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MULTI-DIMENSIONAL VERSIONS OF A THEOREM OF FINE AND WILF AND A FORMULA OF SYLVESTER

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ABSTRACT. Let $\vec{v_0},...,\vec{v_k}$ be vectors in $\mathbf{Z}^k$ which generate $\mathbf{Z}^k$. We show that a body $V \subset \mathbf{Z}^k$ with the vectors $\vec{v_0},...,\vec{v_k}$ as edge vectors is an almost minimal set with the property that every function $f : V \to \mathbf{R}$ with periods $\vec{v_0},...,\vec{v_k}$ is constant. For $k = 1$ the result reduces to the theorem of Fine and Wilf, which is a refinement of the famous Periodicity Lemma.

Suppose $\vec{0}$ is not a non-trivial linear combination of $\vec{v_0},...,\vec{v_k}$ with non-negative coefficients. Then we describe the sector such that every interior integer point of the sector is a linear combination of $\vec{v_0},...,\vec{v_k}$ over $\mathbf{Z}_{\geq 0}$, but infinitely many points on each of its hyperfaces are not. For $k = 1$ the result reduces to a formula of Sylvester corresponding to Frobenius' Coin-changing Problem in the case of coins of two denominations.

1. Introduction

Let p and q be positive integers. Put $d = \gcd(p, q)$. Let $f : I \to \mathbb{R}$ be a function defined on a block of integers $I = \{1, ..., m\}$ such that f is periodic modulo p and modulo q. The theorem of Fine and Wilf says that if $m \geq p + q - d$, then f is periodic modulo d, but if $m < p + q - d$, then f need not be periodic modulo d. The former statement is often called the Periodicity Lemma. The theorem of Fine and Wilf has been extended by Castelli, Mignosi and Restivo [2] to functions $f : I \to \mathbb{R}$ having three periods and by Justin [7] to functions having any number of periods. See also Tijdeman and Zamboni [16]. We shall generalize the Fine and Wilf theorem to functions $f : V \to \mathbb{R}$ with $V \subset \mathbb{Z}^k$ having $k+1$ period vectors $\vec{v_0}, ..., \vec{v_k} \in \mathbb{Z}^k$, in the sense that $f(\vec{v}) = f(\vec{w})$ if $\vec{v}, \vec{w} \in V$ satisfy $\vec{v} - \vec{w} \in \{\vec{v_0}, ..., \vec{v_k}\}$. Suppose that $\vec{v_1}, ..., \vec{v_k} \in \mathbb{Z}^k$ are linearly independent over $\mathbb{Z}$. Let $\vec{v_0} \in \mathbb{Z}^k$ be a vector which can be written as $\mu_1\vec{v_1} + ... + \mu_k\vec{v_k}$ with $\mu_i > 0, \mu_i \in \mathbb{R}$ for $i = 0, ..., k$. Define

$$W = \{\lambda_0\vec{v_0} + \lambda_1\vec{v_1} + ... + \lambda_k\vec{v_k} : 0 \leq \lambda_i \leq 1, \lambda_i \in \mathbb{R} \text{ for } i = 0, ..., k\}.$$

In Theorem 1 we present a set $V \subset W \cap \mathbb{Z}^k$ with cardinality equal to the Lebesgue measure of W such that if $f : V \to \mathbb{R}$ has period vectors $\vec{v_0}, ..., \vec{v_k}$, then f is constant on the cosets of the lattice L generated by $\vec{v_0}, ..., \vec{v_k}$. The assertion remains true if one point of V is removed, but it is no longer true if two points of V are removed

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which are in the same coset of L and do not differ by $\pm\vec{v_i}$ for some i. In the present paper we refer to points of $\mathbb{Z}^k$ as integer points and to points of L as lattice points.

Results comparable with Theorem 1 are in the literature. Giancarlo and Mignosi [4] have given a multi-dimensional generalization of the Fine and Wilf theorem for connected subsets of Cayley graphs. Papers by Amir and Benson [1], Galil and Park [5] and Mignosi, Restivo and Silva [8] provide periodicity lemmas for parallelograms and similar domains in $\mathbb{R}^2$. (A periodicity lemma is a statement that a function f defined on the integer points in some region and having prescribed period vectors has to be constant on the cosets of the lattice generated by these vectors, without indicating how far the region can be reduced without affecting the conclusion.) Regnier and Rostami [10] have provided a framework for the study of periodicity lemmas in case of multi-dimensional patterns. In the Corollary to Theorem 1 we present a periodicity lemma for parallelotopes in any dimension.

In Frobenius' classical Coin-changing Problem, also known as the Postage Stamp Problem and as the Linear Diophantine Problem of Frobenius, we are given positive integers $a_0, ..., a_k$ with greatest common divisor 1, and asked to find the least integer n such that every integer greater than n can be written as a sum of non-negative multiples of $a_0, ..., a_k$. In the case $k = 1$ the answer $n = n_0 := a_0a_1 - a_0 - a_1$ is due to Sylvester [15]. Moreover, Sylvester proved that for $0 \le m \le n_0$ exactly one of the two integers m and $n_0 - m$ is a sum of non-negative multiples of a_0 and a_1. The case $k = 2$ has been settled by Selmer and Beyer [13]; see also Rödseth [11]. For $k > 2$ the answer is only known in special cases and various estimates exist for the general case (cf. [12, 14]).

