SOME REMARKS ON LIOUVILLE TYPE RESULTS
FOR QUASILINEAR ELLIPTIC EQUATIONS

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ABSTRACT. For a wide class of nonlinearities $f(u)$ satisfying

$$f(0) = f(a) = 0, \ f(u) > 0 \text{ in } (0, a) \text{ and } f(u) < 0 \text{ in } (a, \infty),$$

we show that any nonnegative solution of the quasilinear equation $-\Delta_p u = f(u)$ over the entire $\mathbb{R}^N$ must be a constant. Our results improve or complement some recently obtained Liouville type theorems. In particular, we completely answer a question left open by Du and Guo.

1. INTRODUCTION

Consider the semilinear diffusion equation

$$u_t = \Delta u + f(u), \ x \in \mathbb{R}^N, t \in [0, \infty),$$

where f is a C^1 function satisfying

$$f(0) = f(a) = 0, \ f(u) > 0 \text{ in } (0, a) \text{ and } f(u) < 0 \text{ in } (a, \infty).$$

This problem arises in population biology and chemical reaction theory, and has attracted extensive study; we refer to [AW] and the references therein for more details of the background.

If

$$\underline{\lim}_{u \to 0+} \frac{f(u)}{u^{1+2/N}} > 0,$$

then the so-called *hair-trigger* effect of Aronson and Weinberger [AW] implies that for any solution $u(x, t) \in [0, a]$ with $u(x, 0) \not\equiv 0$, one has $\lim_{t \to \infty} u(x, t) = a$ uniformly for $x \in \mathbb{R}^N$. Moreover, it is shown in [AW] that the exponent $1 + 2/N$ is sharp in the sense that the hair-trigger effect fails if $1 + 2/N$ is replaced by any $\xi > 1 + 2/N$ in the above condition.

A simple consequence of the above hair-trigger effect is an interesting Liouville type result for the corresponding elliptic problem, namely, for such $f(u)$, the equation

$$(1.1) \qquad\qquad -\Delta u = f(u), \ x \in \mathbb{R}^N,$$

has only the constant solutions $u = 0$ and $u = a$ satisfying $0 \le u \le a$. Note that since $f(u) < 0$ in (a, ∞), it can be shown that any globally bounded nonnegative solution of (1.1) satisfies $0 \le u \le a$; moreover, with certain further conditions for $f(u)$ near infinity, for example, $\underline{\lim}_{u \to +\infty} \frac{-f(u)}{u^q} \in (0, \infty]$ for some $q > 1$, it can be shown that any nonnegative solution of (1.1) must satisfy $0 \le u \le a$ (see [DM]).

One would like to check the sharpness of the exponent $1 + 2/N$ for this Liouville type result. For $N > 2$ and any $\xi > N/(N-2)$, one can find a C^1 function $f(u)$ such that $f(0) = f(a) = 0$, $f(u) > 0$ on $(0, a)$, $f(u) < 0$ for $u > a$, and $f(u) = cu^\xi$ for small positive u, such that $-\Delta u = f(u)$ has a positive solution satisfying $0 < u < a$ and it decays to zero at infinity. Indeed, if $\xi \in (N/(N-2), (N+2)/(N-2))$, then such an example was constructed near the end of the paper [Da]. If $\xi \ge (N+2)/(N-2)$, it is well known that $-\Delta u = u^\xi$ has a bounded positive solution u_ξ which decays to 0 at infinity, and the required example can be easily constructed by making use of this fact.

So the problem reduces to checking whether (1.1) has a nonconstant solution satisfying $0 \le u \le a$ when $f(u)$ behaves like cu^ξ near 0 with $\xi \in (1+2/N, N/(N-2)]$, the gap left open above.

The present note addresses this question and related problems. In particular, we will show that the answer to the above question is negative.

Theorem 1.1. *Suppose that $f(u)$ is a C^1 function satisfying*

$$f(0) = f(a) = 0, \ f(u) > 0 \ \forall u \in (0, a), \ \underline{\lim}_{u \to 0^+} f(u)/u^{N/(N-2)} \in (0, +\infty].$$

Then any solution of (1.1) satisfying $0 \le u \le a$ must be a constant.

This result and the examples mentioned above imply that the critical exponent for the Liouville theorem is $N/(N-2)$ instead of $1 + 2/N$.

