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ON BEURLING-TYPE THEOREMS IN WEIGHTED l^2 AND BERGMAN SPACES

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ABSTRACT. We prove that analytic operators satisfying certain series of operator inequalities possess the wandering subspace property. As a corollary, we obtain Beurling-type theorems for invariant subspaces in certain weighted l^2 and Bergman spaces.

1. INTRODUCTION

Let X be a Hilbert space of functions analytic in the unit disk $\mathbb{D}$ of the complex plane. Assume that X is invariant with respect to the shift operator S defined as $(Sf)(z) = zf(z)$. We shall say that the Beurling-type theorem holds in X if any S-invariant subspace $I \subset X$ is the smallest S-invariant subspace containing $I \ominus SI$. The classical example is given by the Hardy space H^2 where by Beurling's theorem any S-invariant subspace I has the form $I = \Theta H^2$ for some inner function Θ. In this case $I \ominus SI$ is the one-dimensional subspace generated by Θ. Beurling's description can be considered as a particular case of the Wold-Kolmogorov decomposition theorem for isometries, and in fact it follows from Wold's decomposition that the Beurling-type theorem holds in any space X where S is an isometry.

The first example of spaces X where S is not isometric but the Beurling-type theorem holds was found by S. Richter [6]. He proved that the Beurling-type theorem holds in any space X where the shift operator S satisfies the concavity inequality

$$\|S^2 f\|^2 + \|f\|^2 \leqslant 2\|Sf\|^2, \quad f \in X.$$

In fact, he proved a more general theorem: if H is a Hilbert space and an operator $T \in L(H)$ is such that

$$\bigcap_{n=1}^{\infty} T^n H = \{0\} \tag{1.1}$$

and T satisfies the concavity inequality

$$\|T^2 x\|^2 + \|x\|^2 \leqslant 2\|Tx\|^2, \tag{1.2}$$

then H is the smallest T-invariant subspace containing the wandering subspace $H \ominus TH = \ker T^*$. We shall say that an operator T is *analytic* if (1.1) holds, and that T possesses the *wandering subspace property* if H is the smallest T-invariant

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subspace containing $H \ominus TH$. In other words, Richter's theorem states that analytic operators satisfying the concavity inequality (1.2) possess the wandering subspace property.

The next example of spaces X where the Beurling-type theorem holds was found by A. Aleman, S. Richter and C. Sundberg [1]. This example is the Bergman space $L^2_a(\mathbb{D})$ of functions analytic and square area integrable in $\mathbb{D}$. The original proof in [1] used special tools of function theory in Bergman spaces such as the biharmonic Green function, but later it was shown in [8] that the only special property of the Bergman space $L^2_a(\mathbb{D})$ needed for the Beurling-type theorem is the inequality

$$\|Sf + g\|^2 \leqslant 2(\|f\|^2 + \|Sg\|^2), \quad f, g \in L^2_a(\mathbb{D}). \tag{1.3}$$

More generally, it was proved in [8] that any analytic operator T satisfying the inequality

$$\|Tx + y\|^2 \leqslant 2(\|x\|^2 + \|Ty\|^2), \quad x, y \in H, \tag{1.4}$$

possesses the wandering subspace property.

In the present paper we find one more condition written in terms of operator inequalities which implies that analytic operators possess the wandering subspace property. We deal with analytic contractions T which are left invertible, i.e. such that the operator T^*T is invertible. We define

$$L := (T^*T)^{-1}T^*. \tag{1.5}$$

The following theorem is our main result:

Theorem 1.1. *Assume that T is a left-invertible analytic contraction such that the operator L defined by* (1.5) *has spectral radius one. Assume also that there exists a family $\{\varphi_\alpha\}_{\alpha \in A}$ of functions $\varphi_\alpha \in H^\infty(\mathbb{D})$ such that*

(i) *$\|\varphi_\alpha(T)x\| \geqslant \|x\|$ for any $\alpha \in A$ and $x \in H$ ($\varphi_\alpha(T)$ is taken in the sense of Sz.-Nagy – Foiaş functional calculus);*

(ii) *$\varphi_\alpha(0) = 0$ for any $\alpha \in A$; the functions $\frac{\varphi_\alpha(z)}{z}$ are uniformly bounded and bounded away from zero in $\mathbb{D}$;*

(iii) *for any $z_0 \in \mathbb{D}$ there exists $\alpha_0 \in A$ such that $|\varphi_{\alpha_0}(z_0)| < 1$.*

Then T possesses the wandering subspace property.