Suppose $\vec{v_0}, ..., \vec{v_k}$ defined as above generate $\mathbb{Z}^k$. Then they have the property that $\vec{0}$ cannot be written as a non-trivial non-negative linear combination of $\vec{v_0}, ..., \vec{v_k}$. In other words, the period vectors $\vec{v_0}, ..., \vec{v_k}$ are on the same side of some hyperplane. Let d_0 be the smallest positive integer for which positive integers $d_1, ..., d_k$ exist with

$$d_0\vec{v_0} = d_1\vec{v_1} + ... + d_k\vec{v_k}.$$

By Cramer's rule $d_i = c \cdot |\det(\vec{v_0}, \vec{v_1}, ..., \vec{v_{i-1}}, \vec{v_{i+1}}, ..., \vec{v_k})|$ for $i = 0, ..., k$ where c is some constant. Define $\vec{w} = d_0\vec{v_0} - (\vec{v_0} + ... + \vec{v_k})$ and

$$X = \{s_1\vec{v_1} + ... + s_k\vec{v_k} + \vec{w} : s_1 > 0, ..., s_k > 0\} \cap \mathbb{Z}^k.$$

Theorem 2 implies that every integer point in X can be written as a non-negative linear combination of $\vec{v_0}, ..., \vec{v_k}$, but that for $k > 1$ infinitely many integer points on each hyperface of X cannot be written in that way. We are not aware of a similar result in the literature. For $k = 1$ we obtain the value obtained by Sylvester [15]. The shape of W is essential for the application in Theorem 2.

Both theorems are based on a proposition which shows that $V \cong \mathbb{Z}^k/\Lambda$, where Λ is the lattice generated by $\vec{v_0} + \vec{v_1}, ..., \vec{v_0} + \vec{v_k}$, and that the function ϕ, denoting a translation over $\vec{v_0}$ modulo Λ in V, induces complete cycles of the elements in V belonging to the same coset of the lattice L. In the final section we make some remarks on related complete sets of representatives of $\mathbb{Z}^k/\Lambda$.

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2. Lemmas and a proposition

Let $\vec{v_i} = (v_{i1}, ..., v_{ik})$ $(i = 1, ..., k)$ be vectors in $\mathbb{Z}^k$ which are linearly independent over $\mathbb{Z}$. Let $\vec{v_0} = (v_{01}, ..., v_{0k}) \in \mathbb{Z}^k$ be a vector which can be written as

$$\vec{v_0} = \mu_1 \vec{v_1} + ... + \mu_k \vec{v_k} \text{ with } \mu_i > 0 \text{ for } i = 1, ..., k. \tag{1}$$

(If $\mu_i = 0$, then we have to do with the same situation in a lower dimension; if $\mu_i < 0$, then we may replace $\vec{v_i}$ by $-\vec{v_i}$.) We define the set W by

$$W = \{\lambda_0 \vec{v_0} + ... + \lambda_k \vec{v_k} : 0 \le \lambda_i \le 1, \lambda_i \in \mathbb{R} \text{ for } i = 0, ..., k\}.$$

Furthermore we write W^0 for the interior of W,

$$W_0 := \{\lambda_1 \vec{v_1} + ... + \lambda_k \vec{v_k} : 0 \le \lambda_i < 1 \text{ for } i = 1, ..., k\}$$

and

$$W_j := \{\lambda_0 \vec{v_0} + ... + \lambda_k \vec{v_k} : 0 \le \lambda_i < 1 \text{ for } 0 \le i < j; \lambda_j = 1; 0 < \lambda_i \le 1 \text{ for } j < i \le k\}.$$

By the above choice in each W_i exactly one among two parallel hyperfaces is removed in a suitable way.

Lemma 1. *$W_0, W_1, ..., W_k$ are disjoint and $W^0 \subseteq \bigcup_{j=0}^{k} W_j \subseteq W$.*

Proof. Suppose $\vec{x} \in W^0$. Write

$$\vec{x} = \lambda_0 \vec{v_0} + ... + \lambda_k \vec{v_k} \text{ with } 0 \le \lambda_i \le 1 \text{ for } i = 0, ..., k \tag{2}$$

and λ_0 minimal. Suppose $\vec{x} \notin W_0$. Then there is a smallest h with $\lambda_h = 1$. We claim that $\vec{x} \in W_h$. By the definition of h we have $\lambda_j < 1$ for $0 \le j < h$. Suppose $\lambda_i = 0$ for some $i > h$. Then we see from $\lambda_h = 1, \lambda_i = 0$ that representation (2) is unique. This implies that $\vec{x}$ is a boundary point of W contradicting the hypothesis that x belongs to W^0, the interior of W. Thus $\lambda_i > 0$ for $i > h$ whence $\vec{x} \in W_h$. Obviously $W_i \subset W$ for $i = 0, ..., k$.

It remains to show that $W_0, ..., W_k$ are disjoint. Note that $W_0 \cap W_j = \emptyset$ for $j = 1, ..., k$. Suppose $\vec{x} \in W_h \cap W_j$ for some h, j with $0 < h < j \le k$. Then

$$\vec{x} = \lambda_0 \vec{v_0} + ... + \lambda_k \vec{v_k} = \lambda'_0 \vec{v_0} + ... + \lambda'_k \vec{v_k}$$

with $0 \le \lambda_i < 1$ for $0 \le i < h; \lambda_h = 1; 0 < \lambda_i \le 1$ for $h < i \le k$ and $0 \le \lambda'_i < 1$ for $0 \le i < j; \lambda'_j = 1; 0 < \lambda'_i \le 1$ for $j < i \le k$. By $\lambda_h = 1, \lambda'_h < 1$ we have $\lambda_0 < \lambda'_0$, but by $\lambda_j \le 1, \lambda'_j = 1$ we have $\lambda'_0 \le \lambda_0$. This contradiction completes the proof. $\square$

We set $V_i = W_i \cap \mathbb{Z}^k$ for $i = 0, ..., k$ and $V = \bigcup_{i=0}^{k} V_i$. The case $k = 2$ is illustrated in Figure 1. Generally V is a polytope in k dimensions whose hyperfaces are $(k-1)$-dimensional parallelotopes.

