# RESCALINGS OF FREE PRODUCTS OF $\mathrm{II}_{1}$-FACTORS 

KEN DYKEMA AND FLORIN RĂDULESCU

(Communicated by David R. Larson)


#### Abstract

We introduce the notation $\mathcal{Q}(1) * \cdots * \mathcal{Q}(n) * L\left(\mathbf{F}_{r}\right)$ for von Neumann algebra $\mathrm{II}_{1}$-factors where $r$ is allowed to be negative. This notation is defined by rescalings of free products of $\mathrm{II}_{1}$-factors, and is proved to be consistent with known results and natural operations. We also give two statements which we prove are equivalent to isomorphism of free group factors.


## Introduction

The rescaling $\mathcal{M}_{t}$ of a $\mathrm{II}_{1}$-factor $\mathcal{M}$ by a positive number $t$ was introduced by Murray and von Neumann [5]. In [4, we showed that if $\mathcal{Q}(1), \ldots, \mathcal{Q}(n)$ are $\mathrm{II}_{1}$-factors $(n \in\{2,3, \ldots\})$ and if $0<t<\sqrt{1-1 / n}$, then

$$
\begin{equation*}
(\mathcal{Q}(1) * \cdots * \mathcal{Q}(n))_{t} \cong \mathcal{Q}(1)_{t} * \cdots * \mathcal{Q}(n)_{t} * L\left(\mathbf{F}_{r}\right) \tag{1}
\end{equation*}
$$

where $r=(n-1)\left(t^{-2}-1\right)$. Here $L\left(\mathbf{F}_{r}\right), r>1$, is an interpolated free group factor ([2], [6]). For $\sqrt{1-1 / n} \leq t<1$, we proved a similar formula, where $L\left(\mathbf{F}_{r}\right)$ was replaced by a hyperfinite von Neumann algebra with specified tracial state having free dimension $\left([1)\right.$ equal to $r=(n-1)\left(t^{-2}-1\right) \leq 1$. If one tries to use the formula (1) when $t>1$, one obtains $L\left(\mathbf{F}_{r}\right)$ with $r<0$.

In this note we introduce the notation

$$
\begin{equation*}
\mathcal{Q}(1) * \cdots * \mathcal{Q}(n) * L\left(\mathbf{F}_{r}\right) \quad(n \in \mathbf{N}, 1-n<r \leq \infty) \tag{2}
\end{equation*}
$$

If $r>1$, then $L\left(\mathbf{F}_{r}\right)$ in (2) is an interpolated free group factor, while if $r \leq 1$, then (2) defines a $\mathrm{II}_{1}$ factor which is the rescaling by $t$ of $\mathcal{Q}(1)_{1 / t} * \cdots * \mathcal{Q}(n)_{1 / t}$ if $n=2$ or of $\mathcal{Q}(1)_{1 / t} * L\left(\mathbf{F}_{2}\right)$ if $n=1$ for an appropriate $t>1$. We will prove that this notation is consistent with known results and natural operations involving free products. The notation (2) provides an elegent means of describing rescalings of free products of $\mathrm{II}_{1}$-factors, and is used in [3].

Finally, we show that if the free group factors are isomorphic to each other, then $\mathcal{Q}(1) * \mathcal{Q}(2) \cong \mathcal{Q}(1) * \mathcal{Q}(2) * L\left(\mathbf{F}_{\infty}\right)$ for all $\mathrm{II}_{1}$-factors $\mathcal{Q}(1)$ and $\mathcal{Q}(2)$ and we give one additional equivalent condition. It is conceivable that these conditions may be used to prove nonisomorphism of free group factors.

[^0]
## Rescalings

Recall that the interpolated free group factors rescale as follows:

$$
\begin{equation*}
L\left(\mathbf{F}_{r}\right)_{t} \cong L\left(\mathbf{F}_{1+t^{-2}(r-1)}\right) \quad(1<r \leq \infty, 0<t<\infty) \tag{3}
\end{equation*}
$$

(see [2], 6]).
Lemma 1. Let $n \in \mathbf{N}$, let $\mathcal{Q}(1), \ldots, \mathcal{Q}(n)$ be $I_{1}$-factors, let $1<r \leq \infty$, and let

$$
\mathcal{M}=\mathcal{Q}(1) * \cdots * \mathcal{Q}(n) * L\left(\mathbf{F}_{r}\right)
$$

