THE REMOVAL OF π FROM SOME UNDECIDABLE PROBLEMS INVOLVING ELEMENTARY FUNCTIONS

M. LACZKOVICH

(Communicated by Carl G. Jockusch, Jr.)

ABSTRACT. We show that in the ring generated by the integers and the functions x , $\sin x^n$ and $\sin(x \cdot \sin x^n)$ ($n = 1, 2, \dots$) defined on $\mathbf{R}$ it is undecidable whether or not a function has a positive value or has a root. We also prove that the existential theory of the exponential field $\mathbf{C}$ is undecidable.

1.

Let $\mathcal{S}$ denote the class of expressions generated by the rational numbers and π , the variable x , the operations of addition, multiplication, and composition, and the sine function. It was proved by P. S. Wang [5] (based on the papers of D. Richardson [4] and B. F. Caviness [1]) that the existential theory of $\mathcal{S}$ is undecidable. More precisely, Wang proved that for $f \in \mathcal{S}$ each of the following statements is recursively undecidable: (i) there exists a real number x such that $f(x) > 0$; and (ii) there exists a real number x such that $f(x) = 0$.

Let $\mathcal{S}_1$ denote the ring generated by the integers and by the expressions x , $\sin x^n$ and $\sin(x \cdot \sin x^n)$ ($n = 1, 2, \dots$). In other words, $\mathcal{S}_1$ is the set of expressions obtained by substituting x , $\sin x^n$ and $\sin(x \cdot \sin x^n)$ into an arbitrary polynomial with integer coefficients.

In this note we show that in Wang's theorems the class $\mathcal{S}$ can be replaced by $\mathcal{S}_1$. This improves Wang's result in two ways: it eliminates the use of π , and reduces the number of compositions.

Theorem 1. *For $f \in \mathcal{S}_1$, each of the following statements is recursively undecidable: (i) there exists a real number x such that $f(x) > 0$; and (ii) there exists a real number x such that $f(x) = 0$.*

The proof, which follows Richardson's original argument [4] with some modifications, will be given in the next section.

Let $\mathcal{S}_2$ denote the ring generated by the integers and by the expressions $\sin x^n$ and $\cos x^n$ ($n = 1, 2, \dots$). As the following theorem shows, the statement of Theorem 1 is not very far from being optimal.

Received by the editors February 7, 2002 and, in revised form, February 22, 2002.

2000 *Mathematics Subject Classification.* Primary 03B25, 03D40; Secondary 26A09.

Key words and phrases. Undecidable problems, rings of elementary functions.

This research was partially supported by the Hungarian National Foundation for Scientific Research Grant No. T032042.

Theorem 2. *There is an algorithm that decides, for every $f \in \mathcal{S}_2$, whether or not there exists a real number x such that $f(x) > 0$.*

Proof. Let $P(x_1, \dots, x_n, y_1, \dots, y_n) \in \mathbf{Z}[x_1, \dots, x_n, y_1, \dots, y_n]$, and put

$$Q(x) = P(\sin x, \dots, \sin x^n, \cos x, \dots, \cos x^n).$$

Let A_P denote the statement

$$\begin{aligned} \exists (x_1, \dots, x_n, y_1, \dots, y_n) \in \mathbf{R}^{2n} \text{ such that } x_i^2 + y_i^2 = 1 \ (i = 1, \dots, n) \\ \text{and } P(x_1, \dots, x_n, y_1, \dots, y_n) > 0. \end{aligned}$$

We claim that $\exists x (Q(x) > 0) \iff A_P$. Indeed, $\exists x (Q(x) > 0) \implies A_P$ is obvious. Suppose A_P , and let x_i, y_i be such that $x_i^2 + y_i^2 = 1$ ($i = 1, \dots, n$) and $P(x_1, \dots, x_n, y_1, \dots, y_n) > 0$. It is well-known that the sequence

$$\left(\left\{ \frac{k}{2\pi} \right\}, \left\{ \frac{k^2}{2\pi} \right\}, \dots, \left\{ \frac{k^n}{2\pi} \right\} \right) \ (k = 1, 2, \dots)$$

