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AN ALGEBRAIC PROPERTY OF JOININGS

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Abstract. We show that an ergodic automorphism is semisimple if and only if the set of ergodic self-joinings is a subsemigroup of the semigroup of self-joinings.

1. Introduction

Assume that T is an ergodic automorphism of a probability standard Borel space $(X, \mathcal{B}, \mu)$. By $J(T)$ we denote the set of all self-joinings of T that are all $T \times T$-invariant measures defined on $(X \times X, \mathcal{B} \otimes \mathcal{B})$, both of whose natural projections are equal to μ. On the set $J(T)$ there is a natural structure of a semitopological compact affine semigroup (see the next section for this and some further basic notions and results). By $J^e(T)$ we denote the set of ergodic members of $J(T)$.

In [3], A. del Junco, M.K. Mentzen and the second author introduced a notion of semisimplicity. We say that T is semisimple if for any $\lambda \in J^e(T)$ the automorphism $(T \times T, \lambda)$ is relatively weakly mixing over T (T is given by the projection on the first coordinate). The notion of semisimplicity generalized the notion of minimal self-joinings [7] and of simplicity [4, 9]. Moreover, some Gaussian automorphisms turned out to be semisimple (see [5]). It follows from basic properties of relative products that $J^e(T)$ is stable under composition whenever T is semisimple. The aim of this note is to prove that the converse also holds.

Theorem 1. *Let T be an ergodic automorphism of $(X, \mathcal{B}, \mu)$. Then T is semisimple if and only if the set of ergodic self-joinings is a subsemigroup of $J(T)$.*

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2. Notation and basic results

Suppose that $\pi : (Z, \mathcal{D}, \rho) \to (Y, \mathcal{C}, \eta)$ is a homomorphism of two standard probability spaces. Given $f \in L^1(Z, \rho)$, by $E(f|Y)$ or $E^{\eta}(f|Y)$ we denote the conditional expectation of f with respect to Y, i.e. the function in $L^1(Y, \eta)$ given by

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$E(f|\pi^{-1}(\mathcal{C}))\circ\pi^{-1}$. If

$$\rho=\int_Y \rho_y\,d\eta(y)$$

denotes the disintegration of ρ over η, then $E(\cdot|Y)(y)=\rho_y(\cdot)$ for a.a. $y\in Y$ (see [2], Th. 5.8). If $\pi':(Z',\mathcal{D}',\rho')\to(Y,\mathcal{C},\eta)$ is another homomorphism and $\rho'=\int_Y\rho'_y\,d\eta(y)$, then the measure

$$\rho\otimes_Y\rho'=\int_Y\rho_y\otimes\rho'_y\,d\eta(y)$$

defined on $\mathcal{D}\otimes\mathcal{D}'$ is called the *relative product* of ρ and ρ' over (Y,η) (see [2], Chapter 5, §5). The resulting space will be denoted $(Z\times_Y Z',\rho\otimes_Y\rho')$. In what follows we will need the following.

Lemma 1. *Consider a sequence of homomorphisms* $(Z,\mathcal{D},\rho)\to(Y,\mathcal{C},\eta)\to(X,\mathcal{B},\mu)$. *Whenever* $f,g\in L^2(Z,\rho)$, *then* $E(f\otimes g|Y\times_X Y)=E(f|Y)\otimes E(g|Y)$, $\eta\otimes_X\eta$ *a.s.*

Proof. Let $\rho=\int_Y\rho_y\,d\eta(y)$, $\rho=\int_X\tilde{\rho}_x\,d\mu(x)$ and $\eta=\int_X\eta_x\,d\mu(x)$ stand for the relevant disintegrations. We have

$$\rho(A)=\int_Z\chi_A\,d\rho=\int_Y E(\chi_A|Y)(y)\,d\eta(y)$$
$$=\int_X\left(\int_Y E(\chi_A|Y)(y)\,d\eta_x(y)\right)d\mu(x)=\int_X\left(\int_Y\rho_y(A)\,d\eta_x(y)\right)d\mu(x);$$

thus

$$\tilde{\rho_x}=\int_Y\rho_y\,d\eta_x(y).$$

Hence

$$\tilde{\rho_x}\otimes\tilde{\rho_x}=\int_{Y\times Y}\rho_y\otimes\rho_{y'}\,d\eta_x\otimes\eta_x(y,y').$$