Then for every $0<t<\sqrt{1+(r-1) / n}$,

$$
\mathcal{M}_{t} \cong \mathcal{Q}(1)_{t} * \cdots * \mathcal{Q}(n)_{t} * L\left(\mathbf{F}_{t^{-2} r+(n-1)\left(t^{-2}-1\right)}\right)
$$

Proof. If $t \leq 1$, then this follows from [4] see (11) and (3) above. Suppose $t>1$. Note that $t$ is taken so that $t^{-2} r+(n-1)\left(t^{-2}-1\right)>1$. Applying (1) and (3), we have

$$
\left(\mathcal{Q}(1)_{t} * \cdots * \mathcal{Q}(n)_{t} * L\left(\mathbf{F}_{t^{-2} r+(n-1)\left(t^{-2}-1\right)}\right)\right)_{\frac{1}{t}} \cong \mathcal{Q}(1) * \cdots * \mathcal{Q}(n) * L\left(\mathbf{F}_{r}\right)
$$

Proposition 2. Let $n \in \mathbf{N}$, let $\mathcal{Q}(1), \ldots, \mathcal{Q}(n)$ be $I I_{1}-$ factors, and let

$$
1-n<r \leq 1
$$

Then there is a $I_{1}$-factor $\mathcal{M}$, unique up to isomorphism, such that

$$
\mathcal{M}_{t} \cong \mathcal{Q}(1)_{t} * \cdots * \mathcal{Q}(n)_{t} * L\left(\mathbf{F}_{t^{-2} r+(n-1)\left(t^{-2}-1\right)}\right)
$$

whenever $0<t<\sqrt{1+(r-1) / n}$.
Proof. Let $0<s<t<\sqrt{1+(r-1) / n}$ and let $\mathcal{M}$ and $\widetilde{\mathcal{M}}$ be $\mathrm{II}_{1}$-factors such that

$$
\begin{aligned}
\mathcal{M}_{s} & \cong \mathcal{Q}(1)_{s} * \cdots * \mathcal{Q}(n)_{s} * L\left(\mathbf{F}_{s^{-2} r+(n-1)\left(s^{-2}-1\right)}\right) \\
\widetilde{\mathcal{M}}_{t} & \cong \mathcal{Q}(1)_{t} * \cdots * \mathcal{Q}(n)_{t} * L\left(\mathbf{F}_{t^{-2} r+(n-1)\left(t^{-2}-1\right)}\right)
\end{aligned}
$$

Then using Lemma 1 we have

$$
\widetilde{\mathcal{M}}_{s}=\left(\widetilde{\mathcal{M}}_{t}\right)_{\frac{s}{t}} \cong \mathcal{Q}(1)_{s} * \cdots * \mathcal{Q}(n)_{s} * L\left(\mathbf{F}_{s^{-2} r+(n-1)\left(s^{-2}-1\right)}\right) \cong \mathcal{M}_{s}
$$

Definition 3. We denote the unique factor $\mathcal{M}$ in Proposition 2 by

$$
\mathcal{Q}(1) * \cdots * \mathcal{Q}(n) * L\left(\mathbf{F}_{r}\right)
$$

Proposition 4. Let $n \in \mathbf{N}$, let $\mathcal{Q}(1), \mathcal{Q}(2), \ldots, \mathcal{Q}(n)$ be $I_{1}$-factors, and let $1-n<$ $r \leq \infty$.
(i) If $0<t<\infty$, then
$\left(\mathcal{Q}(1) * \cdots * \mathcal{Q}(n) * L\left(\mathbf{F}_{r}\right)\right)_{t} \cong \mathcal{Q}(1)_{t} * \cdots * \mathcal{Q}(n)_{t} * L\left(\mathbf{F}_{t^{-2} r+(n-1)\left(t^{-2}-1\right)}\right)$.
(ii) If $\sigma$ is a permutation of $\{1,2, \ldots, n\}$, then

$$
\mathcal{Q}(1) * \cdots * \mathcal{Q}(n) * L\left(\mathbf{F}_{r}\right) \cong \mathcal{Q}(\sigma(1)) * \cdots * \mathcal{Q}(\sigma(n)) * L\left(\mathbf{F}_{r}\right)
$$