Theorem 1.1 will follow as a consequence of one of our general results on solutions of p-Laplacian equations of the type

$$(1.2) \qquad\qquad -\Delta_p u = c(x)f(u) \quad \text{in } \mathbb{R}^N \ (N \ge 2)$$

where $\Delta_p u = \text{div}(|Du|^{p-2}Du)$, $p > 1$, $c \in L^\infty(\mathbb{R}^N)$ and $0 < c_1 < c(x) < c_2 < \infty$. By a solution of (1.2) we mean a function $u \in C^1(\mathbb{R}^N)$ satisfying

$$\int_{\mathbb{R}^N} |Du|^{p-2}Du \cdot D\psi dx = \int_{\mathbb{R}^N} c(x)f(u)\psi dx, \ \forall \psi \in C_0^\infty(\mathbb{R}^N).$$

The detailed statements of our main results (Theorem 2.1 and Theorem 2.4) and their proofs are contained in the next section. These results improve or complement recent Liouville type theorems obtained in [BP, DG, DM, SZ]. In particular, Theorem 2.1 completely answers a question left open in [DG]. Our proofs are based upon recent results of Bidaut-Veron and Pohozaev [BP] and Serrin and Zou [SZ].

2. Main results and proofs

We say that $f(s)$ is locally quasi-monotone on $[0, \infty)$ if for any bounded interval $[s_1, s_2] \subset [0, \infty)$, there exists a continuous increasing function $L(s)$ such that $f(s) + L(s)$ is nondecreasing in s for $s \in [s_1, s_2]$. Clearly, this condition is less restrictive than requiring $f(s)$ to be locally Lipschitz continuous on $[0, \infty)$.

Our first main result is the following.

Theorem 2.1. *Let $f(s)$ be continuous and locally quasi-monotone on $[0, \infty)$ and satisfy the following conditions:*

(F_1) *For some $a > 0$,*

$$f(0) = f(a) = 0, \ f(s) > 0 \ \ in \ (0, a), \ f(s) < 0 \ \ in \ (a, \infty).$$

(F_2) *If $N \geq p$, we assume further that for some small $\delta > 0$ and $\xi \in (0, \frac{(p-1)N}{N-p}]$ when $N > p$, $\xi \in (0, \infty)$ when $N = p$, there exists a constant $\sigma > 0$ such that*

$$f(s) \geq \sigma s^\xi, \ \forall s \in (0, \delta).$$

Then any solution of (1.2) satisfying $0 \leq u \leq a$ is a constant.

Remark 2.2.   (i) It is shown in [DG] that if $f(s)$ satisfies (F_1), then any globally bounded nonnegative solution of (1.2) satisfies $0 \leq u \leq a$; moreover, if for some $q > p - 1$,

$$\underline{\lim}_{s \to \infty} \frac{-f(s)}{s^q} \in (0, \infty],$$

then any nonnegative solution of (1.2) satisfies $0 \leq u \leq a$.
 (ii) The example in Remark 1.2 (iii) of [DG] shows that when $N > p$, for any $\xi > (p-1)N/(N-p)$, one can find $c(x)$ satisfying $0 < c_1 \leq c(x) \leq c_2 < \infty$ such that $-\Delta_p u = c(x)u^\xi$ has a positive solution which decays to 0 at infinity. This implies that our condition (F_2) in Theorem 2.1 is sharp.
 (iii) Theorem 2.1 positively answers the conjecture in [DG, Remark 1.2 (iv)]. Note that Theorem 1.1 follows from Theorem 2.1 by letting $p = 2$.

The main ingredient for the proof of Theorem 2.1 is the following recent result due to Bidaut-Veron and Pohozaev [BP] and Serrin and Zou [SZ, Theorem II].

Proposition 2.3.   (i) *If $N < p$, $\Omega \subset \mathbb{R}^N$ is an exterior domain in $\mathbb{R}^N$ (i.e., $\{x \in \mathbb{R}^N : |x| > R\} \subset \Omega$ for some $R > 0$), and $u \in C^1(\Omega)$ is nonnegative and satisfies (in the weak sense) $-\Delta_p u \geq 0$, then u must be a constant.*
 (ii) *If $N \geq p$, $\Omega \subset \mathbb{R}^N$ is an exterior domain, and $u \in C^1(\Omega)$ is nonnegative and satisfies (in the weak sense) $-\Delta_p u \geq u^\xi$ with $\xi \in (0, (p-1)N/(N-p)]$ when $N > p$, $\xi \in (0, \infty)$ when $N = p$, then $u \equiv 0$.*

Proof of Theorem 2.1. By part (i) of Proposition 2.3, we need only consider the case that $N \geq p$. So we assume $N \geq p$ from now on.