We define a function $\phi : V \to V$ by

$$\phi(\vec{v}) = \begin{cases} \vec{v} + \vec{v_0} \text{ if } \vec{v} \in V_0, \\ \vec{v} - \vec{v_h} \text{ if } \vec{v} \in V_h \text{ for some } h > 0. \end{cases}$$

Note that $\vec{v}$ and $\phi(\vec{v})$ are in the same coset of L.

Lemma 2. *$\phi : V \to V$ is a bijection.*

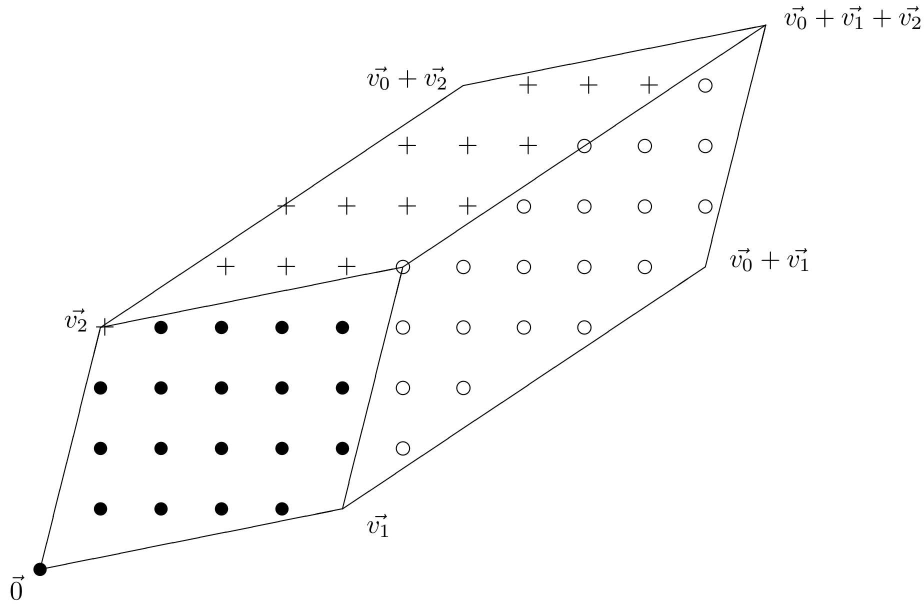

FIGURE 1. The sets V_0 (closed circles), V_1 (open circles) and V_2 (+ signs) formed using $\vec{v_0} = (6, 4)$, $\vec{v_1} = (5, 1)$ and $\vec{v_2} = (1, 4)$. Note that $|V_0| = \left|\begin{smallmatrix} 5 & 1 \\ 1 & 4 \end{smallmatrix}\right| = 19, |V_1| = \left|\begin{smallmatrix} 6 & 1 \\ 4 & 4 \end{smallmatrix}\right| = 20, |V_2| = \left|\begin{smallmatrix} 5 & 6 \\ 1 & 4 \end{smallmatrix}\right| = 14$.

Proof. For $\vec{v} \in V_0$ we have $\phi(\vec{v}) = \vec{v_0} + \lambda_1\vec{v_1} + ... + \lambda_k\vec{v_k}$ with $0 \le \lambda_i < 1$ for $i = 1, ..., k$. For $\vec{v} \in V_h$ with $h > 0$ we have $\phi(\vec{v}) = \lambda_0\vec{v_0} + ... + \lambda_k\vec{v_k}$ with $0 \le \lambda_i < 1$ for $0 \le i < h; \lambda_h = 0; 0 < \lambda_i \le 1$ for $i > h$. In both cases rewrite $\phi(\vec{v})$ as $\lambda'_0\vec{v_0} + ... + \lambda'_k\vec{v_k}$ with $0 \le \lambda'_i \le 1$ for $i = 0, ..., k$ and λ'_0 minimal. Then either there exists a smallest j with $\lambda'_j = 1$ whence $\phi(\vec{v}) \in V_j$, or $\lambda'_0 = 0, \lambda'_i < 1$ for $i = 1, ..., k$, whence $\phi(\vec{v}) \in V_0$. Thus $\phi(V) \subseteq V$.

Next we check that ϕ is injective. It is obvious that $\phi|_{V_i}$ is injective for $i = 0, ..., k$. Suppose $\vec{w} = \phi(\vec{u}) = \phi(\vec{v})$ for some $\vec{u}, \vec{v} \in V, \vec{u} \ne \vec{v}$. If $\vec{u} \in V_0, \vec{v} \in V_j$ for some $j > 0$, then

$$\vec{w} = \lambda_0\vec{v_0} + ... + \lambda_k\vec{v_k} = \lambda'_0\vec{v_0} + ... + \lambda'_k\vec{v_k}$$

with $\lambda_0 = 1; 0 \le \lambda_i < 1$ for $i = 1, ..., k$ and $0 \le \lambda'_i < 1$ for $0 \le i < j; \lambda'_j = 0; 0 < \lambda'_i \le 1$ for $j < i \le k$. Since $\lambda'_j = 0$ we have $\lambda'_0 \ge \lambda_0 = 1$ which yields a contradiction.

If $\vec{u} \in V_h, \vec{v} \in V_j$ for $0 < h < j \le k$, then

$$\vec{w} = \lambda_0\vec{v_0} + ... + \lambda_k\vec{v_k} = \lambda'_0\vec{v_0} + ... + \lambda'_k\vec{v_k}$$

with $0 \le \lambda_i < 1$ for $0 \le i < h; \lambda_h = 0; 0 < \lambda_i \le 1$ for $h < i \le k$ and $0 \le \lambda'_i < 1$ for $0 \le i < j; \lambda'_j = 0; 0 < \lambda'_i \le 1$ for $j < i \le k$. Since $\lambda_h = 0$ we have $\lambda_0 \ge \lambda'_0$. On the other hand, $\lambda_j > 0, \lambda'_j = 0$ imply $\lambda_0 < \lambda'_0$ which also yields a contradiction. Thus ϕ is injective.