(iii) If $\mathcal{Q}(1)=L\left(\mathbf{F}_{s}\right)$ with $1<s \leq \infty$, then

$$
\mathcal{Q}(1) * \cdots \mathcal{Q}(n) * L\left(\mathbf{F}_{r}\right) \cong \begin{cases}\mathcal{Q}(2) * \cdots * \mathcal{Q}(n) * L\left(\mathbf{F}_{r+s}\right) & \text { if } n \geq 2 \\ L\left(\mathbf{F}_{r+s}\right) & \text { if } n=1\end{cases}
$$

(iv) If $n \geq 2$ and if $r>2-n$, then

$$
\mathcal{Q}(1) * \cdots * \mathcal{Q}(n) * L\left(\mathbf{F}_{r}\right) \cong \mathcal{Q}(1) *\left(\mathcal{Q}(2) * \cdots * \mathcal{Q}(n) * L\left(\mathbf{F}_{r}\right)\right)
$$

(v) If $\mathcal{Q}(1)=\mathcal{N}(1) * \mathcal{N}(2)$ where $\mathcal{N}(1)$ and $\mathcal{N}(2)$ are II 1 -factors, then

$$
\mathcal{Q}(1) * \cdots * \mathcal{Q}(n) * L\left(\mathbf{F}_{r}\right) \cong \mathcal{N}(1) * \mathcal{N}(2) * \mathcal{Q}(2) * \cdots * \mathcal{Q}(n) * L\left(\mathbf{F}_{r}\right)
$$

(vi) If $0<\tilde{r} \leq \infty$, then

$$
\left(\mathcal{Q}(1) * \cdots * \mathcal{Q}(n) * L\left(\mathbf{F}_{r}\right)\right) * L\left(\mathbf{F}_{\tilde{r}}\right) \cong \mathcal{Q}(1) * \cdots * \mathcal{Q}(n) * L\left(\mathbf{F}_{r+\tilde{r}}\right)
$$

(vii) If $\tilde{n} \in \mathbf{N}$, if $\widetilde{\mathcal{Q}}(1), \ldots, \widetilde{\mathcal{Q}}(\tilde{n})$ are $I I_{1}$-factors, and if $1-\tilde{n}<\tilde{r} \leq \infty$, then

$$
\begin{aligned}
& \left(\mathcal{Q}(1) * \cdots * \mathcal{Q}(n) * L\left(\mathbf{F}_{r}\right)\right) *\left(\widetilde{\mathcal{Q}}(1) * \cdots * \widetilde{\mathcal{Q}}(\tilde{n}) * L\left(\mathbf{F}_{\tilde{r}}\right)\right) \\
& \quad \cong \mathcal{Q}(1) * \cdots * \mathcal{Q}(n) * \widetilde{\mathcal{Q}}(1) * \cdots * \widetilde{\mathcal{Q}}(\tilde{n}) * L\left(\mathbf{F}_{r+\tilde{r}}\right)
\end{aligned}
$$

(viii) If $n \geq 2$, then

$$
\mathcal{Q}(1) * \cdots * \mathcal{Q}(n) * L\left(\mathbf{F}_{0}\right) \cong \mathcal{Q}(1) * \cdots * \mathcal{Q}(n)
$$

(ix) If $\mathcal{N}$ is a $I I_{1}$-factor and if $\mathcal{A}$ is a von Neumann algebra with specified normal faithful tracial state, where $\mathcal{A} \neq \mathbf{C}$ and $\mathcal{A}$ is either finite dimensional, hyperfinite, an interpolated free group factor or a (possibly countably infinite) direct sum of these, then

$$
\mathcal{N} * \mathcal{A} \cong \mathcal{N} * L\left(\mathbf{F}_{r}\right)
$$

where $r$ is the quantity computed in [1] and (perhaps misleadingly) called the free dimension of $\mathcal{A}$; (in the revised notation of [3, §1], $r$ is such that $\mathcal{A}$ has a generating set of free dimension $r$ ).
Proof. For (i), if $r>1$, then this is Lemma If If $r \leq 1$ but $0<t<\sqrt{1+(r-1) / n}$, then this is Definition 3, Suppose $r \leq 1$ and $\sqrt{1+(r-1) / n} \leq t<\infty$. Let $\lambda>0$ be such that $\lambda t<\sqrt{1+(r-1) / n}$. Then applying Definition 3 twice gives

$$
\begin{aligned}
(\mathcal{Q}(1) & \left.* \cdots * \mathcal{Q}(n) * L\left(\mathbf{F}_{r}\right)\right)_{\lambda t} \\
& \cong \mathcal{Q}(1)_{\lambda t} * \cdots * \mathcal{Q}(n)_{\lambda t} * L\left(\mathbf{F}_{\lambda^{-2} t^{-2} r+(n-1)\left(\lambda^{-2} t^{-2}-1\right)}\right) \\
& \cong\left(\mathcal{Q}(1)_{t} * \cdots * \mathcal{Q}(n)_{t} * L\left(\mathbf{F}_{t^{-2} r+(n-1)\left(t^{-2}-1\right)}\right)\right)_{\lambda}
\end{aligned}
$$