Since $-\Delta_p u \geq 0$ and $u \geq 0$, by the strong maximum principle (see [Va] or [PS, Theorem 1]), either $u \equiv 0$ or $u > 0$ everywhere. Therefore we assume $u > 0$. We need to show that $u \equiv a$.

Let us observe that it suffices to show $u(x) \to a$ as $|x| \to \infty$. Indeed, if this is true but $u \not\equiv a$, then $\inf_{x \in \mathbb{R}^N} u(x) < a$ is achieved at some $x_0 \in \mathbb{R}^N$. Therefore the function $v(x) = u(x) - u(x_0)$ satisfies

$$v \geq 0, \ v(x_0) = 0, \ -\Delta_p v = -\Delta_p u \geq 0.$$

By the strong maximum principle we deduce $v \equiv 0$, that is, $u(x) \equiv u(x_0)$. But this is a contradiction to the assumption that $u(x) \to a$ as $|x| \to \infty$.

To show that $u(x) \to a$ as $|x| \to \infty$, we choose $R > 0$, $\alpha \in (0, \min_{|x|=R} u(x))$ and consider the boundary value problem

(2.1)
$$\begin{aligned} -\Delta_p w &= c_1 f(w) && \text{when } R < |x| < R_n, \\ w &= \alpha && \text{when } |x| = R, \\ w &= 0 && \text{when } |x| = R_n, \end{aligned}$$

where $\{R_n\}$ is an increasing sequence of numbers satisfying $R < R_1$, $R_n \to \infty$ as $n \to \infty$. For each $n \geq 1$, $w = u$ is a super solution to (2.1) while $w = 0$ is a sub-solution. Hence by the well known sub- and super-solution method (see [D]), (2.1) has a minimal and maximal solution satisfying $0 \leq w \leq u$. Let w_n denote the minimal solution. We observe that w_n must be radially symmetric as it is minimal and problem (2.1) is invariant under rotations around the origin. Furthermore, w_{n+1} is a super-solution to (2.1) while 0 is a sub-solution. Hence (2.1) has a solution between 0 and w_{n+1}. It follows that its minimal solution w_n must satisfy $w_n \leq w_{n+1}$. Therefore, we have

$$w_n(x) \leq w_{n+1}(x) \leq u(x) \text{ when } R < |x| < R_n, \ n = 1, 2, \dots.$$

It follows that $w(x) := \lim_{n \to \infty} w_n(x)$ is well-defined for $|x| > R$ and $0 \leq w(x) \leq u(x)$.

Since $0 \leq cf(w_n) \leq c_2 \max_{s \in [0,a]} f(s)$ and $0 \leq w_n \leq u \leq a$, for any given bounded smooth sub-domain Ω of $\{x \in \mathbb{R}^N : |x| > R\}$, applying standard regularity theory (see [To]) to the equation $-\Delta_p w_n = cf(w_n)$ for all large n so that $\Omega \subset A_n := \{x \in \mathbb{R}^N : R < |x| < R_n\}$, we find that $\{u_n|_\Omega\}$ is bounded in $C^{1,\alpha}(\Omega)$ for some α. As $C^{1,\alpha}(\Omega)$ imbeds compactly into $C^1(\overline{\Omega})$, $\{w_n|_\Omega\}$ has a convergent subsequence in $C^1(\overline{\Omega})$. It follows that $w_n \to w$ in $C^1(\overline{\Omega})$ for any bounded smooth domain Ω of $\{x \in \mathbb{R}^N : |x| > R\}$. Therefore, w satisfies

$$-\Delta_p w = c_1 f(w) \text{ when } |x| > R, \ w = \alpha \text{ when } |x| = R.$$

Since each w_n is radially symmetric, so is w: $w(x) = w(r)$, $r = |x|$. Hence we can write

$$-(r^{N-1}|w'|^{p-2}w')' = r^{N-1}c_1 f(w) \text{ for } r > R, \ w(R) = \alpha.$$

Denote $\phi(r) = r^{N-1}|w'(r)|^{p-2}w'(r)$. We find from the above equation that $\phi'(r) = -r^{N-1}c_1 f(w(r)) \leq 0$ for $r > R$. Hence $\phi(r)$ is a nonincreasing function and there are two possibilities:

(i) $\phi(r)$ is negative for large r, or (ii) $\phi(r) \geq 0$ on $[R, \infty)$.