Since V is finite and ϕ is injective, ϕ is also surjective. $\square$

We call $\vec{u}$ and $\vec{v}$ ϕ-adjacent if $\vec{u} = \phi(\vec{v})$ or $\vec{v} = \phi(\vec{u})$. The following lemma shows that V is in a sense minimal with respect to ϕ.

Lemma 3. *If $\vec{u}, \vec{v} \in V$ and $\vec{v} = \vec{u} + \vec{v_i}$ for some i with $0 \le i \le k$, then $\vec{u}$ and $\vec{v}$ are ϕ-adjacent.*

Proof. First suppose that $i > 0$. Write $\vec{u}$ as $\lambda_0 \vec{v_0} + ... + \lambda_k \vec{v_k}$ with $0 \le \lambda_i \le 1$ for $i = 0, ..., k$ and λ_i minimal. Since

$$\vec{v} = \lambda_0 \vec{v_0} + ... + \lambda_{i-1} \vec{v_{i-1}} + (\lambda_i + 1)\vec{v_i} + \lambda_{i+1} \vec{v_{i+1}} + ... + \lambda_k \vec{v_k} \in V \subset W,$$

we have $\lambda_i = 0$. It follows from $\vec{u} \in V$ that $\lambda_j < 1$ for $j < i$ and from $\vec{v} \in V$ that $\lambda_j > 0$ for $j > i$. Thus $\vec{v} \in V_i$ and $\vec{u} = \phi(\vec{v})$.

Now suppose that $\vec{v} = \vec{u} + \vec{v_0}$. Suppose, for the sake of contradiction, that $\vec{u} \in V_j$ for some $j > 0$. Then if $\vec{u} = \lambda'_0 \vec{v_0} + ... + \lambda'_k \vec{v_k}$ with λ'_0 minimal, we must have $\lambda'_0 = 0, \lambda'_j = 1$. Then $\vec{v} = \lambda_0 \vec{v_0} + ... + \lambda_k \vec{v_k}$ where $\lambda_0 = 1$ is minimal. This contradicts that according to the definition of V we can always choose $\lambda_0 < 1$. Hence $\vec{u} \notin V_j$, so $\vec{u} \in V_0$ and $\vec{v} = \vec{u} + \vec{v_0} = \phi(\vec{u})$. Again $\vec{u}$ and $\vec{v}$ are ϕ-adjacent. $\square$

Let d_0 be the smallest positive integer for which positive integers $d_1, ..., d_k$ exist with

$$d_0 \vec{v_0} = d_1 \vec{v_1} + ... + d_k \vec{v_k}. \tag{3}$$

By Cramer's rule $d_i = d^{-1} \cdot |\det(\vec{v_0}, \vec{v_1}, ..., \vec{v_{i-1}}, \vec{v_{i+1}}, ..., \vec{v_k})|$ for $i = 0, ..., k$ where $d = d_0^{-1} |\det(\vec{v_1}, ..., \vec{v_k})|$. By the minimality of d_0 we have $\gcd(d_0, ..., d_k) = 1$, whence $d \in \mathbb{Z}_{>0}$. Note that μ_i defined at the beginning of the Introduction equals d_i/d_0 for $i = 1, ..., k$. Furthermore, $\mu(W_i) = |\det(\vec{v_0}, \vec{v_1}, ..., \vec{v_{i-1}}, \vec{v_{i+1}}, ..., \vec{v_k})| = dd_i > 0$ for $i = 0, ..., k$. Since W_i tiles $\mathbb{R}^k$ and the vertices of W_i are integer points, V_i induces a similar tiling of $\mathbb{Z}^k$ and $|V_i| = \mu(W_i) = dd_i$ for $i = 0, ..., k$ whence $|V| = d(d_0 + ... + d_k)$. Here $|.|$ denotes the cardinality of a set. The set of linear combinations of $\vec{v_0}, ..., \vec{v_k}$ with integer coefficients forms a sublattice L of $\mathbb{Z}^k$ under addition. Let $\vec{w_1}, ..., \vec{w_k}$ form a basis of L. Then $\vec{w_j} = \rho_{0,j} \vec{v_0} + ... + \rho_{k,j} \vec{v_k}$ with $\rho_{0,j}, ..., \rho_{k,j} \in \mathbb{Z}$ for $j = 1, ..., k$ and $\det(L) = \det(\vec{w_1}, ..., \vec{w_k})$. By (3)

$$\vec{w_j} = \left(\rho_{0,j} \frac{d_1}{d_0} + \rho_{1,j}\right) \vec{v_1} + ... + \left(\rho_{0,j} \frac{d_k}{d_0} + \rho_{k,j}\right) \vec{v_k}.$$

Hence $\det(L) \in \frac{\mathbb{Z}}{d_0} \det(\vec{v_1}, ..., \vec{v_k}) = d\mathbb{Z}$. Thus $d | \det(L)$ and L has at least d cosets. The next result implies that L has exactly d cosets in $\mathbb{Z}^k$ and that the elements in V which belong to the same coset of L form a cycle under iteration of ϕ of length $d_0 + ... + d_k$.

