality

$$\forall f \in \mathcal{P}_\Lambda(G),\ \|\hat{f}\|_1 = \sum_{\gamma\in\Gamma} |\hat{f}(\gamma)| \leq C\,[\![f]\!]\,.$$

This led the second author to introduce and study the following notion (we use the terminology "$p$-Rider" although these sets were named "$p$-Sidon ps" in [RP1], [RP2])

**Definition 0.5.** Let $p \in [1,2)$. A subset $\Lambda$ of $\Gamma$ is $p$-Rider if there exists a constant $C$ such that

$$\forall\ f \in \mathcal{P}_\Lambda(G), \qquad \|\hat{f}\|_p \leq C\,[\![f]\!]\,.$$

The lowest constant in the previous inequality is denoted by $\rho_p(\Lambda)$.

The main references on this subject are [RP1] and [RP2]. Some new results on these sets are obtained in [LQR] and [LLQR].

This notion has no interest when $p \geq 2$; any subset of $\Gamma$ is a 2-Rider set because the $\ell^2$ norm of the coefficient is dominated by the norm $[\![.]\!]$.

It is easy to see that the preceding notion is more general than the one of the $p$-Sidon set. Indeed, let us fix $\Lambda$ a $p$-Sidon set. For any trigonometric polynomial $f \in \mathcal{P}_\Lambda$, we have

$$\|\hat{f}\|_p \leq S_p(\Lambda)\|f\|_\infty.$$

Hence, the inequality still holds replacing $f$ by $f^\omega = \sum_{\gamma\in\Lambda} \varepsilon_\gamma(\omega)\hat{f}(\gamma)\gamma$, for any $\omega \in \Omega$. Thus, for every $\omega \in \Omega$, $\|\widehat{f^\omega}\|_p = \|\hat{f}\|_p$. Now, integrating over $\Omega$, we obtain the inequality

$$\|\hat{f}\|_p \leq S_p(\Lambda) [\![ f ]\!]$$

so $\Lambda$ is a $p$-Rider set with $\rho_p(\Lambda) \leq S_p(\Lambda)$.

Concerning the converse, the problem is open in full generality. In her habilitation thesis, M. Déchamps-Gondim ([DG], p. 41) states (without proof) a result privately communicated to her by J. Bourgain: *Let $p < \frac{4}{3}$. Let $\Lambda \subset \Gamma$ such that for every $f \in \mathcal{P}_\Lambda$, $\|\hat{f}\|_p \leq C [\![ f ]\!]$, for some constant $C > 0$. Then $\Lambda$ is a $q$-Sidon set for some $q < 2$.*

In this paper, we shall prove the following theorem, which obviously implies the preceding statement.

**Main Theorem.** *Let $\Lambda \subset \Gamma$, $p < \frac{4}{3}$, $s = \frac{p}{2-p}$, $\rho > 0$ and $\varphi(x) = \frac{x^{1/s}}{1+\log(x)}\cdot$.*

*Let us consider the following assertions:*

(*i*) *$\Lambda$ is a $p$-Rider set with $\rho_p(\Lambda) \leq \rho$.*

(*ii*) *There exists a constant $K > 0$ depending on $\rho$ such that:*

$$\forall\ f \in C_\Lambda(G),\ \forall t > 0,\ \varphi\Big(\mathrm{card}\{\gamma \in \Lambda : |\hat{f}(\gamma)| \geq t\}\Big) \leq \frac{K}{t}\|f\|_\infty.$$

(*iii*) *$\Lambda$ is a $q$-Sidon set for any $q > s$.*

*Then* $(i) \Longrightarrow (ii) \Longrightarrow (iii)$.

*Remark* 0.6. The second statement in the preceding theorem means that if $f \in C_\Lambda(G)$, then $\hat{f}$ belongs to the Lorentz space $\{(a_n)/\sup_{n\geq 1}\varphi(n)a_n^* < \infty\}$, $(a_n^*)$ being the decreasing rearrangement of $\{|a_n|\}_{n\geq 1}$.

