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# RANDOM MENSHOV SPECTRA

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Abstract. We show that the spectra $\Lambda$ of frequencies $\lambda$ obtained by random perturbations of the integers allows one to represent any measurable function $f$ on $\mathbb{R}$ by an almost everywhere converging sum of harmonics:

$$f = \sum_{\Lambda} c_\lambda e^{i\lambda t}.$$

## 1. Introduction

This paper concerns the representation of functions by series of exponentials which converge almost everywhere (a.e.). According to Menshov's theorem (1941, see [1]) every $2\pi$-periodic measurable function $f$ admits a representation as

$$f(t) = \sum_{k\in\mathbb{Z}} c(k) e^{ikt} \quad \text{a.e.} \tag{1}$$

Among the many generalizations and analogs of this fundamental result, there exists a version for the non-periodic case: Davtjan [2] proved that the corresponding representation on $\mathbb{R}$ can be obtained if instead of the sum over integers one considers a "trigonometric integral" which involves all real frequencies.

In our paper [4] it was shown that most of the frequencies are redundant. Namely, by appropriate small perturbations of the integers we constructed a spectrum of frequencies $\Lambda = \{\lambda(k),\, k \in \mathbb{Z}\}$ such that any $f \in L^0(\mathbb{R})$ (that is, a measurable function on $\mathbb{R}$) can be decomposed as

$$f(t) = \sum_{k\in\mathbb{Z}} c(k) e^{i\lambda(k)t} \quad \text{a.e.} \tag{2}$$

The aim of this note is to show that this is not an exceptional feature of the constructed spectrum. In fact, by choosing the perturbations *randomly* one gets such a property with probability 1.

Analogously to the periodic case (see [5]) we introduce the following

**Definition.** A sequence $\Lambda = \{... < \lambda(k) < ... < \lambda(-1) < \lambda(0) < \lambda(1) < ...\}$ is called a Menshov spectrum for $\mathbb{R}$ if for any $f \in L^0(\mathbb{R})$ there are coefficients $\{c(k)\}$ such that the decomposition (2) holds (convergence in (2) is understood in the sense of the limit of symmetric partial sums, i.e. $\lim_{x\to\infty} \sum_{|\lambda|<x}$).

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**Theorem.** *Let $r(n)$ be independent variables uniformly distributed on the segment $[-\frac{1}{2}, \frac{1}{2}]$. Then the sequence*

$$\lambda(n) = n + r(n), \quad n \in \mathbb{Z}, \tag{3}$$

*is almost surely a Menshov spectrum for $\mathbb{R}$.*

The proof is based on the technique used in our recent paper [5]. We refer the reader to this paper for some historical comments and additional references.

## 2. Preliminaries

By a trigonometric polynomial $P$ we mean a finite linear combination of exponentials with real (not necessarily integer) frequencies $... < \lambda(-1) < \lambda(0) < ...$. We call the set of $\lambda$'s involved, the spectrum of $P$ and denote it by $\operatorname{spec} P$. The corresponding coefficients are denoted as $\widehat{P}(\lambda)$, so

$$P = \sum_{\operatorname{spec} P} \widehat{P}(\lambda) e^{i\lambda x}, \quad x \in \mathbb{R} .$$

We denote

$$\deg P = \max_{\operatorname{spec} P} |\lambda| .$$

As usual

$$||\widehat{P}||_1 := \sum |\widehat{P}(\lambda)|, \qquad ||\widehat{P}||_\infty := \max |\widehat{P}(\lambda)| .$$

Let $P^*$ be the (non-symmetric) majorant of partial sums:

$$P^*(x) := \max_{a<b} \left| \sum_{\operatorname{spec} P \cap [a,b]} \widehat{P}(\lambda) e^{i\lambda x} \right| .$$

For a given $P$ and $l \in \mathbb{Z}^+$ we denote by $P_{[l]}$ the "contracted" polynomial:

$$P_{[l]}(x) = P(lx) .$$

The following "special products" are used:

$$H = Q_{[l]} P .$$

If $\operatorname{spec} Q \subset \mathbb{Z}$ and $l > 2 \deg P$, then this product has a simple structure which provides the following estimate (compare with (10) in [5]):

$$H^*(x) \le |P(x)| \cdot ||Q^*||_{L^\infty(-\pi,\pi)} + 2P^*(x) \cdot ||\widehat{Q}||_\infty . \tag{4}$$

We will use the following

**Lemma 1** (see [5], Lemma 2.1)**.** *Given any $\epsilon > 0$, $\delta > 0$, there exists a trigonometric polynomial $P = P_{\epsilon,\delta}$ with integer spectrum such that*

(i) $\widehat{P}(0) = 0$, $||\widehat{P}||_\infty < \delta$;
(ii) $\mathbf{m}\{x \in [-\pi, \pi] : |P(x) - 1| > \delta\} < \epsilon$;
(iii) $||P^*||_\infty < C\epsilon^{-1}$.