Proposition. (i) *If $\vec{u} \in V$ and $\vec{v} \in V$ are in the same coset of L, then $\vec{u} = \phi^m(\vec{v})$ for some m with $0 \le m < d_0 + ... + d_k$.*

(ii) *If $\vec{v}$ is in the same coset as $\vec{0}$, it can be written as*

$$\vec{v} = a_0 \vec{v_0} - a_1 \vec{v_1} - ... - a_k \vec{v_k} \ \textit{with}\ a_i \in \mathbb{Z}, 0 \le a_i \le d_i \ \textit{for}\ i = 0, ..., k.$$

Proof. Since every point in $\mathbb{R}^k$ is a linear combination of vectors $\vec{v_1}, ..., \vec{v_k}$ and since by adding and subtracting vectors $\vec{v_i}$ and so reducing the coefficients modulo 1 we get a point in V_0, each coset of L is represented in V. Since V is finite, for each coset of L there must exist a minimal positive integer n for which some $\vec{v} \in V$ belonging to that coset exists with $\phi^n(\vec{v}) = \vec{v}$. Hence, by the definition of ϕ,

$$\vec{v} = \phi^n(\vec{v}) = \vec{v} + b_0 \vec{v_0} - b_1 \vec{v_1} - ... - b_k \vec{v_k}, \tag{4}$$

with $b_0, ..., b_k \in \mathbb{Z}_{\geq 0}$ and $b_0+b_1+...+b_k = n$. Since $\vec{v_1}, ..., \vec{v_k}$ are linearly independent over $\mathbb{Z}$, we have $b_0 > 0$. Hence

$$\frac{b_1}{b_0}v_{1j} + ... + \frac{b_k}{b_0}v_{kj} = v_{0j} \text{ for } j = 1, ..., k.$$

Solving $\frac{b_i}{b_0}$ from the system of k linear equations in k unknowns, we find from the definition of d_i that

$$\frac{b_i}{b_0} = \frac{d_i}{d_0} \ (i = 1, ..., k).$$

Since $b_i = b_0 d_i / d_0$ is an integer for $i = 1, ..., k$ and $\gcd(d_0, ..., d_k) = 1$, we obtain $d_0 | b_0$, whence $b_0 \geq d_0$ and $b_i \geq d_i$ for $i = 1, ..., k$. Hence

$$n = b_0 + ... + b_k \geq d_0 + ... + d_k.$$

Recall that V has $d(d_0 + ... + d_k)$ elements and splits into at least d cosets. Since every coset of L has a cycle of length at least $d_0 + ... + d_k$ and these d cycles are obviously disjoint, there are exactly d cycles and each has to have minimal length $n = d_0 + ... + d_k$ in view of the total number of elements of V. Thus for any $\vec{v} \in V$ the set $\{\vec{v}, \phi(\vec{v}), ..., \phi^{n-1}(\vec{v})\}$ represents a full coset in V. This proves assertion (i). Statement (ii) follows immediately from (4) and the fact that all the inequalities in the proof turned out to be equalities. □

Remark. Let Λ be the lattice generated by the vectors $\vec{v_0} + \vec{v_1}, ..., \vec{v_0} + \vec{v_k}$. The function ϕ can be considered as adding $\vec{v_0}$ and then, if necessary, subtracting with some $\vec{v_0} + \vec{v_i} \in \Lambda$ to secure that the image is in V. The Proposition implies that no two points of V are equivalent modulo Λ. Since $|\det(\Lambda)| = d(d_0+d_1+...+d_k) = |V|$, this implies that V represents $\mathbb{Z}^k/\Lambda$.

3. Generalization of the Fine and Wilf theorem

The Proposition implies that a function f on V which is periodic with periods $\vec{v_0}, ..., \vec{v_k}$ is constant on each coset in V of L. The following theorem provides a slight refinement. If $k = 1$, it is a theorem of Fine and Wilf ([3] Theorem 1).

Theorem 1 (Fine and Wilf for any dimension)**.** *Suppose $f : V \to \mathbb{R}$ has periods $\vec{v_0}, ..., \vec{v_k}$ in the sense that $f(\vec{v}) = f(\vec{v} + \vec{v_i})$ whenever $\vec{v}, \vec{v} + \vec{v_i} \in V$, for $i = 0, ..., k$. Then f is constant within each coset of L in V. The assertion is still valid when from each coset at most one element is removed, but it is no longer true if from some coset two non-ϕ-adjacent elements are removed.*

Proof. Since $f(\vec{v}) = f(\phi(\vec{v}))$ for every $\vec{v} \in V$, the Proposition implies that f is constant on each coset of L. If we remove $\vec{v} \in V$ and no other element of the same coset, we see from $f(\phi(\vec{v})) = f(\phi^2(\vec{v})) = ... = f(\phi^{n-1}(\vec{v}))$ that f is constant on the remaining elements in V of this coset. (Of course one may also remove $\vec{v}, \phi(\vec{v}), ..., \phi^m(\vec{v})$ at the same time for any m.)

In case we remove two non-ϕ-adjacent elements in V from the same coset, $\vec{u}$ and $\vec{v}$ say, we have $\vec{v} = \phi^i(\vec{u})$ for some i with $1 < i < n-1$. By Lemma 3 a point $\phi^h(\vec{u})$ is non-ϕ-adjacent to a point $\phi^m(\vec{u})$ when $0 < h < i, i < m < n$ and therefore we can give different values to $f(\phi(\vec{u})) = f(\phi^2(\vec{u})) = ... = f(\phi^{i-1}(\vec{u}))$ and to $f(\phi^{i+1}(\vec{u})) = ... = f(\phi^{n-1}(\vec{u}))$. Thus f need not be constant on the coset of L any longer. □

It will rarely happen that a function f has a domain which is exactly a translate of V. It is false that in statement (i) of the Proposition the set V can be replaced by any 'convex' set V^* containing V. A natural requirement is that V^* is connected where two points are joined by an edge if they differ by $\pm\vec{v_i}$ for some $i \in \{0, ..., k\}$. This idea has been worked out by Giancarlo and Mignosi [4]. As stated in the Introduction, Mignosi, Restivo and Silva [8] and others have given sufficient conditions for parallelograms in $\mathbb{R}^2$ to satisfy this connectedness. Here we give a sufficient condition (Periodicity Lemma) for parallelotopes in any dimension.

