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AUTOMORPHISMS OF THE ENDOMORPHISM SEMIGROUP OF A FREE MONOID OR A FREE SEMIGROUP

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ABSTRACT. We determine all isomorphisms between the endomorphism semigroups of free monoids or free semigroups and prove that automorphisms of the endomorphism semigroup of a free monoid or a free semigroup are inner or "mirror inner". In particular, we answer a question of B. I. Plotkin.

INTRODUCTION

One of the venerable algebraic problems, the first instance of which was considered by É. Galois, is (using the words of S. Ułam [7]) "determination of a mathematical structure from a given set of endomorphisms". Let $\mathbf{End}(A)$ and $\mathbf{Aut}(A)$ denote the endomorphism monoid and the automorphism group of an algebraic system A, respectively. What can be said about systems A and B if $\mathbf{End}(A)$ is isomorphic to $\mathbf{End}(B)$? This problem has been considered by numerous authors.

We consider this problem for $\mathbf{End}(X^*)$ and $\mathbf{End}(X^+)$, where X^* and X^+ denote, respectively, the free monoid and the free semigroup generated by a set X. This particular problem about $\mathbf{End}(A)$, for A a free algebra in a certain variety, was raised by B. I. Plotkin [2] in his lectures on universal algebraic geometry. An analogous problem for $\mathbf{End}(F)$ with F a free group was solved by Formanek [1]. Other examples are given in a remark at the end of this paper.

Recall that the elements of X^* are words over X, including the empty word 1. The elements of X^+ are nonempty words. If $u = x_{i_1}...x_{i_k} \in X^*$, then $\overline{u}$ denotes the "opposite" word $x_{i_k}...x_{i_1}$. In particular, $\overline{1} = 1$. Every bijection $f : X \to Y$ induces an isomorphism $\iota_f : X^* \to Y^*$ and an anti-isomorphism $\overline{\iota}_f : X^* \to Y^*$ defined as follows: $\iota_f(u) = f(x_{i_1})...f(x_{i_k})$ and $\overline{\iota}_f(u) = \overline{\iota_f(u)} = f(x_{i_k})...f(x_{i_1})$. Analogous facts are true for X^+ and Y^+.

Let $\iota : S \to T$ be an isomorphism or an anti-isomorphism of a semigroup S onto a semigroup T. Define the mapping $\iota\Box\iota : \mathbf{End}(S) \to \mathbf{End}(T)$ by $\iota\Box\iota(\varphi) = \iota \circ \varphi \circ \iota^{-1}$ for all $\varphi \in \mathbf{End}(S)$. Thus, if $\iota(s_1) = t_1$, $\iota(s_2) = t_2$, and $\varphi(s_1) = s_2$ for some $s_1, s_1 \in S$ and $t_1, t_2 \in T$, then $\iota\Box\iota(\varphi)(t_1) = t_2$. It is easy to see that $\iota\Box\iota$ is an isomorphism of $\mathbf{End}(S)$ onto $\mathbf{End}(T)$. We call it the isomorphism *induced* by ι.

Let $|X|$ denote the cardinality of X and $\mathbb{P}$ the set of prime numbers. A *permutation* of a finite or an infinite set is a bijection of that set onto itself.

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Theorem 1. *Let* $\mathbf{End}(X^*)$ *and* $\mathbf{End}(Y^*)$ *be isomorphic.*

If $|X| = 1$*, then* $|Y| = 1$ *and the isomorphisms of* $\mathbf{End}(X^*)$ *onto* $\mathbf{End}(Y^*)$ *are in a natural one-to-one correspondence with permutations of* $\mathbb{P}$ *(explained in the proof).*