Now the proofs of (ii)-(viii) are obtained by rescaling both sides of the desired isomorphisms by the same $t>0$ which is small enough and applying (i) and perhaps equation (1). For example, to prove (vii) let

$$
0<t<\min \left(\frac{1}{\sqrt{2}}, \sqrt{1+\frac{r-1}{n}}, \sqrt{1+\frac{\tilde{r}-1}{\tilde{n}}}, \sqrt{1+\frac{r+\tilde{r}-1}{n+\tilde{n}}}\right)
$$

and use (i) three times to get

$$
\begin{aligned}
& \left(\left(\mathcal{Q}(1) * \cdots * \mathcal{Q}(n) * L\left(\mathbf{F}_{r}\right)\right) *\left(\widetilde{\mathcal{Q}}(1) * \cdots * \widetilde{\mathcal{Q}}(\tilde{n}) * L\left(\mathbf{F}_{\tilde{r}}\right)\right)\right)_{t} \\
& \quad \cong\left(\mathcal{Q}(1) * \cdots * \mathcal{Q}(n) * L\left(\mathbf{F}_{r}\right)\right)_{t} *\left(\widetilde{\mathcal{Q}}(1) * \cdots * \widetilde{\mathcal{Q}}(\tilde{n}) * L\left(\mathbf{F}_{\tilde{r}}\right)\right)_{t} * L\left(\mathbf{F}_{t^{-2}-1}\right) \\
& \quad \cong \mathcal{Q}(1)_{t} * \cdots * \mathcal{Q}(n)_{t} * \widetilde{\mathcal{Q}}(1)_{t} * \cdots * \widetilde{\mathcal{Q}}(\tilde{n})_{t} * L\left(\mathbf{F}_{t^{-2}(r+\tilde{r})+(n+\tilde{n}-1)\left(t^{-2}-1\right)}\right) \\
& \quad \cong\left(\mathcal{Q}(1) * \cdots * \mathcal{Q}(n) * \widetilde{\mathcal{Q}}(1) * \cdots * \widetilde{\mathcal{Q}}(\tilde{n}) * L\left(\mathbf{F}_{r+\tilde{r}}\right)\right)_{t}
\end{aligned}
$$

For (ix), if $k \in \mathbf{N}$ is large enough, then by [1], $M_{k}(\mathbf{C}) * \mathcal{A}$ is the interpolated free group factor $L\left(F_{r+1-k^{-2}}\right)$. By [1, Thm. 1.2],

$$
\begin{aligned}
(\mathcal{N} * \mathcal{A})_{\frac{1}{k}} & \cong\left(\left(\mathcal{N}_{\frac{1}{k}} \otimes M_{k}(\mathbf{C})\right) * \mathcal{A}\right)_{\frac{1}{k}} \cong \mathcal{N}_{\frac{1}{k}} *\left(M_{k}(\mathbf{C}) * \mathcal{A}\right)_{\frac{1}{k}} \\
& \cong \mathcal{N}_{\frac{1}{k}} * L\left(\mathbf{F}_{r+1-k^{-2}}\right)_{\frac{1}{k}} \cong \mathcal{N}_{\frac{1}{k}} * L\left(\mathbf{F}_{k^{2} r}\right) \cong\left(\mathcal{N} * L\left(\mathbf{F}_{r}\right)\right)_{\frac{1}{k}}
\end{aligned}
$$

Formula (1) can now be extended to all values of $t$.
Theorem 5. Let $n \in\{2,3, \ldots\}$, let $\mathcal{Q}(1), \ldots, \mathcal{Q}(n)$ be $I_{1}$-factors, and let $0<t<$ $\infty$. Then

$$
(\mathcal{Q}(1) * \cdots * \mathcal{Q}(n))_{t}=\mathcal{Q}(1)_{t} * \cdots * \mathcal{Q}(n)_{t} * L\left(\mathbf{F}_{(n-1)\left(t^{-2}-1\right)}\right)
$$