Case (i) cannot occur because otherwise, there exists $R_0 \geq R$ such that $w'(r) < 0$ for $r > R_0$. Thus $w(r) < w(R_0 + 1) < w(R_0) \leq a$ for all $r > R_0 + 1$. Let $\alpha_1 = w(R_0 + 1)$. Then we can use (F_1) and (F_2) to find some $\sigma_1 \in (0, \sigma]$ such that $f(s) \geq \sigma_1 s^\xi$ for $s \in (0, \alpha_1)$. Thus,

$$-\Delta_p w = c_1 f(w) \geq c_1 \sigma_1 w^\xi$$

in the exterior domain $\Omega_{R_0+1} := \{x \in \mathbb{R}^N : |x| > R_0 + 1\}$. We can now easily check that for $\beta = (c_1 \sigma_1)^{-p}$, $v(x) := w(\beta x)$ is a positive solution of $-\Delta_p v \geq v^\xi$ in the exterior domain $\Omega_{(R_0+1)/\beta}$, contradicting Proposition 2.3.

So case (ii) must occur which implies that $w'(r) \geq 0$ for $r > R$. Since $w \leq u \leq a$, $\alpha_2 := \lim_{r \to \infty} w(r)$ exists. If $\alpha_2 < a$, then $w(r) \leq \alpha_2$ for all $r > R$ and we can use the same argument as in case (i) to deduce a contradiction to Proposition 2.3. Thus we necessarily have $\lim_{r \to \infty} w(r) = a$ and hence, due to $a \geq u(x) \geq w(x)$, $\lim_{|x| \to \infty} u(x) = a$, as we wanted. This finishes the proof. $\qquad\square$

Let us note that while condition (F_1) can be regarded as a kind of global restriction on $f(u)$, condition (F_2) is local, it only restricts the behaviour of $f(u)$ near $u = 0$. When condition (F_1) is strengthened, it is possible to relax condition (F_2) and still obtain some Liouville type theorems. Our second main result addresses this point.

Theorem 2.4. *Suppose $N > p > 1$ and $f(u)$ is a C^1 function satisfying (F_1) and furthermore,*

(F_3) *for some $\xi \in (0, \frac{Np}{N-p} - 1)$ and all $u \in (0, a)$, $\frac{d}{du}[f(u)/u^\xi] \leq 0$,*
(F_4) *for some small $\delta_1 > 0$,*

$$\int_{a-\delta_1}^{a} [\int_s^a f(t)dt]^{-1/p}ds = \infty.$$

Then the only solutions to

(2.2) $$-\Delta_p u = f(u) \quad in \; \mathbb{R}^N$$

satisfying $0 \leq u \leq a$ are $u \equiv 0$ and $u \equiv a$.

Let us note that $f(u) = u^\xi - u^q$ satisfies (F_3) whenever $\xi \in (0, \frac{Np}{N-p} - 1)$ and $q > \xi$, but it satisfies (F_2) only if $\xi \in (0, \frac{(N-1)p}{N-p} - 1]$ and $q > \xi$.

Condition (F_4) is satisfied when $f(a) = 0$ and $0 \leq f(s) \leq M|s-a|^{p-1}$ for some $M > 0$ and all $s \in (a - \delta_1, a)$. Thus, if f is C^1, then (F_4) is automatically satisfied provided that $1 < p \leq 2$, but it is an extra restriction when $p > 2$. We suspect that (F_4) is unnecessary in Theorem 2.4.

The main ingredients in our proof of Theorem 2.4 are the following two results.