*Remark* 0.7. The main theorem also holds when we only assume $p \in [1,2)$ but if $p \geq \frac{4}{3}$, then $s \geq 2$ so that the conclusion is not interesting.

The main result may appear more interesting with the following obvious corollary.

**Corollary 0.8.** *Let $\Lambda \subset \Gamma$. Then $\Lambda$ is a $p$-Rider set for every $p > 1$ if and only if $\Lambda$ is a $p$-Sidon set for every $p > 1$.*

## 1. The proof

The proof of the main theorem relies on some results obtained by the second author in [RP1] (see also his thesis [RP2]) and on several lemmas.

**Lemma 1.1.** *Let $\Lambda \subset \Gamma$, $p \in [1,2)$ and $s = \frac{p}{2-p}$. We suppose that $\Lambda$ is a $p$-Rider set.*

*Then there exists a constant $C = C(\rho_p(\Lambda))$ such that, for every quasi-independent subset $Q$ of $\Gamma$, we have*

$$\mathrm{card}(\Lambda \cap [Q]) \leq C\Big(\mathrm{card}(Q).\log\mathrm{card}(Q)\Big)^s.$$

*Proof.* The set $\Lambda \cap [Q]$ is a subset of $\Lambda$ and then a $p$-Rider set with $\rho_p(\Lambda \cap [Q]) \leq \rho_p(\Lambda)$. By Theorem Th.6 of [RP1], there exists a quasi-independent subset $I$ of $\Lambda \cap [Q]$ with cardinal greater than $\delta\big(\mathrm{card}(\Lambda \cap [Q])\big)^{\frac{1}{s}}$, where $\delta$ depends only on $\rho_p(\Lambda)$.

We have $\mathrm{card}(I) = \mathrm{card}(I \cap \Lambda \cap [Q]) = \mathrm{card}(I \cap [Q])$. But the mesh condition (see [K]) for the Sidon set $I$ (with $S_1(I) \leq 8$) gives ($k$ being an absolute constant)

$$\mathrm{card}(I \cap [Q]) \leq k\, \mathrm{card}(Q). \log \mathrm{card}(Q).$$

We then deduce $\big(\mathrm{card}(\Lambda\cap[Q])\big)^{\frac{1}{s}} \leq K \mathrm{card}(Q). \log \mathrm{card}(Q)$ and the result follows. □

The following lemma is well known.

**Lemma 1.2.** *Every quasi-independent set $Q \subset \Gamma$ is an $R$-set of type 2.*

*Proof.* Fix $N \geq 1$ and write $Q = \{\lambda_j |\ j \geq 0\}$. Now we use the Riesz product

$$P = \prod_{j=0}^{N} \big[1 + \frac{1}{2}(\lambda_j + \bar{\lambda}_j)\big].$$

As $P$ is a product of nonnegative terms, it is itself nonnegative, so that $\|P\|_1 = \hat{P}(1_\Gamma)$ (where $1_\Gamma$ is the unit of $\Gamma$). As $Q$ is a quasi-independent set, it is easy to see that $\hat{P}(1_\Gamma) = 1$. Now, for every $\gamma \in \Gamma$,

$$\sum_{k=0}^{N} r_k(\gamma) 2^{-k} = \hat{P}(\gamma) \leq \|P\|_1 = 1. \quad \square$$

**Lemma 1.3.** *Let $\varepsilon \in (0, \frac{1}{2}]$ and $\beta > 0$. There exists a measure $\sigma$ on $[0, \frac{1}{2\beta}]$ such that*

$$\int d|\sigma| \leq c\beta |\log \varepsilon|, \quad \int s d\sigma(s) = 1, \quad \forall\ k \in \mathbb{N} \setminus \{0\}, \Big|\int s^{2k+1} d\sigma(s)\Big| \leq \varepsilon \beta^{-2k}$$

*where $c$ is an absolute constant.*

*Proof.* Let $\sigma_0$ be the measure on $[0, \frac{1}{2}]$ constructed in Lemma 3 [M], which verifies

$$\|\sigma_0\| = \int d|\sigma_0| \leq c_0 |\log \varepsilon|, \ \int s d\sigma_0(s) = 1, \ \forall\ k \in \mathbb{N} \setminus \{0\}, \ \Big|\int s^{2k+1} d\sigma_0(s)\Big| \leq \varepsilon,$$

where $c_0$ is an absolute constant.