In applications, it is convenient to deal with families $\{\varphi_\zeta\}$ of the form $\varphi_\zeta(z) = \varphi(\zeta z)$, where the index ζ runs over $\mathbb{T} = \partial\mathbb{D}$. We have the following corollary:

Corollary 1.2. *Assume that T is as in Theorem 1.1 and there exists a function $\varphi \in H^\infty(\mathbb{D})$ such that*

(i) *$\|\varphi(\zeta T)x\| \geqslant \|x\|$ for any $\zeta \in \mathbb{T}$ and $x \in H$;*

(ii) *$\varphi(0) = 0$ and $\frac{\varphi(z)}{z}$ is bounded away from zero in $\mathbb{D}$;*

(iii) *$|\varphi(r)| < 1$ for any $r \in [0, 1)$.*

Then T possesses the wandering subspace property.

Examples of operators satisfying conditions of Corollary 1.2 are given by the shift operators in certain weighted l^2 and Bergman spaces. For a weight sequence $(w_n)_{n\geqslant 0}$, the space $l^2(w_n)$ consists of sequences $(x_n)_{n\geqslant 0}$ such that

$$\|(x_n)_{n\geqslant 0}\|^2 := \sum_{n\geqslant 0} |x_n|^2 w_n < +\infty.$$

This space can be naturally considered as a space of analytic functions if we identify any sequence $(x_n)_{n\geqslant 0}$ with the function $f(z) = \sum_{n\geqslant 0} x_n z^n$.

For a weight function $\omega(z)$, $z \in \mathbb{D}$, the weighted Bergman space $L^2_a(\mathbb{D}, \omega)$ consists of functions f analytic in $\mathbb{D}$ and such that

$$\|f\|^2_\omega := \int_{\mathbb{D}} |f(z)|^2 \, \omega(z) \, dm_2(z) < +\infty.$$

Here, dm_2 is the normalized area measure in $\mathbb{D}$.

Let $\varphi(z) = z(2 - z)$. Then conditions (ii) and (iii) of Corollary 1.2 are fulfilled. Moreover, as we shall see later, any contraction T satisfying

$$\|T(2\zeta - T)x\|^2 \geqslant \|x\|^2, \quad \zeta \in \mathbb{T}, \ x \in H, \tag{1.6}$$

is left invertible and such that the operator L defined by (1.5) has spectral radius one. Therefore, we get the following:

Corollary 1.3. *If an analytic contraction T satisfies the inequalities* (1.6)*, then T possesses the wandering subspace property.*

Below, we shall analyse the condition (1.6) for the shift operator in spaces $l^2(w_n)$ (see Proposition 3.1). As to weighted Bergman spaces, we have the following result.

Theorem 1.4. *Let $\omega(z)$, $z \in \mathbb{D}$, be such a weight function that*

$$\Delta \log \left(\frac{\omega(z)}{1 - |z|^2} \right) \geqslant 0. \tag{1.7}$$

Then the Beurling-type theorem holds in the space $L^2_a(\mathbb{D}, \omega)$.

This theorem will be obtained as an application of Corollary 1.2 with $\varphi(z) = 1 - (1 - z)^3$. In particular, we obtain that the Beurling-type theorem holds in the standard weighted Bergman spaces corresponding to the weight functions $\omega_\alpha(z) = (\alpha+1)(1-|z|^2)^\alpha$, with $\alpha \in (-1, 1]$. For $\alpha > 1$, it seems that such a theorem fails; at least for $\alpha > 4$ there are examples of zero-based invariant subspaces I in $L^2_a(\mathbb{D}, \omega_\alpha)$ where extremal functions (elements of $I \ominus SI$) may have extra zeros (see [5]). For the class of logarithmically subharmonic weights ω which is smaller than the class determined by condition (1.7), the Beurling-type theorem for $L^2_a(\mathbb{D}, \omega)$ was proved in [8]. In these spaces the shift operator satisfies both inequalities (1.3) and (1.6).

2. Main result

In this section we prove Theorem 1.1. We also prove that under the conditions (i)–(iii) of Corollary 1.2, some additional assumption on the function φ implies that the spectral radius of L is one.

Let T and L be as in Theorem 1.1. Put $E = H \ominus TH = \ker T^*$. Then there exists a Hilbert space $\mathcal{H}$ of E-valued functions analytic in $\mathbb{D}$ and a unitary operator $U : H \mapsto \mathcal{H}$ with the following properties:

• $\mathcal{H}$ is a reproducing kernel Hilbert space and the ($L(E)$-valued) reproducing kernel $k_{\mathcal{H}}$ is such that

$$k_{\mathcal{H}}(z, 0) = I_E \tag{2.1}$$

for any $z \in \mathbb{D}$;

• $UTU^{-1} = \mathcal{S}$ and $ULU^{-1} = \mathcal{L}$, where

$$(\mathcal{S}f)(z) = zf(z) \quad \text{and} \quad (\mathcal{L}f)(z) = \frac{f(z) - f(0)}{z}$$

for any $f \in \mathcal{H}$;

• $\mathcal{E} = UE$ is the subspace of $\mathcal{H}$ consisting of constant E-valued functions, and for $e \in E$ we have $(Ue)(z) = e$.