Proposition 2.5. *Suppose $N > p > 1$. If the C^1 function $f(u)$ satisfies*

(i) $f(0) = 0$, $f(u) \geq 0$ for $u > 0$,
(ii) $\frac{d}{du}[f(u)/u^\xi] \leq 0$ for some $\xi \in (0, \frac{Np}{N-p} - 1)$ and all $u > 0$, and
(iii) $f(u) \geq u^q$ for some $q > p - 1$ and all large u,

then problem (2.2) has only the trivial solution $u \equiv 0$ in $\mathbb{R}^N$.

Proposition 2.6. *Suppose that $N > p > 1$. Let $f(s)$ be continuous and locally quasi-monotone on $[0, \infty)$. Suppose $f(s)$ satisfies (F_1), (F_4) and*

(F_2') *for some small $\delta > 0$, there exist constants $\sigma > 0$ and $\xi \in (p-1, \frac{Np}{N-p} - 1)$ such that*

$$f(s) \geq \sigma s^\xi, \; \forall s \in (0, \delta).$$

Then for any solution of (1.2) satisfying $0 \leq u \leq a$, we have $\sup_{x \in \mathbb{R}^N} u(x) < a$ unless $u \equiv a$.

Proposition 2.5 is due to Serrin and Zou (see [SZ, Theorem II(c)]). To prove Proposition 2.6, we will make use of the following variant of a weak sweeping principle used in [DG], which develops the idea of the well-known sweeping principle due to Serrin [S] for Laplacian equations.

Lemma 2.7 (Weak sweeping principle). *For each $t \in [0, 1]$, let $\zeta_t : \mathbb{R}^N \to \mathbb{R}^N$ be a diffeomorphism of $\mathbb{R}^N$ onto itself, satisfying $\zeta_0(x) = x$ and that $\zeta_t(x) \to \zeta_{t_0}(x)$ as $t \to t_0$ uniformly for x in compact subsets of $\mathbb{R}^N$. Let $\mathcal{D}$ be a bounded smooth domain in $\mathbb{R}^N$ and $\mathcal{D}_t := \zeta_t(\mathcal{D})$. Suppose that $h(x, s)$ is measurable in $x \in \mathbb{R}^N$, continuous and locally quasi-monotone (uniformly for x in compact subsets of $\mathbb{R}^N$) with respect*

to $s \in (-\infty, \infty)$. Let u_t and v_t, $t \in [0,1]$, be functions in $W^{1,p}(\mathcal{D}_t) \cap C(\overline{\mathcal{D}}_t)$ and satisfy in the weak sense, for some $\epsilon > 0$,

(2.3) $-\Delta_p u_t \geq h(x, u_t) + \epsilon$, $\quad -\Delta_p v_t \leq h(x, v_t)$ in $\mathcal{D}_t$, $\forall t \in [0,1]$,

$$u_t \geq v_t + \epsilon \quad \text{on } \partial \mathcal{D}_t, \ \forall t \in [0,1].$$

Moreover, suppose that $u_{t_0} \geq v_{t_0}$ in $\mathcal{D}_{t_0}$ for some $t_0 \in [0,1]$, and $t \to u_t \circ \zeta_t$ and $t \to v_t \circ \zeta_t$ are continuous from $[0,1]$ to $C(\overline{\mathcal{D}})$. Then

$$u_t \geq v_t \quad \text{on } \mathcal{D}_t, \ \forall t \in [0,1].$$

Proof. Denote

$$T = \{t \in [0,1] : u_t \geq v_t \text{ on } \mathcal{D}_t\} = \{t \in [0,1] : u_t \circ \zeta_t \geq v_t \circ \zeta_t \text{ on } \mathcal{D}\}.$$

Clearly T is a closed set with $t_0 \in T$. We show that T is relatively open in $[0,1]$, which implies $T = [0,1]$, as required.

Since $u_t \circ \zeta_t$ and $v_t \circ \zeta_t$ vary continuously with t, it is easily seen that there exist finite numbers $s_1 < s_2$ such that $u_t(x), v_t(x) \in [s_1, s_2]$ for all $x \in \bigcup_{0 \leq t \leq 1} \mathcal{D}_t$ and all $t \in [0,1]$.

Since $h(x, s)$ is locally quasi-monotone in s, we can find a continuous increasing function $L(s)$ such that $\tilde{h}(x, s) := h(x, s) + L(s)$ is nondecreasing in s for all $x \in \bigcup_{0 \leq t \leq 1} \mathcal{D}_t$ and $s \in [s_1, s_2]$.