## 3. Proof of the theorem

3.1. The result is an easy consequence of the following (nonstochastic)

**Proposition.** *Let $\Lambda = \{\lambda(n)\}$, $\lambda(n) = n + r(n)$, $n \in \mathbb{Z}$, and suppose that for every $k \in \mathbb{Z}^+$ there exists a number $l = l(k)$ in $\mathbb{Z}^+$ such that*

$$|r(sl+q) - 2^{-|q|+1}| < \frac{1}{k^2}, \quad 0 < |s| < k,\ |q| < k\ . \tag{5}$$

*Then $\Lambda$ is a Menshov spectrum for $\mathbb{R}$.*

To deduce the theorem from the proposition it is enough to mention that if we fix $k$ and run $l$ over a sufficiently fast increasing sequence $\{l_j\}$, then the events $B_j$ that the inequalities above are fulfilled for $l = l_j$ are mutually independent and each has a positive probability $p(k)$ which does not depend on $j$. So for a random spectrum $H$ the condition of the proposition is true almost surely.

3.2. Now we pass to the proof of the proposition. Denote $I(k) := \{sl(k) + q : 0 < |s| < k, |q| < k\}$ and $M(k) = kl(k) + k$. Clearly (passing to a subsequence if necessary) we may suppose that

$$I(k+1) \cap [-M(k), M(k)] = \emptyset\ .$$

Let $f \in L^0(\mathbb{R})$ be given. We shall define by induction an increasing sequence $\{k_j\}$ and "blocks" of coefficients $\{c(n)\}$, $n \in I(k_j)$; all other coefficients of the expansion (2) will be zero. We denote:

$$\begin{aligned} A_j &:= \sum_{I(k_j)} c(n) \exp(i\lambda(n)x), \\ S_N &:= \sum_{j \le N} A_j \quad N \in \mathbb{Z}^+,\ S_0 := 0. \end{aligned}$$

Fix $N$ and suppose that the polynomials $A_j$ are already defined for $j < N$. Let us describe the $N$'th step of the induction. Set

$$R_N := f - S_{N-1}\ .$$

We need the following result proved in [3]:

if $0 < |r(q)| = o(1)$, then the system of exponentials $\exp i(q + r(q))$, $q \in \mathbb{Z}$, is complete in $L^0(\mathbb{R})$, that is, the set of linear combinations is dense with respect to convergence a.e.

Using this we find a polynomial

$$F_N(x) = \sum a_q \exp i(q + 2^{-|q|+1})x$$

so that

$$\mathbf{m}\left\{x \in [-N\pi, N\pi] : |F_N(x) - R_N(x)| > \frac{1}{N^4}\right\} < \frac{1}{N^2}\ . \tag{6}$$

Next we use Lemma 1 with

$$\delta = \delta_N = \frac{1}{N^4 ||\widehat{F_N}||_1}, \quad \epsilon = \frac{1}{N^3} \tag{7}$$

and find the corresponding polynomial $Q_N$. Fix a number $k_N$ large enough:

$$k_N > k_{N-1},\ 3 \deg F_N,\ \frac{||\widehat{Q_N}||_1}{\delta_N} \tag{8}$$

and set

$$H_N := F_N \cdot (Q_N)_{[l(k_N)]} \ . \tag{9}$$

One can easily see that

$$\operatorname{spec} H \subset \left\{ sl(k_N) + q + 2^{-|q|+1} \ : \ 0 < |s| < k,\ |q| < k \right\} \ ,$$

so we can write

$$H_N = \sum_{\substack{0<|s|<k_N \\ |q|<k_N}} b(N,s,q) \exp i(sl(k_N) + q + 2^{-|q|+1})x \ . \tag{10}$$

Finally we set:

$$A_N := \sum_{\substack{0<|s|<k_N \\ |q|<k_N}} b(N,s,q) \exp i\lambda(sl(k_N)+q)x \equiv \sum_{I(k_N)} c(n) \exp i\lambda(n)x \ . \tag{11}$$