The details of the construction of U and $\mathcal{H}$ are presented in [8]. In brief, we define $(Ux)(z) := P_E(I - zL)^{-1}x$ for $x \in H$, and then $\mathcal{H}$ becomes the Hilbert space of images Ux, $x \in H$, with the norm induced from H. After the passage to the Hilbert space $\mathcal{H}$, the wandering subspace property of T is equivalent to the fact that E-valued polynomials are dense in $\mathcal{H}$. Let $\mathcal{H}_0$ be the closure of E-valued polynomials in $\mathcal{H}$.

Let $H^2(E)$ be the space of E-valued functions $f(z) = \sum_{n\geqslant 0} \hat{f}(n)z^n$, $\hat{f}(n) \in E$, with finite norm

$$\|f\|^2_{H^2(E)} := \sum_{n\geqslant 0} \|\hat{f}(n)\|^2_E.$$

Lemma 2.1. *If T (and hence also $\mathcal{S}$) is a contraction, then $H^2(E) \subset \mathcal{H}_0$ and the inclusion operator is contractive.*

Proof. It suffices to prove that the inequality

$$\|f\|^2_{\mathcal{H}} \leqslant \sum_{n\geqslant 0} \|\hat{f}(n)\|^2_E \tag{2.2}$$

holds for any E-valued polynomial f.

The property (2.1) of the reproducing kernel $k_{\mathcal{H}}$ implies that the subspace $\mathcal{E}$ is orthogonal to $\mathcal{S}H$. Therefore, for an E-valued polynomial $f(z) = \sum_{n\geqslant 0} \hat{f}(n)z^n$ we have

$$\|f\|_{\mathcal{H}} = \|\hat{f}(0)\|^2_{\mathcal{H}} + \|\sum_{n\geqslant 1} \hat{f}(n)z^n\|^2_{\mathcal{H}} = \|\hat{f}(0)\|_E + \|\mathcal{S}Lf\|^2_{\mathcal{H}} \leqslant \|\hat{f}(0)\|^2_E + \|\mathcal{L}f\|^2_{\mathcal{H}}.$$

Repeated application of this inequality leads to (2.2). □

Corollary 2.2. *If $K(z)$, $z \in \mathbb{D}$, is a bounded $L(E)$-valued analytic function, and $e \in E$, then the function $f(z) = K(z)e$ belongs to $\mathcal{H}_0$.*

Corollary 2.3. *Assume that there exists a positive δ such that for any λ_0 with $|\lambda_0| < \delta$ the function $z \mapsto k_{\mathcal{H}}(z, \lambda_0)$ is bounded. Then $\mathcal{H}_0 = \mathcal{H}$.*

Proof. By the preceding corollary, for any $e \in E$ and λ_0 with $|\lambda_0| < \delta$, the function

$$z \mapsto k_{\mathcal{H}}(z, \lambda_0)e \tag{2.3}$$

belongs to $\mathcal{H}_0$. But the formula $(g(\lambda_0), e)_E = (g, k_{\mathcal{H}}(\cdot, \lambda_0)e)_{\mathcal{H}}$ (valid for any $g \in \mathcal{H}$) implies that the family of functions of the form (2.3) is complete in $\mathcal{H}$, which proves the corollary. □

Now, we can accomplish the proof of Theorem 1.1. Take some $\alpha \in A$. The condition (i) of the theorem means that

$$\|\varphi_\alpha f\|_{\mathcal{H}} \geqslant \|f\|_{\mathcal{H}} \tag{2.4}$$

for any $f \in \mathcal{H}$. Let $\mathcal{H}_\alpha$ be the space $\mathcal{H}$ supplied with the norm (equivalent to the original norm in $\mathcal{H}$)

$$\|f\|_{\mathcal{H}_\alpha} := \|\varphi_\alpha f\|_{\mathcal{H}}.$$

Then an explicit calculation shows that the reproducing kernel for the space $\mathcal{H}_\alpha$ is

$$k_{\mathcal{H}_\alpha}(z, \lambda) = \frac{k_{\mathcal{H}}(z, \lambda) - I_E}{\varphi_\alpha(z)\overline{\varphi_\alpha(\lambda)}}.$$

Majorization of norms, given by (2.4), implies the domination of reproducing kernels (see [2]): the function

$$l_\alpha(z,\lambda) := k_{\mathcal{H}}(z,\lambda) - k_{\mathcal{H}_\alpha}(z,\lambda)$$

is positive definite. We obtain

$$\frac{k_{\mathcal{H}}(z,\lambda) - I_E}{\varphi_\alpha(z)\overline{\varphi_\alpha(\lambda)}} \; = \; k_{\mathcal{H}}(z,\lambda) - l_\alpha(z,\lambda), \tag{2.5}$$

whence

$$k_{\mathcal{H}}(z,\lambda) = \frac{I_E - \varphi_\alpha(z)\overline{\varphi_\alpha(\lambda)} l_\alpha(z,\lambda)}{1 - \varphi_\alpha(z)\overline{\varphi_\alpha(\lambda)}}. \tag{2.6}$$