Corollary. *Suppose $\vec{v_1}, ..., \vec{v_k} \in \mathbb{Z}^k$ are linearly independent over $\mathbb{Z}$ and $\vec{v_0} \in \mathbb{Z}^k$ is given by* (1). *Let*

$$V^* = \{\lambda_1\vec{v_1} + ... + \lambda_k\vec{v_k} \in \mathbb{Z}^k : 0 \le \lambda_i < l_i\}$$

where $l_i \ge 1 + \mu_i$ for $i = 1, ..., k$. Let f be periodic modulo $\vec{v_0}, ..., \vec{v_k}$ on V^*. Then f is constant on each coset in V^* of the lattice generated by $\vec{v_0}, ..., \vec{v_k}$.

Proof. Observe that $V \subset V^*$. So by the Proposition it suffices to prove that for every $\vec{v} \in V^*$ there is a $\vec{u} \in V$ in the same coset as $\vec{v}$ with $f(\vec{u}) = f(\vec{v})$. Suppose $\vec{v} = \kappa_1\vec{v_1} + ... + \kappa_k\vec{v_k} \in V^*$. Then

$$f(\vec{v}) = ... = f(\vec{v} - \lfloor\kappa_1\rfloor\vec{v_1}) = ... = f(\vec{v} - \sum_{\kappa=1}^{k}\lfloor\kappa_i\rfloor\vec{v_i}) = f(\vec{u}),$$

where $\vec{u} := \vec{v} - \sum_{\kappa=1}^{k}\lfloor\kappa_i\rfloor\vec{v_i} \in V_0 \subset V$. □

4. Application to the generalized Frobenius problem

Recall that in the Frobenius Coin-changing Problem we are given positive integers $v_0, ..., v_k$ with greatest common divisor 1, and asked to find the least integer n such that every integer $> n$ can be written as a sum of non-negative integer multiples of $v_0, ..., v_k$. In the case $k = 1$ the answer $v_0v_1 - v_0 - v_1$ is due to Sylvester [15]. Here we consider the corresponding question for $\vec{v_0}, ..., \vec{v_k} \in \mathbb{Z}^k$ such that $d = 1$, which means that the vectors generate the full lattice $\mathbb{Z}^k$.

The Proposition says that every point in V can be written as $a_0\vec{v_0} - a_1\vec{v_1} - ... - a_k\vec{v_k}$ with $a_i \in \mathbb{Z},\ 0 \le a_i \le d_i$ for $i = 0, ..., k$ and that

$$d_0\vec{v_0} - d_1\vec{v_1} - ... - d_k\vec{v_k} = \vec{0}. \tag{3}$$

Recall that $\vec{0}$ cannot be written as a positive linear combination of the vectors $\vec{v_0}, ..., \vec{v_k}$. Put

$$\vec{w} = (d_0 - 1)\vec{v_0} - \vec{v_1} - ... - \vec{v_k} = (d_1 - 1)\vec{v_1} + ... + (d_k - 1)\vec{v_k} - \vec{v_0}$$

and

$$X = \{s_1\vec{v_1} + ... + s_k\vec{v_k} + \vec{w} : s_1 > 0, ..., s_k > 0\} \cap \mathbb{Z}^k.$$

Theorem 2 (Sylvester for $k+1$ vectors in $\mathbb{Z}^k$)**.** *Every point in X can be written as $\lambda_0\vec{v_0} + ... + \lambda_k\vec{v_k}$ where $\lambda_0, ..., \lambda_k$ are non-negative integers, but an integer point of the form $s_1\vec{v_1} + ... + s_k\vec{v_k} + \vec{w}$ with $s_1 \ge 0, ..., s_k \ge 0, s_1s_2 \cdots s_k = 0$ can be written in this way unless and only unless $s_1, ..., s_k \in \mathbb{Z}$.*

Proof. The set X for the case $k = 2$ is illustrated in Figure 2.

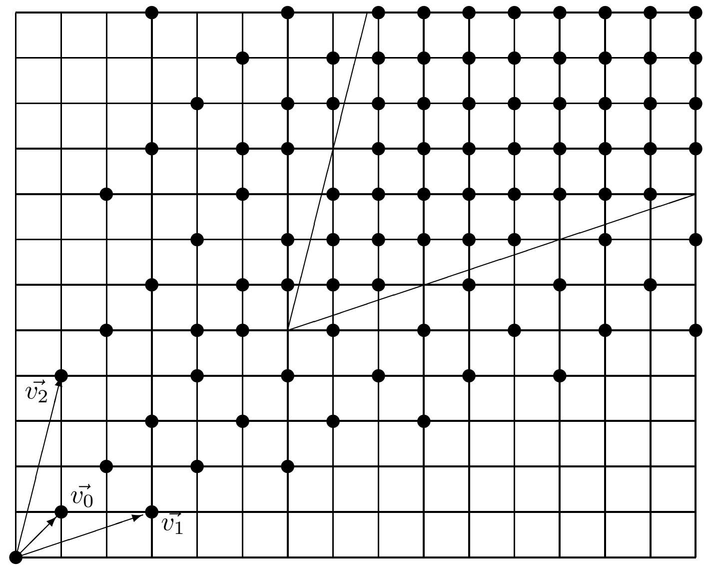

FIGURE 2. The set X (to the northeast of the two line segments) for Sylvester's Theorem in two dimensions formed using $\vec{v_0} = (1,1)$, $\vec{v_1} = (3,1)$ and $\vec{v_2} = (1,4)$.

We first show that every lattice point in the set $\{d_0\vec{v_0} - \vec{v} : \vec{v} \in V\}$ can be written in the required form. Let $\vec{x}$ be an element of this set. Then there exist integers a_i with $0 \le a_i \le d_i$ for $i = 0, ..., k$ such that

$$\vec{x} = d_0\vec{v_0} - a_0\vec{v_0} + a_1\vec{v_1} + ... + a_k\vec{v_k} = (d_0 - a_0)\vec{v_0} + a_1\vec{v_1} + ... + a_k\vec{v_k}$$

which has the required form.

