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SIMILARITY TO AN ISOMETRY OF A COMPOSITION OPERATOR

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ABSTRACT. We study the composition operators which are similar to an isometry on the classical Hardy space $H^2(\mathbb{D})$.

1. Introduction

Throughout this paper, we denote by $\mathbb{D}$ the open unit disc in the complex plane, by $\mathbb{T}$ the circle, and by m the Haar measure on $\mathbb{T}$. We also denote by $H(\mathbb{D})$ the space of holomorphic functions on $\mathbb{D}$, and by $H(\mathbb{D},\mathbb{D})$ the subset of $H(\mathbb{D})$ consisting of all self-maps of $\mathbb{D}$.

We recall that the Hardy Space $H^2(\mathbb{D})$ is the subspace of $H(\mathbb{D})$ consisting of all functions f satisfying

$$\|f\|_2 := \left(\sup_{0\leq r<1}\int_0^{2\pi}\frac{1}{2\pi}|f(re^{i\theta})|^2d\theta\right)^{1/2} < +\infty.$$

Every function $f\in H^2(\mathbb{D})$ has a radial limit, i.e.,

$$\lim_{r\to 1^-} f(re^{i\theta}) = f^*(e^{i\theta})$$

exists for almost every $e^{i\theta}\in\mathbb{T}$. Furthermore, we have $f^*\in L^2$, and $\|f\|_{H^2(\mathbb{D})} = \|f^*\|_{L^2}$. In the sequel, we will write f instead of f^*. A function $\phi\in H(\mathbb{D},\mathbb{D})$ is said to be inner if $|\phi(e^{it})|=1$ almost everywhere.

Let $\phi\in H(\mathbb{D},\mathbb{D})$. We define on $H^2(\mathbb{D})$ the composition operator induced by ϕ by $C_\phi(f)=f\circ\phi$. It is well-known that $C_\phi(f)\in H^2(\mathbb{D})$, and that C_ϕ is continuous on $H^2(\mathbb{D})$. More precisely, the following inequalities hold (see [2], p. 123):

$$\left(\frac{1}{1-|\phi(0)|^2}\right)^{1/2} \leq \|C_\phi\| \leq \left(\frac{1+|\phi(0)|}{1-|\phi(0)|}\right)^{1/2}. \tag{1}$$

In particular, C_ϕ is a contraction if and only if $\phi(0)=0$. Moreover, in Nordgren's classical paper [6], it is shown that if $\phi(0)=0$ and if ϕ is inner, then C_ϕ is an isometry. Our aim in this paper is to prove the following theorem:

Theorem 1. *Let $\phi\in H(\mathbb{D},\mathbb{D})$. The following are equivalent:*

i) ϕ is inner, and has a fixed point in $\mathbb{D}$.
ii) C_ϕ is similar to an isometry.

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This result was first shown by N.Jaoua ([4]) under some additional assumptions on the symbol (ϕ is analytic in a neighborhood of the closed disk). He uses a result of Kamowitz on spectra of composition operators, and a weak form of the Denjoy-Wolff theorem. It turns out that, by a recent result of J.Shapiro ([7]), no extra assumption on ϕ is needed. Indeed, Shapiro proved: Suppose that ϕ is an analytic self map of the disk fixing the origin. If ϕ is *not* inner, then the restriction of C_ϕ to the subspace of functions in H^2 which are 0 at 0 has norm strictly less than 1. Therefore, it is sufficient to prove that ϕ has a fixed point as soon as C_ϕ is similar to an isometry. We will give a proof of this fact later.

The aim of this paper is to give a simple and self-contained proof of Theorem 1, which can be extended to other situations, such as composition operators on Hardy spaces of Dirichlet series. With minor modifications, the same arguments apply, for example, in Theorem 3.5 of [1].

2. Proof of the main theorem

We recall Nagy's criterion ([5]) of similarity to an isometry. We will just use the easier implication.

Theorem 2 (Nagy). *S is similar to an isometry if and only if*

$$\exists k>0,\ \forall x\in X,\ \forall n\in\mathbb{N},\ \frac{1}{k}\|x\|\leq\|S^n x\|\leq k\|x\|.$$

In other words, S is power-bounded, and the ranges of S^n are uniformly closed.