Let $\delta > 0$ be sufficiently small. Then, for any $t \in T$,

$$-\Delta_p u_t + L(u_t) \geq \tilde{h}(x, u_t) + \epsilon \geq \tilde{h}(x, v_t) + \epsilon \geq -\Delta_p v_t + L(v_t) + \epsilon$$
$$\geq -\Delta_p(v_t + \delta) + L(v_t + \delta) \quad \text{in } \mathcal{D}_t,$$

and

$$u_t \geq v_t + \delta \quad \text{on } \partial \mathcal{D}_t.$$

By the weak maximum principle (see, e.g., [D, Theorem 4.9]) we obtain $u_t \geq v_t + \delta$ in $\mathcal{D}_t$, i.e., $u_t \circ \zeta_t \geq v_t \circ \zeta_t + \delta$ in $\mathcal{D}$. Thus for all $\tau \in [0,1]$ with $|\tau - t|$ small, $u_\tau \circ \zeta_\tau \geq v_\tau \circ \zeta_\tau$ in $\mathcal{D}$. This shows that T is relatively open in $[0,1]$. The proof is complete. $\qquad \square$

Remark 2.8. From the above proof, we immediately see that Lemma 2.7 remains true if (2.3) is replaced by

(2.4) $-\Delta_p u_t \geq h(x, u_t)$, $\quad -\Delta_p v_t \leq h(x, v_t) - \epsilon$ in $\mathcal{D}_t$, $\forall t \in [0,1]$.

Proof of Proposition 2.6. Suppose $a = \sup_{x \in \mathbb{R}^N} u(x)$. We want to show that $u \equiv a$. If $u(x_0) = a$ for some $x_0 \in \mathbb{R}^N$, then by condition (F_4), a simple application of the strong maximum principle (see [Va] or [PS, Theorem 1]) shows that $u \equiv a$. Suppose now $u < a$ on $\mathbb{R}^N$. We are going to deduce a contradiction.

Let $x_n \in \mathbb{R}^N$ be such that $u(x_n) \to a$ as $n \to \infty$. Define $u_n(x) = u(x + x_n)$ and $c_n(x) = c(x + x_n)$ for $x \in \mathbb{R}^N$. Then

$$-\Delta_p u_n = c_n f(u_n), \ x \in \mathbb{R}^N.$$

Since $0 \leq c_n f(u_n) \leq c_2 \max_{s \in [0,a]} f(s)$ and $0 \leq u_n \leq a$, by standard regularity theory (see [To]) we find that $\{u_n|_\Omega\}$ is bounded in $C^{1,\alpha}(\Omega)$ for some α and any bounded smooth domain Ω of $\mathbb{R}^N$. Now $C^{1,\alpha}(\Omega)$ imbeds compactly into $C^1(\overline{\Omega})$. Thus $\{u_n|_\Omega\}$ has a convergent subsequence in $C^1(\overline{\Omega})$. From this fact and a standard diagonal process, we can find a subsequence of $\{u_n\}$, say $\{u_{n_k}\}$, that converges to some $u^* \in C^1(\mathbb{R}^N)$ in the C^1 norm on any compact subsets of $\mathbb{R}^N$. Clearly we must have $0 \leq u^* \leq a$ and $u^*(0) = a$.

Since $c_1 \leq c_n(x) \leq c_2$, we have $-\Delta_p u_n \leq c_2 f(u_n)$ and hence, by passing to the limit,

$$-\Delta_p u^* \leq c_2 f(u^*).$$

Since $u^* \leq a$ and $u^*(0) = a$, we can now apply the strong maximum principle to conclude that $u^* \equiv a$. It follows in particular that $u_{n_k} \to a$ uniformly on any compact subsets of $\mathbb{R}^N$ as $k \to \infty$.

Since $\xi \in (p-1, \frac{Np}{N-p} - 1)$, it is well known (see, for example, [Dr]) that the problem

$$-\Delta_p u = u^\xi, \ x \in B_1(0), \ u|_{\partial B_1(0)} = 0$$

has a radially symmetric positive solution $u = u_1$ with $u_1(0) = \max_{x \in B_1(0)} u_1(x)$. Here, and in what follows, we use the notation $B_r(x_0) = \{x \in \mathbb{R}^N : |x - x_0| < r\}$.