Now, let M and m be such positive constants that

$$m \leqslant \left| \frac{\varphi_\alpha(z)}{z} \right| \leqslant M$$

for any $z \in \mathbb{D}$. Put $\delta = 1/2M^2$ and fix some λ_0 with $|\lambda_0| < \delta$. Then we have for any $z \in \mathbb{D}$

$$\begin{aligned} \|k_{\mathcal{H}}(z,\lambda_0)\| &\leqslant 2\|I_E - \varphi_\alpha(z)\overline{\varphi_\alpha(\lambda_0)} l_\alpha(z,\lambda_0)\| \\ &\leqslant 2\left(1 + \frac{1}{2}\|l_\alpha(z,\lambda_0)\|\right) \leqslant 2 + \|l_\alpha(z,z)\|^{1/2}\|l_\alpha(\lambda_0,\lambda_0)\|^{1/2} \end{aligned} \tag{2.7}$$

(the last inequality follows from the property that l_α is positive definite).

Now, we fix $z_0 \in \mathbb{D}$ with $|z_0| \geqslant 1/2$ and we choose such $\alpha_0 \in A$ that $|\varphi_{\alpha_0}(z_0)| < 1$. The substitution $z = \lambda = z_0$ and $\alpha = \alpha_0$ to (2.6) yields

$$0 \; \leqslant \; k_{\mathcal{H}}(z_0,z_0) = \frac{I_E - |\varphi_{\alpha_0}(z_0)|^2 \, l_{\alpha_0}(z_0,z_0)}{1 - |\varphi_{\alpha_0}(z_0)|^2},$$

whence

$$l_{\alpha_0}(z_0,z_0) \leqslant \frac{1}{|\varphi_{\alpha_0}(z_0)|^2} I_E \leqslant \frac{4}{m^2} I_E,$$

and hence

$$\|l_{\alpha_0}(z_0,z_0)\| \leqslant \frac{4}{m^2}. \tag{2.8}$$

To estimate $\|l_{\alpha_0}(\lambda_0,\lambda_0)\|$ we just note that

$$l_{\alpha_0}(\lambda_0,\lambda_0) \leqslant k_{\mathcal{H}}(\lambda_0,\lambda_0)$$

and hence

$$\|l_{\alpha_0}(\lambda_0,\lambda_0)\| \leqslant \|k_{\mathcal{H}}(\lambda_0,\lambda_0)\|. \tag{2.9}$$

Substituting (2.8) and (2.9) to (2.7), we get

$$\|k_{\mathcal{H}}(z_0,\lambda_0)\| \leqslant 2 + \frac{2}{m}\|k_{\mathcal{H}}(\lambda_0,\lambda_0)\|^{1/2}. \tag{2.10}$$

The right-hand side of this estimate does not depend on $z_0 \in \mathbb{D}$, and the application of Corollary 2.3 accomplishes the proof of Theorem 1.1.

It turns out that for some functions $\varphi \in H^\infty(\mathbb{D})$ conditions (i)-(iii) of Corollary 1.2 imply that the spectral radius of L is one.

Proposition 2.4. *Assume that an analytic contraction T and a function $\varphi \in H^\infty(\mathbb{D})$ are such that the conditions* (i)-(iii) *of Corollary 1.2 are fulfilled and in addition for any $r \in [0,1)$ the function $\frac{\varphi(z)-\varphi(r)}{z-r}$ is bounded away from zero in $\mathbb{D}$. Then the spectral radius of L is one.*

Proof. Let R_0 be the spectral radius of L and $r_0 = R_0^{-1}$. We will prove that under the assumption that $r_0 < 1$,

$$\sup_{r_0/2 \leqslant |\lambda| < r_0} \|(I - \lambda L)^{-1}\| < +\infty,$$

which is impossible. The case $r_0 > 1$ is also impossible since L is an operator left inverse to a contraction T. Hence we must have $r_0 = 1$.

So, we assume that $r_0 < 1$. As in the proof of Theorem 1.1, we shall use the Hilbert space $\mathcal{H}$ where T is modelled by the shift operator $\mathcal{S}$ and L is modelled by the backward shift $\mathcal{L}$. By the construction of $\mathcal{H}$ (see [8]), functions $f \in \mathcal{H}$ are defined and analytic in the disk $r_0\mathbb{D} = \{z \in \mathbb{C} : |z| < r_0\}$. In view of the rotational symmetry of our arguments, it suffices to estimate

$$\sup_{r_0/2 \leqslant r < r_0} \|(I - r\mathcal{L})^{-1}\|.$$

Consider the operator $\mathcal{L}_\varphi$ defined as

$$(\mathcal{L}_\varphi f)(z) := \frac{f(z) - f(0)}{\varphi(z)}.$$

It follows from condition (i) of Corollary 1.2 and the identity

$$\|f(\cdot) - f(0)\|^2_{\mathcal{H}} = \|f(\cdot)\|^2_{\mathcal{H}} - \|f(0)\|^2_E$$

that $\mathcal{L}_\varphi$ is a contraction. Further, we claim that for $r \in [0, r_0)$

$$\left[(I - \varphi(r)\mathcal{L}_\varphi)^{-1}\right] f(z) = \frac{\varphi(z)f(z) - \varphi(r)f(r)}{\varphi(z) - \varphi(r)}. \tag{2.11}$$