If $|X| \geq 1$*, then every isomorphism* $\alpha : \mathbf{End}(X^*) \to \mathbf{End}(Y^*)$ *is induced either by the isomorphism* ι_f *or by the anti-isomorphism* $\bar{\iota}_f$ *of* X^* *onto* Y^* *for a uniquely determined bijection* $f : X \to Y$*. In other words, either* $\alpha = \iota_f \Box \iota_f$ *or* $\alpha = \bar{\iota}_f \Box \bar{\iota}_f$*.*

The same results hold for every isomorphism $\alpha : \mathbf{End}(X^+) \to \mathbf{End}(Y^+)$*.*

Let $\mathbf{Aut}(X)$ and C_2 denote the symmetric group on X and a 2-element group, respectively. Also, μ is the so-called mirror automorphism (see Definition 1.4(iii)).

Theorem 2. *The groups* $\mathbf{Aut}(\mathbf{End}(X^+))$ *and* $\mathbf{Aut}(\mathbf{End}(X^*))$ *are isomorphic.*

If $|X| > 1$*, every automorphism of* $\mathbf{End}(X^*)$ *and of* $\mathbf{End}(X^+)$ *is either inner or a product of an inner automorphism and the mirror automorphism* μ*. In this case* $\mathbf{Aut}(\mathbf{End}(X^*))$ *is isomorphic to the direct product* $\mathbf{Aut}(X) \times C_2$*.*

If $|X| = 1$*,* $\mathbf{Aut}(\mathbf{End}(X^*))$ *is isomorphic to the symmetric group on a countably infinite set.*

In Theorem 2 an automorphism is *inner* if it is of the form ι_f, where f is a permutation of X.

1. Notations and preliminaries

We give the proofs in the case of the free monoid X^*. The proofs in the case of the free semigroup X^+ are almost *verbatim* the same, so we give only a few remarks in the case of free semigroups. Each endomorphism φ of X^* and of X^+ is uniquely determined by a mapping $X \to X^*$ or, respectively, $X \to X^+$. To define φ, it suffices to define $\varphi(x)$ for all $x \in X$. The mapping $\varphi \mapsto \varphi^*$ such that $\varphi^*(x) = \varphi(x)$ and $\varphi^*(1) = 1$ defines an injective homomorphism of $\mathbf{End}(X^+)$ into $\mathbf{End}(X^*)$. For simplicity we identify $\mathbf{End}(X^+)$ with a subsemigroup of $\mathbf{End}(X^*)$.

Clearly, φ is an automorphism precisely when its restriction to X is a permutation of X. Thus the automorphism groups $\mathbf{Aut}(X^*)$ and $\mathbf{Aut}(X^+)$ of X^* and X^+ are isomorphic to the symmetric group $\mathbf{Aut}(X)$ of all permutations of X.

Definition 1.1. (i) Let $u = x_{i_1}...x_{i_k} \in X^*$. Denote the length k of u by $|u|$. The empty word 1 has length 0.

(ii) Let $\mathbf{c}(u)$ be the set of all letters of u.

(iii) An endomorphism $\varphi \in \mathbf{End}(X^*)$ is *linear* if $\varphi(x) \in X \cup \{1\}$ for every $x \in X$. In the case of $\mathbf{End}(X^+)$, φ is linear when $\varphi(x) \in X$ for all $x \in X$.

(iv) If $u \in X^*$ is a fixed word, let c_u be the endomorphism of X^+ such that $c_u(x) = u$ for all $x \in X$. We call c_u a *constant* endomorphism. Observe that the range of c_u does not consist of a single word, unless $u = 1$. Clearly, c_1 is the zero element of $\mathbf{End}(X^*)$. We denote it by 0, that is, $c_1 = 0$. If $v \in X^*$, then $c_u(v) = u^{|v|}$.

(v) An endomorphism $\varphi \in \mathbf{End}(X^*)$ is called *full* if $\varphi\psi = 0 \Rightarrow \psi = 0$ for all $\psi \in \mathbf{End}(X^*)$.