It follows that

$$\rho\otimes_X\rho=\int_X\tilde{\rho_x}\otimes\tilde{\rho_x}\,d\mu(x)$$
$$=\int_X\left(\int_{Y\times Y}\rho_y\otimes\rho_{y'}\,d\eta_x\otimes\eta_x(y,y')\right)d\mu(x)=\int_{Y\times Y}\rho_y\otimes\rho_{y'}\,d\eta\otimes_X\eta(y,y').$$

Hence, if $f,g\in L^2(Z,\rho)$, then

$$E(f\otimes g|Y\times_X Y)(y,y')=\int_{Y\times Y}f\otimes g\,d\rho_y\otimes\rho_{y'}$$
$$=\int_Y f\,d\rho_y\int_Y g\,d\rho_{y'}=E(f|Y)(y)\cdot E(g|Y)(y')$$

and therefore $E(f\otimes g|Y\times_X Y)=E(f|Y)\otimes E(g|Y)$ a.s. with respect to $\eta\otimes_X\eta$. □

Assume now that T is an ergodic automorphism on a standard probability Borel space $(X,\mathcal{B},\mu)$. To each element $\lambda\in J(T)$ we associate a Markov operator $\Phi_\lambda:L^2(X_1,\mu_1)\to L^2(X_2,\mu_2)$ (where $(X_i,\mu_i)=(X,\mu)$, for $i=1,2$) given by

$$\int_{X_2}\Phi_\lambda(f)\overline{g}\,d\mu_2=\int_{X_1\times X_2}f\overline{g}\,d\lambda.$$

By Markov property we mean that Φ_λ is positive and $\Phi_\lambda 1=\Phi_\lambda^*1=1$. We also have $\Phi_\lambda\circ T=T\circ\Phi_\lambda$. Moreover, for each $f\in L^2(X,\mu)$

$$(\Phi_\lambda f)(x_2)=E^\lambda(f|X_2)(x_2). \tag{1}$$

Furthermore, each Markov operator on $L^2(X,\mu)$ that commutes with T is necessarily of the form Φ_λ (see e.g. [5] or [8]). The latter observation introduces a semigroup law on $J(T)$ by the formula $\Phi_{\lambda_2\circ\lambda_1} = \Phi_{\lambda_2}\circ\Phi_{\lambda_1}$. Together with the weak topology and the natural simplex structure on $J(T)$ we obtain that $J(T)$ is a compact semitopological affine semigroup.

Suppose now $\lambda_1, \lambda_2 \in J(T)$. We will treat λ_1 as defined on $X_1 \times X_2$, while λ_2 is defined on $X_2 \times X_3$. By λ_2^* we mean the joining corresponding to $\Phi^*_{\lambda_2}$, that is, the self-joining given by

$$\lambda_2^*(A_2 \times A_3) = \lambda_2(A_3 \times A_2).$$

Disintegrate λ_1 and λ_2^* over the common factor X_2:

$$\lambda_1 = \int_{X_2} \lambda_{1,x_2}\, d\mu_2(x_2), \quad \lambda_2^* = \int_{X_2} \lambda^*_{2,x_2}\, d\mu_2(x_2).$$

Consider the relative product of λ_1 and λ_2^* over the common factor X_2 that is the measure defined on $X_1 \times X_2 \times X_3$ given by

$$\lambda_1 \otimes_{X_2} \lambda_2^* = \int_{X_2} \lambda_{1,x_2} \otimes \lambda^*_{2,x_2}\, d\mu_2(x_2).$$