Indeed, the right-hand side of this formula is well-defined, since

$$\begin{aligned}\frac{\varphi(z)f(z) - \varphi(r)f(r)}{\varphi(z) - \varphi(r)} &= \frac{z-r}{\varphi(z) - \varphi(r)} \cdot \frac{\varphi(z)f(z) - \varphi(r)f(r)}{z-r} \\ &= \frac{z-r}{\varphi(z) - \varphi(r)} \cdot \left[\mathcal{L}(I - r\mathcal{L})^{-1}\right](\varphi f)(z)\end{aligned}$$

and the function $\frac{z-r}{\varphi(z)-\varphi(r)}$ is bounded in $\mathbb{D}$. Then an explicit computation shows that the operator $I - \varphi(r)\mathcal{L}_\varphi$ applied to the right-hand side of (2.11) gives us $f(z)$.

The formula (2.11) leads to the estimate

$$\begin{aligned}\|\varphi(\cdot)f(\cdot) - \varphi(r)f(r)\|_{\mathcal{H}} &= \|(\varphi(\cdot) - \varphi(r))\left[(I - \varphi(r)\mathcal{L}_\varphi)^{-1}\right] f(\cdot)\|_{\mathcal{H}} \\ &\leqslant \frac{2\|\varphi\|_\infty}{1 - |\varphi(r)|}\|f\|_{\mathcal{H}},\end{aligned}$$

whence we obtain that the estimate $\|f(r)\| \leqslant C(r_0)\|f\|_{\mathcal{H}}$ holds for any $f \in \mathcal{H}$ and $r \in [r_0/2, r_0)$. The formula

$$\left[(I - r\mathcal{L})^{-1}\right] f(z) = \frac{zf(z) - rf(r)}{z-r}$$

implies the estimate

$$\begin{aligned}(1-|\varphi(r)|)\cdot\|\left[(I-r\mathcal{L})^{-1}\right]f\|_{\mathcal{H}} &\leqslant \|\left(\varphi(z)-\varphi(r)\right)\left[(I-r\mathcal{L})^{-1}\right]f(z)\|_{\mathcal{H}}\\ &= \|\frac{\varphi(z)-\varphi(r)}{z-r}\cdot\left(zf(z)-rf(r)\right)\|_{\mathcal{H}}\\ &\leqslant \frac{2\|\varphi\|_\infty}{1-r}\cdot\|zf(z)-rf(r)\|_{\mathcal{H}}\\ &\leqslant \frac{2\|\varphi\|_\infty}{1-r}(1+C(r_0))\|f\|_{\mathcal{H}}.\end{aligned}$$

This inequality implies that $\sup_{r_0/2\leqslant r<r_0}\|(I-r\mathcal{L})^{-1}\|<+\infty$ which accomplishes the proof. □

Combining this Proposition with Corollary 1.2, we get the following.

Theorem 2.5. *Assume that an analytic contraction T and a function $\varphi\in H^\infty(\mathbb{D})$ are such that*
(i) $\|\varphi(\zeta T)x\|\geqslant\|x\|$ *for any* $\zeta\in\mathbb{T}$ *and* $x\in H$*;*
(ii) $\varphi(0)=0$ *and* $\frac{\varphi(z)-\varphi(r)}{z-r}$ *is bounded away from zero in* $\mathbb{D}$ *for any* $r\in[0,1)$*;*
(iii) $|\varphi(r)|<1$ *for any* $r\in[0,1)$.
Then T possesses the wandering subspace property.

The particular choice $\varphi(z)=z(2-z)$ in this theorem gives us Corollary 1.3.

3. Applications

We turn to the applications of the general theorems to weighted l^2 and Bergman spaces. The following proposition gives a necessary and sufficient condition for the weight sequence $(w_n)_{n\geqslant 0}$ which guarantees that the shift operator in $l^2(w_n)$ satisfies inequalities (1.6).

Proposition 3.1. *Let $(w_n)_{n\geqslant0}$ be a weight sequence. Then the following conditions are equivalent:*
(1) $\|S(2I-S)x\|\geqslant\|x\|$ *for any* $x\in l^2(w_n)$*. Here, S is the shift operator.*
(2) *The sequence $(\beta_n)_{n\geqslant1}$, defined recursively as*

$$\beta_1:=4w_1+w_2-w_0,$$
$$\beta_{n+1}:=4w_{n+1}+w_{n+2}-w_n-\frac{4w_{n+1}^2}{\beta_n},$$

is well-defined (i.e., it never happens that $\beta_n=0$) and all β_n, $n\geqslant1$, are positive.