We then define $\sigma$ by $\sigma(A) = \beta\sigma_0(\beta A)$, for any Borel set $A$. The measure $\sigma$ has norm $\|\sigma\| = \beta\|\sigma_0\|$ and for every $n \in \mathbb{N} \setminus \{0\}$, $\displaystyle\int s^n d\sigma(s) = \beta^{1-n} \int t^n d\sigma_0(t)$. The result follows. □

The following proposition is a generalization of a result due to J.F. Méla [M], which was stated only for dissociated sets. We shall use it when $Q$ is a quasi-independent set (see Lemma 1.2: take $\beta = 2$ and $B = 1$).

**Proposition 1.4.** *Let $Q \subset \Gamma$ be an $R$-set of type $\beta$ and $\varepsilon \in (0, \frac{1}{2}]$.*

*There exists a measure $\mu \in M(G)$ such that*

*(i) $\|\mu\| \leq c\beta B |\log \varepsilon|$,*

*(ii) $\forall \gamma \notin [Q]$, $\hat{\mu}(\gamma) = 0$,*

*(iii) $\forall \gamma \notin Q$, $|\hat{\mu}(\gamma)| \leq \beta B \varepsilon$,*

*(iv)* $\forall \gamma \in Q$, $|\hat{\mu}(\gamma)| \geq 1 - \beta B \varepsilon$,
*where* $c$ *is an absolute constant.*

*Proof.* The proof essentially uses the ideas of [M] and we adapt it to the present arithmetical framework. For the sake of completeness, we give the whole proof for $R$-sets.

We write $Q = \{\gamma_j : j \geq 1\}$ and fix an integer $N \geq 1$. For every $s \in [0, \frac{1}{2\beta}]$, the following partial products (with spectrum contained in $[Q]$) are bounded in $L^1(G)$ for every $x \in \mathbb{T}$:

$$\nu_{s,N} = \prod_{j=1}^{N} \Big[1 + s(e_1(x)\gamma_j + e_1(-x)\bar{\gamma}_j)\Big].$$

Indeed, it is a product of nonnegative terms, so its $L^1$-norm coincides with its Fourier coefficient at $1_\Gamma$, the unit of $\Gamma$. Hence the $L^1$-norm is less than

$$\sum_{k=0}^{N} r_k(1_\Gamma) s^k \leq \sum_{k=0}^{N} r_k(1_\Gamma) \beta^{-k} \leq B,$$

where the constant $B$ is the one in Definition 0.3.

We have for every $x \in \mathbb{T}$, $\|\nu_{s,N}(.,x)\|_{M(G)} \leq B$, so that $\|\nu_{s,N}\|_{M(G\times\mathbb{T})} \leq B$.

First, we note that for any $\gamma \in \Gamma$ and any $n \in \mathbb{Z}$, the Fourier coefficient $\widehat{\nu_{s,N}}(\gamma, n)$ is nonzero only if $n = \sum_1^N \varepsilon_j$ for some $(\varepsilon_j)_{1 \leq j \leq N} \in \{-1, 0, 1\}^N$ and $\gamma$ has the form

$$\gamma = \prod_1^N \gamma_j^{\varepsilon_j}.$$

In this case,

$$\widehat{\nu_{s,N}}(\gamma, n) = \sum_{\substack{\gamma = \prod \gamma_j^{\alpha_j} \\ n = \sum \alpha_j}} s^{\sum |\alpha_j|}$$

where the sum covers all the ways to write $\gamma$ as $\prod_1^N \gamma_j^{\alpha_j}$ and $n = \sum_1^N \alpha_j$ with $\alpha_j \in \{-1, 0, 1\}$.