3.3. Now we show that

$$S_N \to f \quad \text{a.e.} \tag{12}$$

For this first we get from (ii) of Lemma 1:

$$\begin{aligned} &\mathbf{m}\left(\left\{ x \in [-N\pi, N\pi] \ : \ |H_N(x) - F_N(x)| > \frac{1}{N^4} \right\}\right) \\ &\qquad \le N \cdot \mathbf{m}\left(\{x \in [-\pi,\pi] \ : \ |Q_N - 1| \ge \delta_N\}\right) = O(N^{-2}) \ . \end{aligned} \tag{13}$$

Further, (9), (8) and (7) imply:

$$||\widehat{H_N}||_1 = ||\widehat{F_N}||_1 \cdot ||\widehat{Q_N}||_1 < ||\widehat{F_N}||_1 k_N \delta_N = \frac{k_N}{N^4} \ , \tag{14}$$

so we can estimate, using (10), (11), (5), (14), (3) and (8):

$$\begin{aligned} &||A_N - H_N||_{L^\infty(-\pi N, \pi N)} \\ &\quad \le \quad ||\widehat{H_N}||_1 \cdot \max_{\substack{0<|s|<k_N \\ |q|<k_N}} ||\exp i(r(sl(k_N)+q) - 2^{-|q|+1})x - 1||_\infty \\ &\quad < \quad \frac{k_N}{N^4} \cdot \frac{1}{k_N^2} \cdot \pi N = O(N^{-4}) \ . \end{aligned} \tag{15}$$

Finally, we have from (6), (13) and (15):

$$\mathbf{m}\left\{ x \in [-N\pi, N\pi] \ : \ |A_N(x) - R_N(x)| > \frac{C}{N^4} \right\} = O(N^{-2}) \ ,$$

so

$$R_{N+1} = R_N - A_N = O(N^{-4}) \quad \text{a.e.,} \tag{16}$$

and (12) follows.

3.4. At last:

$$A_N^* \to 0 \quad \text{a.e.} \tag{17}$$

Indeed, estimating as in (15) we see:

$$A_N^* < H_N^* + O(N^{-4}) \quad \text{a.e.}$$

Since $l(k_N) > k_N > 3 \deg F_N$ we can use (4) and get:

$$H_N^*(x) < |F_N(x)| \cdot ||Q_N^*||_{L^\infty(-\pi,\pi)} + 2||\widehat{Q}||_\infty \cdot ||\widehat{F}||_1 \ .$$

The first term on the right hand side is $O(N^{-1})$ a.e. due to (6), (16) and (iii). The last term is $O(N^{-4})$ because of (i) and (7). Clearly (12) and (17) imply the decomposition (2) and this completes the proof. □

## 4. Remarks

4.1. One can see that the result holds for $r(n)$ uniformly distributed on any fixed neighbourhood of zero. Moreover, it holds true for $r(n)$ uniformly distributed on $[-d_n, d_n]$ if $d_n$ decrease slowly enough. This allows one to cover in full generality the result from [4] where a Menshov spectrum $\{n + o(1)\}$ was constructed. But $d_n$ really must decay slowly. In contrast to the completeness property which occurs for **any** (nonzero) perturbation $r(n) = o(1)$ (the result from [3] used above), the following simple observation is true:

*If $\lambda(n) = n + O(n^{-\alpha})$, $\alpha > 0$, then it is not a Menshov spectrum.*

Indeed, if $\Delta f := f(x) - f(x - 2\pi)$, then

$$\Delta^k \left( \sum c_n e^{i\lambda_n x} \right) = \sum c_n O(n^{-\alpha k}) e^{i\lambda_n x}.$$

Hence for $k$ sufficiently large (depending only on $\alpha$) the representation (2) implies that $\Delta^k f$ equals a smooth function a.e., so $\lambda(n)$ cannot be a Menshov spectrum for $\mathbb{R}$.

4.2. In [5] we studied Menshov spectra in the periodic case. The main results of that paper can be extended to Menshov spectra in $\mathbb{R}$. For example, Menshov spectra for $\mathbb{R}$ may be quite sparse, up to "almost Hadamarian lacunarity". More precisely:

*For any $\epsilon(n)$ decreasing to zero one can construct a (symmetric) Menshov spectrum $\Lambda$ for $\mathbb{R}$ such that $\lambda(n+1)/\lambda(n) > 1 + \epsilon(n)$, $n \in \mathbb{Z}^+$.*

This is an analog of Theorem 1 from [5] and the proof is basically the same.

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