Cima, Thomson and Wogen proved in [3] that the composition operator C_ϕ on $H^2(\mathbb{D})$ has closed range if and only if

$$\exists c>0,\ \forall B\subset\mathbb{T},\ m(\phi^{-1}(B)\cap\mathbb{T})\geq cm(B),$$

where B is a Borel set of $\mathbb{T}$ and $\phi^{-1}(B)=\{z\in\overline{\mathbb{D}},\ \phi(z)\in B\}$. On the other hand, a composition operator is injective, so it has closed range if and only if there exists $k>0$ such that, for all $f\in H^2(\mathbb{D})$, we have $k\|f\|\leq\|C_\phi(f)\|$. In the next lemma, we use arguments similar to those in [3] to give a quantitative relation between c and k.

Lemma 1. *Let C_ϕ be a composition operator on $H^2(\mathbb{D})$, and $k>0$ be a positive constant. We suppose that, for all $f\in H^2(\mathbb{D})$, we have $k\|f\|\leq\|C_\phi(f)\|$. Then, for every Borel set $B\subset\mathbb{T}$, one has*

$$m(\phi^{-1}(B)\cap\mathbb{T})\geq k^2 m(B).$$

Proof. Let B be such a set, and $A=\phi^{-1}(B)\cap\mathbb{T}$. We take a function $f\in H^\infty(\mathbb{D})$ such that

$$|f|=\begin{cases}1 & \text{on } B,\\ 1/2 & \text{on } \mathbb{T}-B.\end{cases}$$

The existence of such a function follows from Beurling's theory of outer functions. It is clear that $\|f^n\|_{H^2}^2\to m(B)$. On the other hand, we have

$$\|C_\phi(f^n)\|_{H^2}^2 = \int_A |f^n\circ\phi|^2dm+\int_{\mathbb{T}-A}|f^n\circ\phi|^2dm.$$

But, $\int_A |f^n\circ\phi|^2dm=m(A)$, whereas $|f\circ\phi(z)|<1$ for $z\in\mathbb{T}-A$. Lebesgue's theorem then implies that

$$\|C_\phi(f^n)\|^2\to m(A).$$

Passing to the limit in the inequality $k^2\|f^n\|^2 \leq \|C_\phi(f^n)\|^2$, we get that $m(A) \geq k^2 m(B)$. □

Our principal result follows from this lemma:

Proof of Theorem 1. We already know that if ϕ is inner and $\phi(0) = 0$, then C_ϕ is an isometry. Now, if $\phi(a) = a$, let ψ be an automorphism of $\mathbb{D}$ which maps a to 0. Let $\varphi = \psi^{-1} \circ \phi \circ \psi$. Then C_φ is an isometry. C_ϕ, which is similar to C_φ, is in particular similar to an isometry.

Conversely, suppose that C_ϕ is similar to an isometry. Let ϕ_j be the function $\phi \circ \cdots \circ \phi$ (j times). By the assumption, there exist $K, k > 0$ such that

$$\forall j \in \mathbb{N},\ \forall f \in H^2(\mathbb{D}),\ K\|f\| \geq \|C_\phi^j(f)\| = \|C_{\phi_j}(f)\| \geq k\|f\|. \tag{2}$$

Now, suppose that C_ϕ has no fixed point in $\mathbb{D}$. By a weak form of the Denjoy-Wolff theorem, we have $|\phi_n(0)| \to 1$. Therefore

$$\|C_\phi^n\| = \|C_{\phi_n}\| \geq (1 - |\phi_n(0)|)^{-1/2}.$$

This inequality contradicts (2). We now prove that ϕ is inner. Using the lemma, for all Borel sets B of $\mathbb{T}$, and all $j \in \mathbb{N}$, we have $m(\phi_j^{-1}(B) \cap \mathbb{T}) \geq k^2 m(B)$. If ϕ were not inner, there would exist $B \subset \mathbb{T}$ such that $m(B) > 0$, and $|\phi(z)| < 1$ if $z \in B$. Let us define $B_0 = B$, and $B_j = \phi_j^{-1}(B) \cap \mathbb{T}$. We have by Lemma 1:

$$m(B_j) \quad \geq \quad k^2 m(B). \tag{3}$$

We now observe that the B_j's are disjoint:

$$\text{If } i \neq j,\ B_i \cap B_j = \varnothing. \tag{4}$$

Indeed, if for example $i < j$ and $z \in B_i$, $\phi_i(z) \in B$, so $|\phi_j(z)| = |\phi_{j-i}(\phi_i(z))| < 1$. As $B \subset \mathbb{T}$, $\phi_j(z) \notin B$, and this implies $z \notin B_j$. But it follows from (4) that $\sum m(B_j) \leq m(\mathbb{T}) \leq 1$, and this fact is in contradiction with (3). This ends the proof of Theorem 1. □

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