Let us define the measure $\mu_{s,N} = \int_{\mathbb{T}} \nu_{s,N}(.,x)\overline{e_1(x)}dx \in M(G)$, for $s \in [0, \frac{1}{2\beta}]$. Obviously, $\|\mu_{s,N}\| \leq B$.

Moreover, for every $\gamma \in \Gamma$, we have $\widehat{\mu_{s,N}}(\gamma) = \widehat{\nu_{s,N}}(\gamma, 1)$. Hence, the Fourier coefficient $\widehat{\mu_{s,N}}(\gamma)$ is nonzero only if $\gamma = \prod_1^N \gamma_j^{\varepsilon_j}$, where $\varepsilon_j \in \{-1, 0, 1\}$ and $\sum_1^N \varepsilon_j = 1$.

If $\gamma = \prod_1^N \gamma_j^{\varepsilon_j}$, where $\varepsilon_j \in \{-1, 0, 1\}$ with $\sum_1^N |\varepsilon_j|$ even, then $\widehat{\mu_{s,N}}(\gamma) = 0$. Indeed, the condition $\sum_1^N \varepsilon_j = 1$, i.e. $|\{\varepsilon_j = 1\}| - |\{\varepsilon_j = -1\}| = 1$, implies that $\sum_1^N |\varepsilon_j| = |\{\varepsilon_j = 1\}| + |\{\varepsilon_j = -1\}|$ is odd.

We now use Lemma 1.3 and define the measure $\mu_N = \displaystyle\int \mu_{s,N} d\sigma(s)$.

We have $\|\mu_N\| \leq c\beta B|\log\varepsilon|$.

For every $\gamma \in \Gamma$, $\widehat{\mu_N}(\gamma) = \displaystyle\int \widehat{\mu_{s,N}}(\gamma) d\sigma(s)$. More precisely, if $\gamma = \displaystyle\prod_1^N \gamma_j^{\varepsilon_j}$, then

$$\widehat{\mu_N}(\gamma) = \sum_{\substack{\gamma=\prod \gamma_j^{\alpha_j} \\ \alpha_j \in \{-1,0,1\}}} \int s^{\sum|\alpha_j|} d\sigma(s).$$

Testing this in the special case $\gamma = \gamma_i \in \{\gamma_1, \dots, \gamma_N\}$, we get

$$\widehat{\mu_N}(\gamma_i) = \int s d\sigma(s) + \sum_{\substack{\gamma=\prod \gamma_j^{\alpha_j} \\ \alpha_j \in \{-1,0,1\}}} \int s^{\sum|\alpha_j|} d\sigma(s)$$

where in the right-hand term, we necessarily have $3 \leq \sum|\alpha_j| \leq N$.

Then

$$\widehat{\mu_N}(\gamma_i) = \int s d\sigma(s) + \sum_{\substack{\gamma=\prod \gamma_j^{\alpha_j} \\ 3 \leq \sum|\alpha_j| = 2k+1 \leq N}} \int s^{2k+1} d\sigma(s).$$

By Lemma 1.3, we then have

$$\left|\widehat{\mu_N}(\gamma_i)\right| \geq 1 - \sum_{\substack{\gamma=\prod \gamma_j^{\alpha_j} \\ \sum|\alpha_j|=2k+1; k\geq 1}} \left|\int s^{2k+1} d\sigma(s)\right| \geq 1 - \sum_{k\geq 1} r_{2k+1}(\gamma_i)\varepsilon\beta^{-2k}.$$

Since $Q$ is an $R$-set of type $\beta$, we get

$$\left|\widehat{\mu_N}(\gamma_i)\right| \geq 1 - \beta B\varepsilon.$$

In the same way, when $\gamma \notin \{\gamma_1, \dots, \gamma_N\}$,

$$\widehat{\mu_N}(\gamma) = \sum_{\substack{\gamma=\prod \gamma_j^{\alpha_j} \\ 3 \leq \sum|\alpha_j| = 2k+1 \leq N}} \int s^{2k+1} d\sigma(s) = \sum_{3 \leq 2k+1 \leq N} r_{2k+1}(\gamma) \int s^{2k+1} d\sigma(s)$$

so

$$\left|\widehat{\mu_N}(\gamma)\right| \leq \beta B\varepsilon.$$

As the sequence $(\mu_N)_N$ is bounded in $M(G)$ by $c\beta B|\log\varepsilon|$, we deduce that there is a subsequence converging (for the $w^*$-topology of $M(G)$) to a measure $\mu$, which has all the required properties. □

Note that the preceding property is still true when we assume that $Q$ is a finite union of $R$-sets (with a similar proof).