Now consider an arbitrary point $\vec{x} = s_1\vec{v_1} + ... + s_k\vec{v_k} + \vec{w}$ in X. Put $t_i = \lceil s_i \rceil - 1$ for $i = 1, ..., k$. Consider the lattice point

$$\begin{aligned} \vec{x} - t_1\vec{v_1} - ... - t_k\vec{v_k} &= \vec{w} + (s_1 - t_1)\vec{v_1} + ... + (s_k - t_k)\vec{v_k} \\ &= d_0\vec{v_0} - \vec{v_0} - (\lceil s_1 \rceil - s_1)\vec{v_1} - ... - (\lceil s_k \rceil - s_k)\vec{v_k}. \end{aligned}$$

Now $\vec{v_0} + (\lceil s_1 \rceil - s_1)\vec{v_1} + ... + (\lceil s_k \rceil - s_k)\vec{v_k}$ is in V, as we have shown in the beginning of the proof of Lemma 2. Hence $\vec{x} - t_1\vec{v_1} - ... - t_k\vec{v_k}$ is of the form $d_0\vec{v_0} - \vec{v}$ $(\vec{v} \in V)$. By the first part of the proof there exist non-negative integers $\lambda_0, ..., \lambda_k$ such that

$$\vec{x} - t_1\vec{v_1} - ... - t_k\vec{v_k} = \lambda_0\vec{v_0} + ... + \lambda_k\vec{v_k}.$$

Thus

$$\vec{x} = \lambda_0\vec{v_0} + (\lambda_1 + t_1)\vec{v_1} + ... + (\lambda_k + t_k)\vec{v_k}$$

as required.

Now suppose for instance $s_1 = 0$. (The other cases are similar.) We have to show that no lattice point of the form $s_2\vec{v_2} + ... + s_k\vec{v_k} + \vec{w}$ with $s_2, ..., s_k$ non-negative integers can be written as $\lambda_0\vec{v_0} + ... + \lambda_k\vec{v_k}$ with $\lambda_0, ..., \lambda_k$ non-negative integers. By (3) we have

$$\vec{v_0} = \frac{d_1}{d_0}\vec{v_1} + ... + \frac{d_k}{d_0}\vec{v_k}.$$

Substituting this into both representations and assuming that they are equal we obtain

$$\sum_{i=1}^{k} s_i\vec{v_i} + \sum_{i=1}^{k}(d_i-1)\vec{v_i} - \sum_{i=1}^{k}\frac{d_i}{d_0}\vec{v_i} = \lambda_0(\sum_{i=1}^{k}\frac{d_i}{d_0}\vec{v_i}) + \sum_{i=1}^{k}\lambda_i\vec{v_i}.$$

Since $\vec{v_1}, ..., \vec{v_k}$ are linearly independent, this implies that

$$s_i + d_i - 1 - \frac{d_i}{d_0}(1+\lambda_0) - \lambda_i = 0 \ (i = 1, ..., k).$$

Let $\gcd(d_0, 1+\lambda_0) = g$ and write $d_0 = gD_0$ and $1+\lambda_0 = gL_0$. The set of equations then becomes

$$s_i + d_i - 1 - \frac{d_i}{D_0}L_0 - \lambda_i = 0 \ (i = 1, ..., k).$$

Since all the terms here are integers and $\gcd(D_0, L_0) = 1$, this implies that D_0 divides d_i for all i. By $\gcd(d_0, ..., d_k) = 1$, we must have $D_0 = 1$. We now obtain a contradiction by noting that $s_1 = 0, \lambda_0 \geq 0, \lambda_1 \geq 0, L_0 > 0$ and hence

$$0 = s_1 + d_1 - 1 - \frac{d_1}{D_0}L_0 - \lambda_i = d_1(1-L_0) - 1 - \lambda_1 < 0.$$

Finally, consider a lattice point $\vec{v}$ of the form $s_2\vec{v_2} + ... + s_k\vec{v_k} + \vec{w}$ with $s_2 \geq 0, ..., s_k \geq 0$ which cannot be written as $\lambda_0\vec{v_0} + ... + \lambda_k\vec{v_k}$ with $\lambda_0, ..., \lambda_k$ non-negative integers. We know that $\vec{v} + \vec{v_0}$ is in X and can therefore be written as $e_0\vec{v_0} + ... + e_k\vec{v_k}$ with $e_i \in \mathbb{Z}_{\geq 0}$ for all i. We have $e_0 = 0$ since otherwise $\vec{v} = (e_0-1)\vec{v_0} + e_1\vec{v_1} + ... + e_k\vec{v_k}$ would contradict the definition of $\vec{v}$. Hence

$$\vec{v} - \vec{w} = s_2\vec{v_2} + ... + s_k\vec{v_k} = (e_1 - d_1 + 1)\vec{v_1} + ... + (e_k - d_k + 1)\vec{v_k}.$$

Since $\vec{v_1}, ..., \vec{v_k}$ are linearly independent over $\mathbb{Z}$, we obtain $s_i = e_i - d_i + 1$ for $i = 2, ..., k$. Thus $s_i \in \mathbb{Z}$ for $i = 2, ..., k$. $\square$

Remark. The obvious analogue of Sylvester's result mentioned in the Introduction that for $0 \leq m \leq n_0$ exactly one among m and $n_0 - m$ is a non-negative linear combination of a_0 and a_1 is false. If $\vec{v_0} = (2,2), \vec{v_1} = (3,0), \vec{v_2} = (1,5)$, then $\vec{w} = (24,23)$ and both $(5,8)$ and $\vec{w} - (5,8) = (19,15)$ cannot be written as linear combinations of $\vec{v_0}, \vec{v_1}, \vec{v_2}$ over $\mathbb{Z}_{\geq 0}$.