Lemma 1.2. (i) φ *is a constant endomorphism of* $\mathbf{End}(X^*)$ *if and only if* $\varphi\alpha = \varphi$ *for all* $\alpha \in \mathbf{Aut}(X^*)$*. The same holds for* $\mathbf{End}(X^+)$*;*

(ii) $\varphi c_u = c_{\varphi(u)}$ *for all* $\varphi \in \mathbf{End}(X^*)$*;*

(iii) $\varphi \in \mathbf{End}(X^*)$ *is a constant idempotent if and only if either* $\varphi = c_x$ *for some* $x \in X$ *or* $\varphi = 0$*;*

(iv) $\varphi \in \mathbf{End}(X^*)$ *is full if and only if* $\varphi(x) \neq 1$ *for all* $x \in X$.

Proof. (i) If $x \in X$, then $\alpha(x) = y \in X$. Varying α, we obtain $c_u\alpha(x) = c_u(y) = u$ for all $y \in X$. Thus $c_u\alpha = c_u$. Conversely, let $\varphi\alpha = \varphi$ for all automorphisms α. If $\alpha(x) = y$ for some $x, y \in X$, then $\varphi(x) = \varphi\alpha(x) = \varphi(y)$. Varying α, we obtain $\varphi(x) = \varphi(y)$ for all $x, y \in X$. It follows that $\varphi = c_u$, where $u = \varphi(x)$ for any $x \in X$.

(ii) $\varphi c_u(x) = \varphi(u)$ for every $x \in X$. Thus $\varphi c_u = c_{\varphi(u)}$.

(iii) If φ is a constant idempotent, that is, $\varphi = c_u$ for some $u \in X^*$ and $c_u c_u = c_u$, then, by (ii), $c_u = c_u c_u = c_{c_u(u)} = c_{u^{|u|}}$. Thus $u^{|u|} = u$, that is, $|u| \leq 1$. The converse is obvious.

(iv) If $\varphi(x) = 1$ for some $x \in X$, then $c_x \neq 0$ but $\varphi c_x = 0$. Thus φ is not full.

Conversely, let $\varphi(x) \neq 1$ for all $x \in X$. Thus $\varphi(u) \neq 1$ for all $u \in X^+$. If $\psi \neq 0$, then $\psi(x) = u \neq 1$ for some $x \in X$, and hence $\psi c_x = c_{\psi(x)} = c_u \neq 0$. Thus $\varphi\psi c_x = \varphi c_u = c_{\varphi(u)} \neq 0$, so that $\varphi\psi \neq 0$. It follows that φ is full. □

Lemma 1.3. *The mapping* $\mu(\varphi) = \overline{\varphi}$ *is an automorphism of* $\mathbf{End}(X^*)$.

Proof. Obviously, $f : u \to \overline{u}$ is an antiautomorphism of X^* and $f^{-1} = f$. Thus $f^{-1}\varphi f = \overline{\varphi}$, and Lemma 1.3 follows from this equality. □

Definition 1.4. (i) Let $\alpha : \mathbf{End}(X^*) \to \mathbf{End}(Y^*)$ be an isomorphism. By parts (i) and (iii) of Lemma 1.2, for every $x \in X$ there exists $y \in Y$ such that $\alpha(c_x) = c_y$. Define a bijection $f : X \to Y$ by $f(x) = y$. We say that f is *induced* by α. We make an analogous definition for $\mathbf{End}(X^+) \to \mathbf{End}(Y^+)$.

(ii) An automorphism α of $\mathbf{End}(X^*)$ or of $\mathbf{End}(X^+)$ is *stable* if it induces the identity permutation of X, that is, $\alpha(c_x) = c_x$ for all $x \in X$.

(iii) The mapping μ of Lemma 1.3 is the *mirror automorphism* of $\mathbf{End}(X^*)$.