*Proof of the main theorem.* $(i) \Longrightarrow (ii)$. It is sufficient to prove it for a function $f \in C_\Lambda(G)$ such that $\|f\|_\infty = 1$. For every $t > 0$, the set $A_t = \{\gamma \in \Lambda : |\hat{f}(\gamma)| \geq t\}$ is finite (as $\hat{f} \in c_0(\Gamma)$). Applying Theorem Th.6 [RP1], as $\Lambda$ is a $p$-Rider set, there exists a quasi-independent set $Q \subset A_t$ such that $\mathrm{card}(Q) \geq \delta \mathrm{card}(A_t)^{\frac{1}{s}}$, where $\delta$ depends only on $\rho$.

Now, we fix for a moment $\varepsilon \in (0, \frac{1}{4}]$. Proposition 1.4 (applied for the quasi-independent set $Q$ with $\beta = 2$ and $B = 1$) provides a measure $\mu$, whose spectrum is a subset of $[Q]$, hence is finite.

We then have

$$\begin{aligned}2c|\log\varepsilon| &\geq \|\mu\|.\|f\|_\infty \geq \|f*\mu\|_\infty \\ &\geq \|\sum_{\gamma\in Q}\hat{f}(\gamma)\hat{\mu}(\gamma)\gamma\|_\infty - \|\sum_{\gamma\in\Lambda\cap[Q]\setminus Q}\hat{f}(\gamma)\hat{\mu}(\gamma)\gamma\|_\infty.\end{aligned}\tag{1}$$

But $Q$ is a Sidon set (with constant less than 8) so

$$\|\sum_{\gamma\in Q}\hat{f}(\gamma)\hat{\mu}(\gamma)\gamma\|_\infty \geq \frac{1}{8}\|\hat{f}\hat{\mu}\|_{\ell^1(Q)}.$$

Using the fact that $|\hat{\mu}|\geq 1-2\varepsilon$ and $|\hat{f}|\geq t$ on $Q\subset A_t$, we get

$$\begin{aligned}\|\sum_{\gamma\in Q}\hat{f}(\gamma)\hat{\mu}(\gamma)\gamma\|_\infty &\geq \frac{1}{8}t(1-2\varepsilon)\mathrm{card}(Q)\geq\frac{\delta}{8}t(1-2\varepsilon)\mathrm{card}(A_t)^{\frac{1}{s}} \\ &\geq \frac{t\delta}{16}\mathrm{card}(A_t)^{\frac{1}{s}}.\end{aligned}\tag{2}$$

On the other hand, by the properties of $\mu$,

$$\begin{aligned}\|\sum_{\gamma\in\Lambda\cap[Q]\setminus Q}\hat{f}(\gamma)\hat{\mu}(\gamma)\gamma\|_\infty &\leq \sup\nolimits_{\gamma\in[Q]\setminus Q}|\hat{\mu}(\gamma)|.\|f\|_\infty\mathrm{card}(\Lambda\cap[Q]) \\ &\leq 2\varepsilon\mathrm{card}(\Lambda\cap[Q]).\end{aligned}\tag{3}$$

Injecting inequalities (2) and (3) into (1), we get

$$2c|\log\varepsilon| + 2\varepsilon\mathrm{card}(\Lambda\cap[Q]) \geq \frac{t\delta}{16}\mathrm{card}(A_t)^{\frac{1}{s}}.$$

We now choose $\varepsilon=\Big(2\mathrm{card}(\Lambda\cap[Q])\Big)^{-1}$ (we may and do suppose that $\mathrm{card}(A_t)$ is large enough to have $\mathrm{card}(\Lambda\cap[Q])\geq 2$).