Proof. Use part (viii) followed by part (i) of Proposition 4.
We know from [6] (see also [2]) that the interpolated free group factors $\left(L\left(\mathbf{F}_{t}\right)\right)_{1<t \leq \infty}$ are either all isomorphic to each other or all mutually nonisomorphic.
Theorem 6. The following are equivalent:
(a) $L\left(F_{s}\right) \cong L\left(F_{t}\right)$ for some, and then for all, $1<s<t \leq \infty$;
(b) for every $I I_{1}$-factor $\mathcal{Q}$ and every $r>0$,

$$
\mathcal{Q} * L\left(\mathbf{F}_{r}\right) \cong \mathcal{Q} * L\left(\mathbf{F}_{\infty}\right)
$$

(c) for all $I I_{1}$-factors $\mathcal{Q}(1)$ and $\mathcal{Q}(2)$,

$$
\mathcal{Q}(1) * \mathcal{Q}(2) \cong \mathcal{Q}(1) * \mathcal{Q}(2) * L\left(\mathbf{F}_{\infty}\right)
$$

Proof. For $(\mathrm{a}) \Longrightarrow(\mathrm{b})$, if $0<t<\sqrt{r}$, then by part (i) of Proposition 4

$$
\left(\mathcal{Q} * L\left(\mathbf{F}_{r}\right)\right)_{t} \cong \mathcal{Q}_{t} * L\left(\mathbf{F}_{t^{-2} r}\right) \cong \mathcal{Q}_{t} * L\left(\mathbf{F}_{\infty}\right) \cong\left(\mathcal{Q} * L\left(\mathbf{F}_{\infty}\right)\right)_{t}
$$

while $(\mathrm{b}) \Longrightarrow(\mathrm{a})$ can be seen by choosing $\mathcal{Q}=L\left(\mathbf{F}_{2}\right)$. For $(\mathrm{a}) \Longrightarrow(\mathrm{c})$, if $0<t<$ $1 / \sqrt{2}$, then using Lemma 1 ,

$$
\begin{aligned}
(\mathcal{Q}(1) * \mathcal{Q}(2))_{t} & \cong \mathcal{Q}(1)_{t} * \mathcal{Q}(2)_{t} * L\left(\mathbf{F}_{t^{-2}-1}\right) \\
& \cong \mathcal{Q}(1)_{t} * \mathcal{Q}(2)_{t} * L\left(\mathbf{F}_{\infty}\right) \cong\left(\mathcal{Q}(1) * \mathcal{Q}(2) * L\left(\mathbf{F}_{\infty}\right)\right)_{t}
\end{aligned}
$$

Taking $\mathcal{Q}(1) \cong \mathcal{Q}(2) \cong L\left(\mathbf{F}_{2}\right)$ shows $(\mathrm{c}) \Longrightarrow(\mathrm{a})$.

## References

[1] K. Dykema, Free products of hyperfinite von Neumann algebras and free dimension, Duke Math. J. 69 (1993), 97-119. MR 93m:46071
[2] ——, Interpolated free group factors, Pacific J. Math. 163 (1994), 123-135. MR 95c:46103
[3] _, Free subproducts and free scaled products of $I I_{1}$-factors, J. Funct. Anal. (to appear).
[4] K. Dykema, F. Rădulescu, Compressions of free products of von Neumann algebras, Math. Ann. 316 (2000), 61-82. MR 2001f: 46100
[5] F.J. Murray and J. von Neumann, Rings of operators. IV, Ann. of Math. 44 (1943), 716-808. MR 5:101a
[6] F. Rădulescu, Random matrices, amalgamated free products and subfactors of the von Neumann algebra of a free group, of noninteger index, Invent. Math. 115 (1994), 347-389. MR 95c:46102

Department of Mathematics, Texas A\&M University, College Station, Texas 778433368

E-mail address: Ken.Dykema@math.tamu.edu
Department of Mathematics, University of Iowa, Iowa City, Iowa 52242-1466
E-mail address: radulesc@math.uiowa.edu


[^0]:    Received by the editors April 3, 2001 and, in revised form, January 18, 2002.
    2000 Mathematics Subject Classification. Primary 46L09.
    The first author was partially supported by NSF grant DMS-0070558.
    The second author was partially supported by NSF grant DMS-9970486. Both authors also thank the Mathematical Sciences Research Institute, where they were engaged in this work. Research at MSRI is supported in part by NSF grant DMS-9701755.