Take $f, g \in L^2(X,\mu)$. Using (1) we then have

$$\begin{aligned}
&\int_{X_1\times X_3} f(x_1)g(x_3)\, d\lambda_1 \otimes_{X_2} \lambda_2^*(x_1,x_2,x_3)\\
&= \int_{X_2}\left(\int_{X_1\times X_3} f(x_1)g(x_3)\, d\lambda_{1,x_2}\otimes\lambda^*_{2,x_2}(x_1,x_3)\right) d\mu_2(x_2)\\
&= \int_{X_2} (\Phi_{\lambda_1} f)(x_2)(\Phi_{\lambda_2^*} g)(x_2)\, d\mu_2(x_2) = \int_{X_3} (\Phi_{\lambda_2}\circ\Phi_{\lambda_1})(f) g\, d\mu_3.
\end{aligned}$$

We have shown the following:

$$\lambda_2 \circ \lambda_1 = \lambda_1 \otimes_{X_2} \lambda_2^*|_{X_1\times X_3}. \tag{2}$$

In particular, if $\lambda \otimes_{X_2} \lambda^*$ is ergodic, then $\lambda\circ\lambda$ is ergodic and the key observation for the proof of Theorem 1 is that the converse is also true (see Proposition 1 below).

Let T acting on $(X,\mathcal{B},\mu)$ be a factor of an ergodic automorphism S acting on $(Y,\mathcal{C},\eta)$. Following [2] (see condition C5 on p. 132), we say that S is a *compact* extension of T if for each $0 \neq f \in L^2(Y,\eta)$ the limit of ergodic averages of $f\otimes\overline{f}$ for $S\times S$ acting on $(Y\times_X Y, \eta\otimes_X \eta)$ is also non-zero.

Remark 1. Usually a compact extension is defined in terms of relative eigenvectors (see [1, 10]). R. Zimmer proved in [10] that S is a compact extension of T if and only if S is an isometric extension of T. Another proof of Zimmer's result follows easily from the joining characterization of isometric extensions given in [6].

Assume that R acting on $(Z,\mathcal{D},\rho)$ is an ergodic extension of T acting on $(X,\mathcal{B},\mu)$. Then (see [2], Chapter 6):

(A) there exists a biggest factor, called the *relative Kronecker factor*, S acting on $(Y,\mathcal{C},\eta)$ between R and T such that S is a compact extension of T;

(B) the relative Kronecker factor S is trivial (i.e. $S = T$) iff the relative product $R\times R$ on $(Z\times_X Z, \rho\otimes_X \rho)$ is ergodic (the latter condition means that R is a *relatively weakly mixing extension* of T).

Finally, recall that an ergodic automorphism T on $(X,\mathcal{B},\mu)$ is called *semisimple* ([3]) if for each $\lambda\in J^e(T)$, the relative product $\lambda\otimes_{X_2}\lambda^*$ is ergodic, that is (using (B)), $T\times T$ on $(X_1\times X_2,\lambda)$ is a relatively weakly mixing extension of X_2.

3. Proof of Theorem 1

We will need a lemma which is a simple consequence of the L^1-convergence in the pointwise ergodic theorem.

Lemma 2. *Let S be an automorphism on $(Y,\mathcal{C},\eta)$. Denote by $\mathcal{I}$ the σ-algebra of S-invariant sets. Assume that $\mathcal{E}\subset\mathcal{C}$ is a factor of S. Then*:

(i) *If the action of S on $\mathcal{E}$ is ergodic, then $E(f|\mathcal{I})=\int_Y f\,d\eta$ for each $f\in L^1(\mathcal{E})$.*

(ii) *If $f\in L^1(Y,\eta)$ and the sequence $(\frac{1}{n}\sum_{i=0}^{n-1}f\circ S^i)_{n\geq 1}$ converges to a constant $c(=\int_Y f\,d\eta)$, then $E(E(f|\mathcal{E})|\mathcal{I})=c$.*

Proof. (i) Since $f\in L^1(Y,\eta)$, $\frac{1}{n}\sum_{i=0}^{n-1}f\circ S^i$ converges to $E(f|\mathcal{I})$ in $L^1(\mathcal{C})$ by the ergodic theorem. However f is measurable with respect to $\mathcal{C}$ which is S-invariant. The result follows by the ergodicity of S on $\mathcal{E}$.