Proof. Assume first that all β_n in condition (2) are well-defined and positive. The condition (1) is equivalent to the inequality

$$\sum_{n\geqslant1}|2y_n-y_{n-1}|^2w_n\geqslant\sum_{n\geqslant0}|y_{n+1}|^2w_n \tag{3.1}$$

for any finitely supported sequence $(y_n)_{n\geqslant 0}$ with $y_0 = 0$. We then have

$$\begin{aligned}
&\sum_{n\geqslant 1} |2y_n - y_{n-1}|^2 w_n - \sum_{n\geqslant 0} |y_{n+1}|^2 w_n \\
&\quad = \sum_{n\geqslant 1}(4w_n + w_{n+1} - w_{n-1})|y_n|^2 - 2\,\mathrm{Re}\left(\sum_{n\geqslant 1} 2w_{n+1} y_n \overline{y_{n+1}}\right) \\
&\quad = \beta_1|y_1|^2 - 2\,\mathrm{Re}\,(2w_2 y_1 \overline{y_2}) + \sum_{n\geqslant 2}(4w_n + w_{n+1} - w_{n-1})|y_n|^2 \\
&\qquad - 2\,\mathrm{Re}\left(\sum_{n\geqslant 2} 2w_{n+1} y_n \overline{y_{n+1}}\right) \qquad (3.2)\\
&\quad = \left|\sqrt{\beta_1}y_1 - \frac{2w_2}{\sqrt{\beta_1}}y_2\right|^2 + \beta_2|y_2|^2 - 2\,\mathrm{Re}(2w_3 y_2 \overline{y_3}) \\
&\qquad + \sum_{n\geqslant 3}(4w_n + w_{n+1} - w_{n-1})|y_n|^2 - 2\,\mathrm{Re}\left(\sum_{n\geqslant 3} 2w_{n+1} y_n \overline{y_{n+1}}\right) \\
&\quad = \ldots = \sum_{l=1}^{m-1}\left|\sqrt{\beta_l}y_l - \frac{2w_{l+1}}{\sqrt{\beta_l}}y_{l+1}\right|^2 + \beta_m|y_m|^2,
\end{aligned}$$

which is positive. Here, m is the greatest integer such that $y_m \neq 0$.

Conversely, assume that condition (2) fails. If m is such a positive integer that $\beta_m < 0$ and $\beta_l > 0$ for $1 \leqslant l \leqslant m-1$, then we choose such a sequence $y = (y_l)_{l\geqslant 0}$ that $y_0 = 0$, the sum of squares in the right hand side of (3.2), vanishes, $y_m \neq 0$, and $y_j = 0$ for $j \geqslant m+1$. We then get that (3.1) fails. In the case where $\beta_m = 0$ and $\beta_l > 0$ for $1 \leqslant l \leqslant m-1$, we choose $y = (y_l)_{l\geqslant 0}$ such that $y_0 = 0$, the sum of squares in the right-hand side of (3.1), vanishes, $y_m = 1$, $y_{m+1} > 0$ and $y_j = 0$ for $j \geqslant m+2$. We then get by the same calculation as in (3.2),

$$\begin{aligned}
&\sum_{n\geqslant 1} |2y_n - y_{n-1}|^2 w_n - \sum_{n\geqslant 0} |y_{n+1}|^2 w_n \\
&\qquad\qquad = -4w_{m+1}y_{m+1} + (4w_{m+1} + w_{m+2} - w_m)y_{m+1}^2
\end{aligned}$$

which is negative if y_{m+1} is sufficiently small. □

Corollary 3.2. *Assume that the weight sequence $(w_n)_{n\geqslant 0}$ is decreasing and satisfies condition* (2) *of Proposition 3.1. Then the Beurling-type theorem is valid in the space $l^2(w_n)$.*

It can be shown that condition (2) of Proposition 3.1 is fulfilled if w_n is sufficiently close to the constant sequence. This fact leads to the following natural question: if an abstract analytic contraction is a very small (in some sense) perturbation of an isometry, is it true that it possesses the wandering subspace property?

Our next example is weighted Bergman spaces.

Proposition 3.3. *Let the weight function $\omega(z)$, $z \in \mathbb{D}$, be such that*

$$\Delta \log\left(\frac{\omega(z)}{1-|z|^2}\right) \geqslant 0, \quad |z| < 1. \qquad (3.3)$$

Then for any function $f \in L^2_a(\mathbb{D}, \omega)$

$$\|\varphi f\|_\omega \geqslant \|f\|_\omega, \tag{3.4}$$

where $\varphi(z) = 1 - (1-z)^3$.