2. The case $|X| = 1$

If $|X| > 1$, let x and y be distinct elements of X. Then $c_{xy}(c_x(x)) = c_{xy}(x) = xy \neq xx = c_x(xy) = c_x c_{xy}(x)$, and hence $c_{xy}c_x \neq c_x c_{xy}$. Thus $\mathbf{End}(X^*)$ and $\mathbf{End}(X^+)$ are not commutative.

If $X = \{x\}$, a singleton, then $X^+ = \{x, x^2, x^3, ...\}$ consists of all powers of x and is isomorphic to the additive semigroup $(\mathbb{N}, +)$ of positive integers. Every element of $\mathbf{End}(X^+)$ corresponds to $\varphi_k = \left(\begin{smallmatrix} x \\ x^k \end{smallmatrix}\right)$ for some $k \in \mathbb{N}$, and $\mathbf{End}(X^+)$ is isomorphic to the multiplicative semigroup $(\mathbb{N}, \cdot)$ of positive integers. $(\mathbb{N}, \cdot)$ is a free commutative monoid with the countably infinite set $\mathbb{P}$ of free generators that are prime numbers. Therefore, $\mathbf{End}(X^+)$ is a free commutative semigroup and $\{\varphi_k\}_{k \in \mathbb{P}}$ is its set of free generators. If $\alpha : \mathbf{End}(X^+) \to \mathbf{End}(Y^+)$ is an isomorphism, then $\mathbf{End}(Y^+)$ is commutative, and hence $|Y| = 1$. Thus α is uniquely determined by a bijection between the free generators of $\mathbf{End}(X^+)$ and $\mathbf{End}(Y^+)$. These bijections (and hence the isomorphisms) are in a one-to-one correspondence with permutations of $\mathbb{P}$. The elements of $\mathbf{Aut}(\mathbf{End}(X^+))$ correspond to permutations of generators, and thus $\mathbf{Aut}(\mathbf{End}(X^+))$ is an infinite group isomorphic to the symmetric group $\mathbf{Aut}(\mathbb{P})$ of all permutations of $\mathbb{P}$, and also isomorphic to $\mathbf{Aut}(\mathbb{P}^+)$ because $\mathbb{P}$ is countably infinite.

Similarly, X^* is isomorphic to the additive semigroup $(\mathbb{N}_0, +)$ of nonnegative integers, $\mathbf{End}(X^*)$ is isomorphic to the multiplicative semigroup $(\mathbb{N}_0, \cdot)$ of nonnegative integers, and hence $\mathbf{Aut}(\mathbf{End}(X^*))$ is isomorphic to $\mathbf{Aut}(\mathbf{End}(X^+))$.

3. Stable automorphisms of $\mathbf{End}(X^*)$ and $\mathbf{End}(X^+)$

Lemma 3.1. (i) *If α is a stable automorphism of $\mathbf{End}(X^*)$, $\varphi \in \mathbf{End}(X^*)$, and $x \in X$, then $\alpha(c_{\varphi(x)}) = c_{\alpha(\varphi)(x)}$.*

(ii) *If φ is linear, then $\alpha(\varphi) = \varphi$.*

Proof. (i) By Lemma 1.2(ii), $\alpha(c_{\varphi(x)}) = \alpha(\varphi c_x) = \alpha(\varphi)\alpha(c_x) = \alpha(\varphi)c_x = c_{\alpha(\varphi)(x)}$.

(ii) If φ is linear, then $\varphi(x) \in X \cup \{1\}$, and hence $c_{\alpha(\varphi)(x)} = \alpha(c_{\varphi(x)}) = c_{\varphi(x)}$. Therefore, $\alpha(\varphi)(x) = \varphi(x)$ for every $x \in X$, and so $\alpha(\varphi) = \varphi$. □

Lemma 3.2. *If $\alpha(c_u) = c_v$, where α is a stable automorphism of $\mathbf{End}(X^*)$ and $u, v \in X^*$, then $\mathbf{c}(u) = \mathbf{c}(v)$.*