(ii) Put $g=E(f|\mathcal{E})$. Then by the ergodic theorem, $\frac{1}{n}\sum_{i=0}^{n-1}g\circ S^i$ converges to $E(g|\mathcal{I})$ in $L^1(\mathcal{C})$ and hence in $L^1(\mathcal{E})$. Therefore for all $h\in L^\infty(\mathcal{E})$,

$$\frac{1}{n}\sum_{i=0}^{n-1}\int f\circ S^i\cdot\overline{h}\,d\eta\to c\int\overline{h}\,d\eta.$$

We have

$$\frac{1}{n}\sum_{i=0}^{n-1}\int f\circ S^i\cdot\overline{h}\,d\eta=\frac{1}{n}\sum_{i=0}^{n-1}\int E(f\circ S^i|\mathcal{E})\cdot\overline{h}\,d\eta=\frac{1}{n}\sum_{i=0}^{n-1}\int E(f|\mathcal{E})\circ S^i\cdot\overline{h}\,d\eta;$$

thus $\frac{1}{n}\sum_{i=0}^{n-1}g\circ S^i$ converges weakly to c in $L^1(\mathcal{E})$. Hence $E(g|\mathcal{I})=c$. □

The following lemma is a direct consequence of our definition of compact extension, the L^1-convergence of ergodic averages and the fact that $f\otimes\overline{f}\in L^1(Y\times_X Y)$ whenever $f\in L^2(Y,\eta)$.

Lemma 3. *Let S be an ergodic automorphism on $(Y,\mathcal{C},\eta)$. Suppose that S is a compact extension of T acting on $(X,\mathcal{B},\mu)$ and $f\in L^2(Y)$. Then $f=0$ if and only if $E(f\otimes\overline{f}|\mathcal{I})=0$ in the relative product $Y\times_X Y$.*

The following result is of independent interest.

Proposition 1. *Assume that T is an ergodic automorphism of $(X,\mathcal{B},\mu)$ and let $\lambda\in J^e(T)$. If $\lambda\circ\lambda$ is ergodic, then $\lambda\otimes_X\lambda^*$ is ergodic.*

Proof. Given a real function $f\in L^2(X,\mu)$ put $f\otimes f(x_1,x_2,x_3)=f(x_1)f(x_3)$. We have $f\otimes f\in L^1(X_1\times X_2\times X_3,\lambda_1\otimes_{X_2}\lambda_2^*)$, where $\lambda_1=\lambda_2=\lambda$. If $\mathcal{I}$ denotes the σ-algebra of $T\times T\times T$-invariant sets in the relative product, then our ergodicity assumption on $\lambda\circ\lambda$ and (2) give rise to

$$E(f\otimes f|\mathcal{I})=\int f\otimes f\,d\lambda\otimes_{X_2}\lambda^*. \tag{3}$$

Let $(Y, \mathcal{C}, \eta)$ denote the relative Kronecker factor of $T \times T$ on $(X_1 \times X_2, \lambda_1)$ over X_2 (see (A)). Then:

$$\begin{tikzcd} & (X \times X, \lambda) \arrow[dl] \arrow[dd] \\ Y \arrow[dr] & \\ & X \end{tikzcd}$$

Fix a real function $f \in L^2(X, \mu)$. We will show that

$$E(f(x_1)|Y) = E(f(x_1)|X_2). \tag{4}$$

Let $g = E(f(x_1)|Y) - E(f(x_1)|X_2)$. By Lemma 3 it is enough to prove that $E(g \otimes g|\mathcal{I}) = 0$ with respect to $\lambda_1 \otimes_{X_2} \lambda_2^*$. We have