Proof. Consider the space $L^2_a(\mathbb{D}, \omega_0)$, where $\omega_0(z) = 2(1-|z|^2)$. Its reproducing kernel is

$$k(z,\lambda) = \frac{1}{(1-\bar{\lambda} z)^3}.$$

For $a \in \mathbb{D}$, $a \neq 0$, let $I_a = \{f \in L^2_a(\mathbb{D}, \omega_0) \,:\, f(a) = 0\}$. The extremal function for the subspace I_a (the normalized projection of $\mathbf{1}$ to I_a in $L^2_a(\mathbb{D}, \omega_0)$) is

$$G_a(z) = \frac{1 - \left(\frac{1-|a|^2}{1-\bar{a}z}\right)^3}{\sqrt{1-(1-|a|^2)^3}}.$$

It was proved in [3] that for each $a \in \mathbb{D}$ there exists a function Φ_a positive and C^2-smooth in $\overline{\mathbb{D}}$ which satisfies

$$\begin{cases} \Phi_a(\zeta) = \nabla\Phi_a(\zeta) = 0, & |\zeta| = 1, \\ \Delta\Phi_a(z) = \left(|G_a(z)|^2 - 1\right)(1-|z|^2), & |z| < 1. \end{cases}$$

Now let $\omega_1(z) = \omega(z)/(1-|z|^2)$ and assume that ω_1 is sufficiently smooth (say, $\omega_1 \in C^2(\overline{\mathbb{D}})$) and f is analytic in $\overline{\mathbb{D}}$. Then

$$\begin{aligned} \int_{\mathbb{D}} \left(|G_a(z)|^2 - 1\right) |f(z)|^2 \omega(z)\, dm_2(z) \;&=\; \int_{\mathbb{D}} \Delta\Phi_a(z)\, |f(z)|^2 \omega_1(z)\, dm_2(z) \\ &= \int_{\mathbb{D}} \Phi_a(z)\, \Delta\left[|f(z)|^2\omega_1(z)\right]\, dm_2(z) \;\geqslant 0. \end{aligned} \tag{3.5}$$

A standard change of variables in this inequality shows that the inequality

$$\|\varphi_a f\|^2_\omega \geqslant \|f\|^2_\omega$$

holds for the function

$$\varphi_a(z) = G_a\left(\frac{a-z}{1-\bar{a}z}\right) = \frac{1-(1-\bar{a}z)^3}{\sqrt{1-(1-|a|^2)^3}}.$$

Letting $a \to 1$, we obtain (3.4) in the case where ω_1 is sufficiently smooth and f is analytic in $\overline{\mathbb{D}}$. The standard procedure of smoothing the weight (see, e.g. [4], Ch. 9.3) shows that (3.4) is valid for arbitrary ω satisfying (3.3) and f is analytic in $\overline{\mathbb{D}}$. For arbitrary $f \in L^2_a(\mathbb{D}, \omega)$, we obtain (3.4), letting $r \to 1-0$ in the inequality

$$\int_{r\mathbb{D}} \Big(|\varphi(z/r)|^2 - 1\Big)\; |f(z)|^2 \omega(z)\, dm_2(z) \;\geqslant 0.$$

$\square$

Remark. By rotational symmetry, we also have inequalities

$$\|\varphi(\zeta z) f(z)\|_\omega \geqslant \|f(z)\|_\omega, \quad \zeta \in \mathbb{T}. \tag{3.6}$$

The function φ from the preceding proposition satisfies conditions of Theorem 2.5. We immediately obtain Theorem 1.4 of the Introduction.

We also have the following corollary of inequality (2.10).

Theorem 3.4. *There exist positive constants $c(r)$ and $C(r)$ depending on $r \in [0,1)$ such that for any weight function ω satisfying* (3.3) *and the reproducing property*

$$\int_{\mathbb{D}} f\omega\, dm_2 \;=\; f(0) \tag{3.7}$$

for $f \in L^2_a(\mathbb{D},\omega)$, we have

$$c(|\lambda|) \leqslant |k_\omega(z,\lambda)| \leqslant C(|\lambda|), \quad z \in \mathbb{D}, \tag{3.8}$$

where k_ω is the reproducing kernel for $L^2_a(\mathbb{D},\omega)$.