Using Lemma 1.1, we have

$$\mathrm{card}(\Lambda\cap[Q]) \leq C\big(\mathrm{card}(Q).\log(\mathrm{card}(Q))\big)^s \leq C\big(\mathrm{card}(A_t)\big)^{2s}.$$

We get the following inequality:

$$1+2c\log\Big(2C\big(\mathrm{card}(A_t)\big)^{2s}\Big) \geq \frac{t\delta}{16}\mathrm{card}(A_t)^{\frac{1}{s}}.$$

Then, with $M=\max(1+2c\log(2C),4cs)$, we finally obtain

$$\varphi\big(\mathrm{card}(A_t)\big) \leq \frac{16M}{\delta t}$$

which is the first part of the theorem.

$(ii)\implies(iii)$. The second part is very easy: Let $f\in C_\Lambda(G)$ with $\|f\|_\infty=1$. Note that for any $q>s$ and $r\in(s,q)$, there exists a constant $k_r>0$ such that $\varphi(x)\geq k_r x^{\frac{1}{r}}$ for $x\geq 1$, hence $\big(\mathrm{card}(A_t)\big)^{\frac{1}{r}}\leq\frac{K_r}{t}$, where $K_r$ depends only on $\rho$ and $r$.

For every $n\geq 0$, let

$$\Lambda_n=\{\gamma\in\Lambda:\ |\hat{f}(\gamma)|\in(2^{-(n+1)},2^{-n}]\}.$$

We have $\big(\mathrm{card}(\Lambda_n)\big)^{\frac{1}{r}}\leq K_r2^{n+1}$.

We now compute

$$\|\hat{f}\|_q^q = \sum_{n\geq 0}\sum_{\gamma\in\Lambda_n} |\hat{f}(\gamma)|^q \leq \sum_{n\geq 0} 2^{-nq}\mathrm{card}(\Lambda_n) \leq \sum_{n\geq 0} 2^{-nq}.K_r^r 2^{r(n+1)}.$$

As $q > r$, the right-hand term converges and the theorem is proved. □

## 2. Applications

Is the union of two $p$-Sidon sets still a $p$-Sidon set? The problem is still open in full generality. It was proved by Woodward [W] that the union of a $p$-Sidon set and a Sidon set is still a $p$-Sidon set. On the other hand, it is obvious that the union of two $p$-Rider sets is still $p$-Rider (this is due to the unconditionality of the character basis in the space $C^{a \cdot s}(G)$). Our main theorem implies

**Corollary 2.1.** *Let $p < \frac{4}{3}$ and $s = \frac{p}{2-p}\cdot$ The union of two $p$-Sidon sets is a $q$-Sidon set for all $q > s$.*

Indeed, the union of two $p$-Sidon sets is a $p$-Rider set hence $q$-Sidon.

As a special case, the class of subsets of $\Gamma$ which are $p$-Sidon for every $p > 1$, is stable under finite union.

On the other hand, the ideas used to prove the main theorem give a similar result for the space $M_\Lambda(G)$ when $\Lambda$ is a $p$-Rider set. These results are very close to the ones obtained for $p$-Sidon sets in [L] (see Corollary 2.4).

**Theorem 2.2.** *Let $p \in [1, 2)$ and $\alpha = \frac{2p}{2-p}\cdot$ Let $\Lambda \subset \Gamma$, $\rho > 0$ and $\psi(x) = \frac{x^{1/\alpha}}{1+\log(x)}\cdot$*
*Let us consider the following assertions:*

*(i) $\Lambda$ is a $p$-Rider set with $\rho_p(\Lambda) \leq \rho$.*

*(ii) There exist a constant $K > 0$ depending on $\rho$ such that*

$$\forall\ m \in M_\Lambda(G), \forall t > 0, \psi\Big(\mathrm{card}\{\gamma\in\Lambda : |\hat{m}(\gamma)| \geq t\}\Big) \leq \frac{K}{t}\|m\|_{M(G)}.$$