Proof. If $z \in \mathbf{c}(u)\backslash\mathbf{c}(v)$, choose $x \in X$, $\varphi \in \mathbf{End}(X^*)$, and $g \in \mathbf{End}(X^+)$ such that $x \neq z$, $\varphi(x) = u$, $g(z) = x$, and $g(y) = y$ for all $y \neq z$, $y \in X$. Then g is linear, $g(v) = v$, $\alpha(g) = g$ by Lemma 3.1(ii), and $\alpha(c_u) = c_v = c_{g(v)} = gc_v = \alpha(g)\alpha(c_u) = \alpha(gc_u) = \alpha(c_{g(u)})$. By injectivity of α, $c_u = c_{g(u)}$, so that $u = g(u)$, which is not true. Thus $\mathbf{c}(u)\backslash\mathbf{c}(v) = \varnothing$. Similarly, $\mathbf{c}(v)\backslash\mathbf{c}(u) = \varnothing$. Therefore, $\mathbf{c}(u) = \mathbf{c}(v)$. □

Lemma 3.3. *If α is a stable automorphism of $\mathbf{End}(X^*)$, then $|\varphi(x)| = |\alpha(\varphi)(x)|$ for all $\varphi \in \mathbf{End}(X^*)$ and $x \in X$.*

Proof. Suppose that $|\varphi_1(x)| = |\varphi_2(x)| = m$, $|\alpha(\varphi_1)(x)| = k$, and $|\alpha(\varphi_2)(x)| = l$. Then $c_x\varphi_1 c_x = c_x\varphi_2 c_x = c_{x^m}$. Also, $\alpha(c_x) = c_x$. Therefore, $\alpha(c_{x^m}) = \alpha(c_x\varphi_1 c_x) = c_x\alpha(\varphi_1)c_x = c_{x^k}$ and $\alpha(c_{x^m}) = \alpha(c_x\varphi_2 c_x) = c_x\alpha(\varphi_2)c_x = c_{x^l}$. Thus $k = l$.

If Y is a finite n-element subset of X and m a nonnegative integer, define $End_Y^m(x) = \{\varphi \in \mathbf{End}(X^*) \mid |\varphi(x)| = m \text{ and } \mathbf{c}(\varphi(x)) = Y\}$. By the previous paragraph and Lemma 3.2, for every m there exists k such that $\alpha(End_Y^m(x)) \subseteq End_Y^k(x)$. There are n^k words of length k and n^m words of length m over Y. Since α is injective, then $k \geq m$. Since α is surjective, $k = m$. Thus $\alpha(End_Y^m(x)) = End_Y^m(x)$, and hence $|\varphi(x)| = |\alpha(\varphi)(x)|$ for all $\varphi \in \mathbf{End}(X^*)$ and $x \in X$. □

Corollary 3.4. *Choose two distinct elements $x_1, x_2 \in X$. If α is a stable automorphism of $\mathbf{End}(X^*)$, then $\alpha(c_{x_1x_2})$ is either $c_{x_1x_2}$ or $c_{x_2x_1}$.*

Proof. By Lemma 1.2(i), $\alpha(c_{x_1x_2}) = c_u$ for some $u \in X^*$. By Lemma 3.2, $\mathbf{c}(u) = \{x_1, x_2\}$. By Lemma 3.3, $|u| = 2$. Thus u is either x_1x_2 or x_2x_1. □

Lemma 3.5. *If α is a stable automorphism of $\mathbf{End}(X^*)$ and $\alpha(c_{x_1x_2}) = c_{x_1x_2}$ for some $x_1, x_2 \in X$, $x_1 \neq x_2$, then α is the identity automorphism of $\mathbf{End}(X^*)$.*