Proof. The reproducing property (3.7) means that $k_\omega(z,0) \equiv 1$. Therefore, (3.8) follows from a more general statement: for any weight function ω satisfying (3.3) and $a, b \in \mathbb{D}$ we have

$$c(\rho(a,b)) \leqslant \left|\frac{k_\omega(z,a)}{k_\omega(z,b)}\right| \leqslant C(\rho(a,b)), \quad z \in \mathbb{D}, \tag{3.9}$$

where $\rho(a,b) = \left|\frac{a-b}{1-\bar{a}b}\right|$ is the pseudohyperbolic distance between a and b. It suffices to check (3.9) only in the case where $\rho(a,b) < \delta$ for appropriate positive δ and then to iterate the obtained inequality. Moreover, in view of conformal invariance it suffices to consider only the case $b = 0$ and to obtain only the estimate from above. We have, therefore, to prove that

$$\left|\frac{k_\omega(z,\lambda)}{k_\omega(z,0)}\right| \leqslant C$$

holds for all ω satisfying (3.3) and λ with $|\lambda| < \delta$. For given ω, we introduce a new weight

$$\omega'(z) := \frac{|k_\omega(z,0)|^2}{k_\omega(0,0)}\,\omega(z)$$

which now satisfies both conditions (3.3) and (3.7), and for which

$$k_{\omega'}(z,\lambda) = \frac{k_\omega(z,\lambda)\,k_\omega(0,0)}{k_\omega(z,0)\,k_\omega(0,\lambda)}.$$

The space $X = L^2_a(\mathbb{D},\omega')$ is invariant with respect to both forward and backward shift operators $\mathcal{S}$ and $\mathcal{L}$, and its reproducing kernel satisfies $k_X(\cdot,0) \equiv \mathbf{1}$. Therefore, X is identical to the space $\mathcal{H}$ (of scalar functions) from the proof of Theorem 1.1 associated with the operator $\mathcal{S}$ in $L^2_a(\mathbb{D},\omega)$. Moreover, the shift operator $\mathcal{S}$ in X satisfies $\|\varphi(\mathcal{S})f\| \geqslant \|f\|$ (where $\varphi(z) = 1-(1-z)^3$). The inequality (2.10) gives us

$$|k_{\omega'}(z,\lambda)| \leqslant 2 + \frac{2}{m}\,|k_{\omega'}(\lambda,\lambda)|^{1/2}, \quad z \in \mathbb{D},\ |\lambda| < \delta,$$

and hence

$$\left|\frac{k_\omega(z,\lambda)}{k_\omega(z,0)}\right| \leqslant 2\,\frac{|k_\omega(0,\lambda)|}{k_\omega(0,0)} \;+\; \frac{2}{m}\cdot\frac{(k_\omega(\lambda,\lambda))^{1/2}}{(k_\omega(0,0))^{1/2}} \;\leqslant\; (2+\frac{2}{m})\left(\frac{k_\omega(\lambda,\lambda)}{k_\omega(0,0)}\right)^{1/2}.$$

It remains to apply the following easy lemma.

Lemma 3.5. *If the weight function ω satisfies* (3.3)*, then there exists a sufficiently small δ and an absolute constant A such that for any λ with $|\lambda| < \delta$*

$$k_\omega(\lambda,\lambda) \leqslant A\,k_\omega(0,0).$$

Proof. We can assume without loss of generality that $k_\omega(0,0) = 1$. The same arguments which were used for the proof of inequality (3.5) show that

$$\|zf\|_\omega^2 \geqslant \frac{1}{3}\|f\|_\omega^2.$$

Therefore, we get for the backward shift $\mathcal{L}$ in $L_a^2(\mathbb{D},\omega)$ the estimate

$$\|\mathcal{L}\| \leqslant \sqrt{3}\left(1+(k_\omega(0,0))^{1/2}\right) = 2\sqrt{3}$$

(we use the fact that $(k_\omega(0,0))^{1/2}$ is the norm of the evaluation functional $f \mapsto f(0)$ in $L_a^2(\mathbb{D},\omega)$). Since

$$\left[\mathcal{L}(I-\lambda\mathcal{L})^{-1}\right] f(z) = \frac{f(z)-f(\lambda)}{z-\lambda},$$

we have the estimate (provided that $\|f\| = 1$)

$$\begin{aligned} |f(\lambda)| \leqslant \|f-f(\lambda)\|_\omega + \|f\|_\omega \leqslant (1+|\lambda|)\cdot\left\|\frac{f(z)-f(\lambda)}{z-\lambda}\right\|_\omega + \|f\|_\omega \\ \leqslant (1+|\lambda|)\frac{2\sqrt{3}}{1-2\sqrt{3}\,|\lambda|} + 1 \leqslant A \quad \text{if } |\lambda| \text{ is sufficiently small,} \end{aligned}$$

which shows that $k_\omega(\lambda,\lambda) \leqslant A^2$.

□

The off-diagonal estimates of reproducing kernels of the form (3.8) as well as Beurling-type theorems are important for the study of approximate spectral synthesis. In particular, it can be derived from Theorems 1.4 and 3.4 that any S-invariant subspace of index one in $L_a^2(\mathbb{D},\omega)$ with radial ω satisfying (3.3) is a limit of zero-based invariant subspaces and any S^*-invariant subspace is a lower limit of finitely dimensional S^*-invariant subspaces. See [7] and the discussion of Theorem 4.6 in [8] for more details.

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