*(iii) For every $m \in M_\Lambda(G)$, we have $\hat{m} \in \ell^a$, for any $a > \alpha$.*

*Then (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii).*

*Proof.* We adapt the argument given in the main theorem. Let us fix a function $m \in L^1_\Lambda(G)$, normalized by $\|m\|_1 = 1$. For every $t > 0$, the set $A_t = \{\gamma \in \Lambda : |\hat{m}(\gamma)| \geq t\}$ is finite (as $\hat{m}$ tends to zero). Applying Theorem Th.6 [RP1], there exists a quasi-independent set $Q \subset A_t$ such that $\mathrm{card}(Q) \geq \delta \mathrm{card}(A_t)^{\frac{1}{s}}$, where $\delta$ depends only on $\Lambda$.

Now, we take $\varepsilon = \Big(2\mathrm{card}(\Lambda\cap[Q])\Big)^{-\frac{1}{2}}$ and we can assume $\varepsilon \in (0, \frac{1}{4}]$. Proposition 1.4 provides us with a measure $\mu$. We then have

$$2c|\log\varepsilon| \geq \|\mu\|.\|m\|_1 \geq \|m*\mu\|_1 \geq \|\sum_{\gamma\in Q}\hat{m}(\gamma)\hat{\mu}(\gamma)\gamma\|_1 - \|\sum_{\gamma\in\Lambda\cap[Q]\setminus Q}\hat{m}(\gamma)\hat{\mu}(\gamma)\gamma\|_1.$$

But $Q$ is a Sidon set (with constant less than 8) and then a so-called $\Lambda(2)$ set, i.e., $\|\sum_{\gamma\in Q}\hat{m}(\gamma)\hat{\mu}(\gamma)\gamma\|_1 \geq k\|\hat{m}\hat{\mu}\|_{\ell^2(Q)}$, where $k > 0$ is an absolute constant. We get

$$\|\sum_{\gamma\in Q}\hat{m}(\gamma)\hat{\mu}(\gamma)\gamma\|_1 \geq kt(1-2\varepsilon)\mathrm{card}(Q)^{\frac{1}{2}} \geq \frac{1}{2}k\delta t\,\mathrm{card}(A_t)^{\frac{1}{\alpha}}.$$

On the other hand, we have

$$\|\sum_{\gamma\in\Lambda\cap[Q]\setminus Q}\hat{m}(\gamma)\hat{\mu}(\gamma)\gamma\|_1 \le 2\varepsilon\mathrm{card}(\Lambda\cap[Q])^{\frac{1}{2}} = 1.$$

We conclude that there is some constant $K$ (depending only on $\rho$) such that

$$\psi\big(\mathrm{card}(A_t)\big) \le \frac{K}{t}$$

Now, to conclude for every $m \in M(G)$, it suffices to apply the preceding result to $F_N * m$, where $F_N$ is an approximate identity. We then conclude $\psi(\mathrm{card}(B)) \le K/t$ for arbitrary finite subsets $B$ of $\{\gamma\in\Lambda :\ |\hat{m}(\gamma)| \ge t\}$, therefore for the whole set. This gives the first part of the theorem.

The second part follows the scheme of the proof of the second part of the main theorem and is left to the reader. □

The preceding theorem gives some results on the regularity of measures with spectrum in a $p$-Rider set. Using a principle due to J. Fournier and L. Pigno [FP] (see also [L]), we get that every $p$-Rider subset of $\Gamma$ is a set of continuity: for every $\varepsilon > 0$, there exists $\delta > 0$ such that for every $\mu \in M(G)$ with $\|\mu\| = 1$,

$$\overline{\lim_{\Gamma\setminus\Lambda}}|\hat{\mu}(\gamma)| < \delta \Rightarrow \overline{\lim_{\Lambda}}|\hat{\mu}(\gamma)| < \varepsilon.$$

As an immediate consequence, every $p$-Rider set is a so-called Rajchman set: every measure with spectrum in a $p$-Rider set has a Fourier transform whose coefficients tend to zero at infinity.

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