Of course, in general at most one of the vectors $\vec{v}$ and $\vec{w} - \vec{v}$ can be written as a non-negative linear combination, since by Theorem 2 the sum $\vec{w}$ cannot be written in this way. A valid analogue of Sylvester's result will be described in [9]. Theorem 2 implies another complementarity property, viz. that if $\vec{w} + \vec{v}$ is an integer point on the boundary of X, then exactly one among $\vec{v}$ and $\vec{w} + \vec{v}$ is a linear combination of $\vec{v_0}, ..., \vec{v_k}$ over $\mathbb{Z}_{\geq 0}$,

5. Related representations of $\mathbb{Z}^k/\Lambda$

Remark 1. The parallelepiped with the disjoint sums of vectors $\vec{v_0} + \vec{v_i}$ $(i = 1, ..., k)$ as vertices is the simplest fundamental domain of $\mathbb{R}^k/\Lambda$. One may wonder whether in this parallelepiped the integer points in a coset of L also form a cycle when points differing by $\pm\vec{v_i}$ for some $i \in \{0, ..., k\}$ are joined. The following example shows that this is not true and that one has to introduce one more period to restore the cycle along the representatives. Take $k = 2, \vec{v_0} = (4,3), \vec{v_1} = (7,-3), \vec{v_2} = (-4,4)$. Then Λ has as generating vectors $\vec{u_1} := \vec{v_0} + \vec{v_1} = (11,0)$ and $\vec{u_2} := \vec{v_0} + \vec{v_2} = (0,7)$. Hence U consists of the set $\{(x,y) \in \mathbb{Z}^2 : 0 \leq x < 11, 0 \leq y < 7\}$. After joining

h	i	k	b	d	g	i	j	a	c	e
b	d	g	i	k	b	d	f	h	i	k
i	k	b	d	g	i	j	l	b	d	g
d	g	i	j	a	c	e	h	i	k	b
k	b	d	f	h	i	k	b	d	g	i
g	i	j	l	b	d	g	i	k	b	d
a	c	e	h	i	k	b	d	g	i	j

FIGURE 3. Distinct components for a function with period vectors $(4,3), (7,-3), (-4,4)$ are indicated by distinct letters. If a period vector $(7,4)$ is added, only one component is left.

entries as indicated above, the integer points fall apart in 12 components indicated by letters in Figure 3, since the 4-by-3 points in the upper right corner have only one adjacent point and all others have at most 2. If we introduce an extra period, that is, also connect entries which differ by $\pm(\vec{v_0} + \vec{v_1} + \vec{v_2}) = \pm(7,4)$, then the points in U form a cycle again.

Remark 2. Both Pierre Arnoux and Laurent Vuillon have remarked that the approach in this paper is related to a method used by Ito and Kimura [6] in their analysis of the Rauzy fractal. The fundamental difference between both approaches is the shape of the resulting representation of $\mathbb{Z}^k/\Lambda$. By using substitutions as Ito and Kimura do, the representation becomes non-convex and has a boundary like a fractal. The iterated use of ϕ in our approach results in the polytope V whose faces are $(k-1)$-dimensional parallelotopes. The shape of V is essential for the application in the Corollary and in Theorem 2.

References

1. A. Amir, G.E. Benson, *Alphabet independent two-dimensional pattern matching*, Proc. 24th ACM Symp. Theory on Comp. (1992) pp. 59-68.
2. M.G. Castelli, F. Mignosi, A. Restivo, *Fine and Wilf's theorem for three periods and a generalization of Sturmian words*, Theor. Comput. Sci. **218** (1999), 83-94. MR **2000c:**68110
3. N.J. Fine, H.S. Wilf, *Uniqueness theorem for periodic functions*, Proc. Amer. Math. Soc. **16** (1965), 109-114. MR **30:**5124
4. R. Giancarlo, F. Mignosi, *Generalizations of the periodicity theorem of Fine and Wilf*, CAAP '94, LNCS **787** (1994), 130-141. MR **95g:**68091
5. Z. Galil, K. Park, *Truly alphabet independent two-dimensional pattern matching*, Proc. 33rd IEEE Symp. on Foundations of Computer Science (1992), 247-256.
6. Sh. Ito, M. Kimura, *On Rauzy fractal*, Japan J. Indust. Appl. Math. **8** (1991), 461-486. MR **93d:**11084
7. J. Justin, *On a paper by Castelli, Mignosi, Restivo*, RAIRO: Theoret. Informatics Appl. **34** (2000), 373-377. MR **2002d:**68089
8. F. Mignosi, A. Restivo, P.V. Silva, *On Fine and Wilf's theorem for bidimensional words*, to appear in Theor. Computer Sc.
9. H. Ray, R.J. Simpson, *The Frobenius problem in two dimensions*, in preparation.
10. M. Regnier, L. Rostami, *A unifying look at d-dimensional periodicities and space coverings*, Proc. 4th Symp. on Combinatorial Pattern Matching LNCS **684** (1993), 215-227. MR **94j:**68310
11. Ö. Rödseth, *On a linear diophantine problem of Frobenius*, J. reine angew. Math. **301** (1978), 171-178. MR **58:**27741
12. E.S. Selmer, *On the linear diophantine problem of Frobenius*, J. reine angew. Math. **293/294** (1977), 1-17. MR **56:**246

13. E.S. Selmer, Ö. Beyer, *On the linear diophantine problem of Frobenius in three variables*, J. reine angew. Math. **301** (1978), 161-170. MR **58:**27740
14. C. Smoryński, *Skolem's solution to a problem of Frobenius*, Math. Intelligencer **3** (1981), 123-132. MR **83b:**03031
15. J.J. Sylvester, *Mathematical questions, with their solutions, Educational Times* **41** (1884), 21.
16. R. Tijdeman and L. Zamboni, *The Fine and Wilf theorem for any number of periods*, in preparation.

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