Denote $\alpha = -p/(\xi - p + 1)$ and $c = (c_1\sigma/2)^{\alpha/p}$. Then for any $R > 0$, it is easily checked that $v_R(x) := cR^\alpha u_1(x/R)$ satisfies

$$-\Delta_p v_R = (c_1\sigma/2)v_R^\xi, \ x \in B_R(0), \ v_R|_{\partial B_R(0)} = 0.$$

Clearly $v_R(x) \leq v_R(0) = cR^\alpha u_1(0)$. We now fix R large enough such that $cR^\alpha u_1(0) = \delta/2$, where δ is given in (F_2'). Then $v_R \leq \delta/2$ on $B_R(0)$.

We next show that $u(x) \geq \delta/2$ over $\mathbb{R}^N$. To this end, we fix an arbitrary $x_0 \in \mathbb{R}^N$. Since $u_{n_k}(x) \to a$ as $k \to \infty$ uniformly for $x \in B_R(0)$, we can find $m = n_k$ with large enough k so that $u_m(x) > \delta$ on $B_R(0)$. We now make use of the weak sweeping principle. For $t \in [0, 1]$ we define $\zeta_t(x) = x + t(x_m - x_0)$, $\mathcal{D}_t = \zeta_t(\mathcal{D})$ with $\mathcal{D} = B_R(x_0)$. Clearly $\mathcal{D}_0 = \mathcal{D}$ and $\mathcal{D}_1 = B_R(x_m)$.

Since $-\Delta_p u \geq 0$ on $\mathbb{R}^N$, by the strong maximum principle we know that $u(x) > 0$ for all x. By our previous assumption, $u(x) < a$ for all x. Hence there exists $\delta_1 \in (0, \delta/2)$ such that $a - \delta_1 \geq u(x) \geq \delta_1$ for all x in the compact set $\bigcup_{t \in [0,1]} \overline{\mathcal{D}}_t$. Define

$$v_t(x) = v_R[x - x_0 - t(x_m - x_0)].$$

Then clearly

$$-\Delta_p v_t(x) = (c_1\sigma/2)v_t^\xi(x) \leq (1/2)c(x)f(v_t(x)), \ \forall x \in \mathcal{D}_t, \forall t \in [0, 1].$$

Define $u_t = u$ for $t \in [0, 1]$. Since $a - \delta_1 \geq u(x) \geq \delta_1$ for $x \in \bigcup_{t \in [0,1]} \overline{\mathcal{D}}_t$, we can find $\epsilon \in (0, \delta_1)$ such that $(c_1/2)f(u(x)) \geq \epsilon$ for all such x. Hence,

$$-\Delta_p u_t = c(x)f(u_t) \geq (1/2)c(x)f(u_t) + \epsilon, \ \forall x \in \mathcal{D}_t, \forall t \in [0, 1],$$

$$u_t(x) \geq \delta_1 \geq v_t(x) + \epsilon, \ \forall x \in \partial\mathcal{D}_t, \forall t \in [0, 1].$$

Note also that on $\mathcal{D}_1 = B_R(x_m)$, $u(x) \geq \delta$ and $v_1(x) \leq \delta/2$. Therefore, we can use Lemma 2.7 to conclude that

$$u(x) \geq v_t(x), \ \forall x \in \mathcal{D}_t, \forall t \in [0, 1].$$

In particular, $u(x) \geq v_0(x)$ on $\mathcal{D}_0 = B_R(x_0)$ and $u(x_0) \geq v_0(x_0) = v_R(0) = \delta/2$. As $x_0 \in \mathbb{R}^N$ is arbitrary, this implies that $u(x) \geq \delta/2$ on $\mathbb{R}^N$. It now follows from Lemma 2.3 of [DG] that $u(x) \geq a$ on $\mathbb{R}^N$. Hence $u \equiv a$. But this contradicts our previous assumption that $u < a$ on $\mathbb{R}^N$. $\qquad\square$

Proof of Theorem 2.4. Suppose that u is a solution to (2.2) satisfying $0 \leq u \leq a$. We may assume that $\|u\|_\infty < a$ for otherwise $u \equiv a$ by Proposition 2.6. By our assumption we can write $f(u) = u^\xi g(u)$ with $g(u)$ positive, C^1 and nonincreasing in $(0, a)$. Now we define a C^1 function $\tilde{g}(u)$ for $u > 0$ such that

$\tilde{g}(u) = g(u)$ for $u \in (0, \|u\|_\infty]$,

$\tilde{g}(u) = $ positive constant for $u > (1/2)(\|u\|_\infty + a)$,

$\tilde{g}'(u) \leq 0$ for $u \in (\|u\|_\infty, (1/2)(\|u\|_\infty + a))$.