Proof. First we prove that $\alpha(c_u) = c_u$ for all $u \in X^*$ by induction on $|u|$. The base of induction is obvious for $|u| = 0$ and follows from the stability of α for $|u| = 1$. Let the claim hold for $|u| < k$, and let $|u| = k$, where $u = u_1y$. If $\varphi(x_1) = u_1, \varphi(x_2) = y$, and $\varphi(x) = x$ for other $x \in X$, then $\varphi c_{x_1x_2} = c_u$. By the induction hypothesis, $\alpha(c_{u_1}) = c_{u_1}$. Also, $\alpha(c_x) = c_x$ for every $x \in X$. By Lemma 3.1(i), $\alpha(\varphi)(x) = \varphi(x)$ for all $x \in X$. Thus $\alpha(\varphi) = \varphi$, and hence $\alpha(c_u) = \alpha(\varphi c_{x_1x_2}) = \varphi c_{x_1x_2} = c_u$.

By Lemma 3.1(i), $c_{\varphi(x)} = \alpha(c_{\varphi(x)}) = c_{\alpha(\varphi)(x)}$, and so $\varphi(x) = \alpha(\varphi(x))$ for all $x \in X$. Thus $\varphi = \alpha(\varphi)$ for all $\varphi \in \mathbf{End}(X^*)$. □

Lemma 3.6. *If α is a stable automorphism of* $\mathbf{End}(X^*)$ *and* $\alpha(c_{x_1x_2}) = c_{x_2x_1}$ *for some* $x_1, x_2 \in X$, $x_1 \neq x_2$, *then* $\alpha = \mu$.

Proof. First we prove that $\alpha(c_u) = c_{\bar{u}}$ for all $u \in X^*$ by induction on $|u|$. The base of induction is obvious for $|u| = 0$ and follows from the stability of α for $|u| = 1$. Let the claim hold for $|u| < k$, and let $|u| = k$, where $u = u_1 y$. If $\varphi(x_1) = u_1, \varphi(x_2) = y$, and $\varphi(x) = x$ for other $x \in X$, then $\varphi c_{x_1x_2} = c_u$. By the induction hypothesis, $\alpha(c_{u_1}) = c_{\bar{u}_1}$. Also, $\alpha(c_x) = c_x$ for every $x \in X$. By Lemma 3.1(i), $\alpha(\varphi)(x) = \overline{\varphi(x)}$ for all $x \in X$. Thus $\alpha(\varphi) = \overline{\varphi}$, and hence $\alpha(c_u) = \alpha(\varphi c_{x_1x_2}) = \overline{\varphi} c_{x_2x_1} = c_{y\bar{u}_1} = c_{\bar{u}}$.

By Lemma 3.1(i), $\alpha(\varphi) = \overline{\varphi}$ for all $\varphi \in \mathbf{End}(X^*)$. Thus $\alpha = \mu$. □

We skip the proof of the following proposition (it is obtained as an obvious simplification of our proof of Theorem 2).

Proposition 3.7. *Let $F(X)$ be the free commutative monoid with a set X, $|X| > 1$, of free generators. Every automorphism of* $\mathbf{End}(F(X))$ *is inner.*

Remark 3.8. Similarly, automorphisms of a free commutative semigroup are inner.

4. Group of automorphisms of $\mathbf{End}(X^*)$

Proof of Theorem 1. For $|X| = 1$ Theorem 1 was proved in Section 2. Suppose that $|X| > 1$, $\alpha : \mathbf{End}(X^*) \to \mathbf{End}(Y^*)$ is an isomorphism, and $f : X \to Y$ is a bijection induced by α, that is, $\alpha(c_x) = c_{f(x)}$ for every $x \in X$. Then f extends to an isomorphism $\iota_f : X^* \to Y^*$ and ι_f induces an isomorphism $\alpha_f = \iota_f \square \iota_f$: $\mathbf{End}(X^*) \to \mathbf{End}(Y^*)$. Consider the automorphism $\beta = \alpha_f^{-1}\alpha$ of $\mathbf{End}(X^*)$. Then $\beta(c_x) = \alpha_f^{-1}(\alpha(c_x)) = \alpha_f^{-1}(c_{f(x)}) = c_{f^{-1}(f(x))} = c_x$ for every $x \in X$ and β is stable. By Corollary 3.4 and Lemmas 3.5 and 3.6, β is an identity automorphism Δ or a mirror automorphism μ. In the former case, $\alpha_f^{-1}\alpha = \Delta$, and hence $\alpha = \alpha_f$, the automorphism induced by ι_f. In the latter case, $\alpha_f^{-1}\alpha = \mu$, whence $\alpha = \alpha_f\mu$, the automorphism induced by $\bar{\iota}_f$. □