$$\begin{aligned} &E(g \otimes g|\mathcal{I}) \\ &= E((E(f(x_1)|Y_1) - E(f(x_1)|X_2)) \otimes (E(f(x_3)|Y_2) - E(f(x_3)|X_2))|\mathcal{I}) \\ &= E(E(f(x_1)|Y_1) \otimes E(f(x_3)|Y_2)|\mathcal{I}) - E(E(f(x_1)|Y_1) \cdot E(f(x_3)|X_2)|\mathcal{I}) \\ &\quad - E(E(f(x_1)|X_2) \cdot E(f(x_3)|Y_2)|\mathcal{I}) + E(E(f(x_1)|X_2) \cdot E(f(x_3)|X_2)|\mathcal{I}) \\ &= E(E(f(x_1)|Y_1) \otimes E(f(x_3)|Y_2)|\mathcal{I}) - \int_{Y_1} E(f(x_1)|Y_1) \cdot E(f(x_3)|X_2)\, d\eta_1 \\ &\quad - \int_{Y_2} E(f(x_1)|X_2) \cdot E(f(x_3)|Y_2)\, d\eta_2 + \int_{X_2} E(f(x_1)|X_2) \cdot E(f(x_3)|X_2)\, d\mu_2 \end{aligned}$$

by Lemma 2(i) and the fact that X_2 is a factor of Y and Y is ergodic. By taking in the latter three summands the conditional expectation with respect to X_2, we obtain

$$\begin{aligned} &E(g \otimes g|\mathcal{I}) \\ &= E(E(f(x_1)|Y_1) \otimes E(f(x_3)|Y_2)|\mathcal{I}) - \int_{X_2} E(f(x_1)|X_2) \cdot E(f(x_3)|X_2)\, d\mu_2. \end{aligned}$$

Using consecutively Lemma 1 and (3), together with Lemma 2(ii), and finally the definition of the relative product, we obtain that

$$\begin{aligned} &E(E(f(x_1)|Y_1) \otimes E(f(x_3)|Y_2)|\mathcal{I}) \\ &= E(E(f \otimes f|Y_1 \times_{X_2} Y_2)|\mathcal{I}) = \int_{X \times X \times X} f(x_1) f(x_3)\, d\lambda \otimes_X \lambda^*(x_1, x_2, x_3) \\ &= \int_{X_2} E(f(x_1)|X_2) \cdot E(f(x_3)|X_2)\, d\mu_2. \end{aligned}$$

We hence have proved $E(g \otimes g|\mathcal{I}) = 0$ and (4) directly follows.

If $h = h(x_2)$ is in $L^2(X_2)$, then h is Y-measurable and by (4) we have

$$\begin{aligned} &E(f(x_1) \cdot h(x_2)|Y) = h(x_2) \cdot E(f(x_1)|Y) \\ &= h(x_2) \cdot E(f(x_1)|X_2) = E(f(x_1) \cdot h(x_2)|X_2). \end{aligned}$$

Since the family of the function of the form $f \otimes h$ as above forms a linearly dense subset in $L^2(X \times X, \lambda)$, $E(F|Y) = E(F|X_2)$ for all $F \in L^2(X \times X, \lambda)$. Hence $Y = X_2$ and the relative Kronecker factor of $(X_1 \times X_2, \lambda_1)$ over X_2 is trivial. In view of (B), it follows that $\lambda \otimes_{X_2} \lambda^*$ is ergodic. $\square$

Proof of Theorem 1. At first, assume that T is semisimple. Consider $\lambda_1, \lambda_2 \in J^e(T)$. Then, by using Proposition 6.3 from [2], $\lambda_1 \otimes_{X_2} \lambda_2^*$ is ergodic by semisimplicity of T. Therefore $\lambda_2 \circ \lambda_1$ remains ergodic.

If $J_2^e(T)$ is a subsemigroup, then directly from Proposition 1 it follows that T is semisimple.

Remark 2. The proof of Proposition 1 gives a slightly more general result: Assume that λ is an ergodic joining of S (acting on $(Y, \mathcal{C}, \eta)$) and T (acting on $(X, \mathcal{B}, \mu)$). Then the relative product $\lambda \otimes_X \lambda$ is ergodic if and only if the measure $\lambda^* \circ \lambda$ on $Y \times Y$ (given by the Markov operator $\Phi_\lambda^* \circ \Phi_\lambda$ on $L^2(Y, \eta)$) is ergodic. Therefore we obtain an answer to the question by Ryzhikov from [8], p. 95.

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