Then clearly $\tilde{f}(u) := u^\xi \tilde{g}(u)$ satisfies the conditions (i)–(iii) of Proposition 2.5, and u solves $-\Delta_p u = \tilde{f}(u)$. It follows that $u \equiv 0$. $\square$

Remark 2.9. By using a more involved proof for Proposition 2.6, condition (F_2') can be removed. As this point does not improve our main result, Theorem 2.4, we only give a brief sketch of its proof here. The idea is to construct the lower solutions v_t in a different way. Indeed, let $u > 0$ be a solution of (1.2) as given in Proposition 2.6. By minimizing the corresponding functional of the problem

$$-\Delta_p v = \lambda f(v), \; v|_{\partial B_1(0)} = 0$$

over the order interval $[0, u|_{B_1(0)}]$ one finds that it has a radial solution v satisfying $0 < v \leq u$ provided that λ is large enough. Then v_t can be obtained by a rescaling and shifting of v much as before.

Remark 2.10. The conclusion of Proposition 2.6 can be strengthened. We can show that there is a positive δ such that any nonconstant solution in Proposition 2.6 satisfies $u(x) \leq a - \delta$ on R^N. This is useful for studying solutions on bounded domains when the diffusion rate is small. Note also that the results here could be used to simplify the proof of Theorem 4 in [Da].

Let us end this note with some discussions of a simple yet illustrative example. From Theorem 2.4 and Remark 2.2 (i), we immediately see that when $1 < p \leq 2$, if $\xi \in (p-1, \frac{Np}{N-p} - 1)$ and $q > \xi$, then the only nonnegative entire solutions of

$$(2.5) \qquad\qquad\qquad -\Delta_p u = u^\xi - u^q$$

are $u \equiv 0$ and $u \equiv 1$. The restriction that $p \leq 2$ is due to condition (F_4); we believe that this restriction can be removed. Note that if $\xi > \frac{Np}{N-p} - 1$ and $q > \xi$, then it is known that (2.5) has radial ground state solutions (see [KMPT] for the case $p = 2$ and [T] for the general case $p > 1$). When $\xi = \frac{Np}{N-p} - 1$ and $q > \xi$, it follows from [T, Theorem (i)] that (2.5) has no radial ground states. It would be interesting to know whether there can be other types of nonconstant positive solutions in this latter case.

In the case $p = 2$, this question has a negative answer. Indeed, by Theorem 3 of Bianchi [B], if for $s \in (0, \infty)$, the function $g(s)$ is locally Lipschitz, is nonnegative and $g(s)/s^{(N+2)/(N-2)}$ is nonincreasing, then any positive C^2 solution of $-\Delta u = g(u)$ in $\mathbb{R}^N$ ($N > 2$) is either a constant and g vanishes on that constant, or

$$u(x) = \frac{k}{(|x - x_0|^2 + h^2)^{(N-2)/2}}$$

for some $x_0 \in \mathbb{R}^N$ and positive constants k and h, and $g(s)$ is a suitable scalar multiple of $s^{(N+2)/(N-2)}$ for $s \in (0, \max_{\mathbb{R}^N} u]$.

If we take $g(s) = s^{(N+2)/(N-2)} - s^q$ with $q > (N+2)/(N-2)$ for $s \in [0, 1]$ and $g(s) = 0$ for $s > 1$, then Bianchi's result implies that

$$(2.6) \qquad\qquad\qquad -\Delta u = u^{(N+2)/(N-2)} - u^q$$

has no solution in $\mathbb{R}^N$ satisfying $0 < u \leq 1$ except $u \equiv 1$. On the other hand, by Remark 2.2 (i), we know that any positive solution of (2.6) satisfies $0 < u \leq 1$. Thus $u \equiv 1$ is the only positive solution of (2.6).

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