Proof of Theorem 2. If $X = Y$, Theorem 1 becomes the first part of Theorem 2. Thus we prove here the second part of Theorem 2 for $|X| > 1$.

Obviously, $\{\Delta, \mu\}$ is a two-element group of automorphisms of $\mathbf{End}(X^*)$. It is isomorphic to C_2. Define a mapping $i : \mathbf{Aut}(\mathbf{End}(X^*)) \to \mathbf{Aut}(X) \times C_2$ as follows: if $\alpha \in \mathbf{Aut}(\mathbf{End}(X^*))$, then $i(\alpha) = (f, \Delta)$ for $\alpha = \alpha_f$, and $i(\alpha) = (f, \mu)$ for $\alpha = \alpha_f\mu$. Clearly, i is a bijection. Also, i is a homomorphism because $\alpha_g \circ \alpha_f = \alpha_{g\circ f}$ and $\mu \circ \alpha = \alpha \circ \mu$ for all $f, g \in \mathbf{Aut}(X)$ and $\alpha \in \mathbf{Aut}(\mathbf{End}(X^*))$. Thus i is an isomorphism of $\mathbf{Aut}(\mathbf{End}(X^*))$ onto $\mathbf{Aut}(X) \times C_2$. □

Remark 4.1. Let $\mathbf{L}$ be the variety of all left zero semigroups (that is, the semigroups satisfying the identity $x_1x_2 = x_1$). It is easily seen that every semigroup in this variety is free. If X is a left zero semigroup, then every transformation $X \to X$ is an endomorphism of X, and hence $\mathbf{End}(X)$ is merely the full transformation semigroup $\mathcal{T}_X$ of the set X. Then $\mathbf{Aut}(\mathbf{End}(X)) = \mathbf{Aut}(\mathcal{T}_X)$. Schreier [5] proved that all automorphisms of $\mathcal{T}_X$ are inner and $\mathbf{Aut}(\mathcal{T}_X)$ is isomorphic to $\mathbf{Aut}(X)$, the symmetric group on X.

Let $\mathbf{Z}$ be the variety of all zero semigroups (that is, the semigroups satisfying the identity $x_1x_2 = x_3x_4$). They are semigroups $X^0 = X \cup \{0\}$, $0 \notin X$, with zero 0 such that $xy = 0$ for all $x, y \in X^0$. Every semigroup in this variety is free. Every partial transformation f of X corresponds bijectively to an endomorphism $\bar{f}$ of X^0:

for every $s \in X^0$, $\bar{f}(s) = f(s)$ if $f(s)$ is defined, and $\bar{f}(s) = 0$ otherwise. Clearly, $\mathfrak{F}_X$, the semigroup of all partial transformations of X, is naturally isomorphic to $\mathbf{End}(X^0)$ under the correspondence $f \mapsto \bar{f}$, and hence $\mathbf{Aut}(\mathbf{End}(X))$ is isomorphic to $\mathbf{Aut}(\mathfrak{F}_X)$. As proved by Shutov [6], all automorphisms of $\mathfrak{F}_X$ are inner and $\mathbf{Aut}(\mathfrak{F}_X)$ is isomorphic to $\mathbf{Aut}(X)$, the symmetric group on X.

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