is uniformly distributed in $[0, 1]^n$. (See [2, Theorem 6.3 on p. 48 and Theorem 3.2 on p. 27].) In particular, this sequence is everywhere dense in $[0, 1]^n$. Choose α_i such that $(x_i, y_i) = (\sin \alpha_i, \cos \alpha_i)$ for every $i = 1, \dots, n$. Then, for every $\varepsilon > 0$, we can select an integer k such that $|k^i - \alpha_i - 2m_i\pi| < \varepsilon$ ($i = 1, \dots, n$), where $m_0, \dots, m_n$ are also integers. Clearly, if ε is chosen small enough, then $Q(k) > 0$ will be satisfied. By Tarski's theorem, there is an algorithm that decides the truth of A_P for every $P \in \mathbf{Z}[x_1, \dots, x_n, y_1, \dots, y_n]$. Then the same algorithm decides whether or not $P(\sin x, \dots, \sin x^n, \cos x, \dots, \cos x^n) > 0$ holds for a suitable real number x . $\square$

Question. Does there exist an algorithm that decides, for every $f \in \mathcal{S}_2$, whether or not f has a real root? Does there exist such an algorithm for the ring generated by $\sin x$ and $\sin x^2$?

The problem of eliminating π also appears in the context of the undecidability of the exponential field $\mathbf{C}$. In [3, Theorem 20, p. 201] A. J. Macintyre states that the existential theory of the exponential field $\mathbf{C}$, with π as a distinguished constant, is undecidable. Then he poses the problem of removing the use of π . Unfortunately, Theorem 1 above does not solve this problem, because the notion of real numbers cannot be formulated in $\mathbf{C}$. As Macintyre remarks, the theory of the rationals can be reduced to $\mathbf{C}$. Indeed, we have

$$x \in \mathbf{Q} \iff \exists u \exists v [(e^u = e^v = 1) \wedge (v \neq 0) \wedge (vx = u)].$$

Now the theory of $\mathbf{Q}$ is undecidable by a well-known theorem of J. Robinson. However, the undecidable scheme of statements obtained from J. Robinson's theorem contains universal quantifiers, and the question of whether or not the *existential* theory of $\mathbf{Q}$ is undecidable seems to be open.

Still, we can solve the problem posed by Macintyre using the simple observation that a rational number x is an integer if and only if 2^x is rational.

Theorem 3. *The existential theory of the exponential field $\mathbf{C}$ is undecidable.*

Proof. We claim that for every $x \in \mathbf{Q}$,

$$x \in \mathbf{Z} \iff \exists z [(e^z = 2) \wedge (e^{zx} \in \mathbf{Q})].$$

Indeed, the formula on the right-hand side is true if and only if

$$e^{(\log 2 + 2k\pi i)x} = 2^x (\cos 2k\pi x + i \cdot \sin 2k\pi x)$$

is rational for a suitable integer k . But this happens if and only if $\sin 2k\pi x = 0$ and $2^x \cdot \cos 2k\pi x = \pm 2^x$ is rational; that is, when x is an integer.

By the Davis-Putnam-Robinson-Matijasevic theorem, there exists a polynomial $P(y, x_1, \dots, x_n)$ with integer coefficients such that the set

$$H = \{y \in \mathbf{Z} : \exists (k_1, \dots, k_n) \in \mathbf{Z}^n (P(y, k_1, \dots, k_n) = 0)\}$$

is not recursive. We shall use the abbreviations

$$R(x, u, v) = [(e^u = e^v = 1) \wedge (v \neq 0) \wedge (vx = u)]$$

and

$$I(x, u, v, z) = [(e^z = 2) \wedge R(e^{zx}, u, v)].$$

We put $x = (x_1, \dots, x_n)$, etc., and denote by $Q(y, x, u, v, s, t, z)$ the expression

$$R(x_1, u_1, v_1) \wedge \dots \wedge R(x_n, u_n, v_n) \\ \wedge I(x_1, s_1, t_1, z_1) \wedge \dots \wedge I(x_n, s_n, t_n, z_n) \wedge (P(y, x_1, \dots, x_n) = 0).$$

Then Q is an expression of the exponential field $\mathbf{C}$ for every fixed integer y . Clearly, the formula $\exists x \exists u \exists v \exists s \exists t \exists z Q$ is true if and only if there are integers $x_1, \dots, x_n$ such that $P(y, x_1, \dots, x_n) = 0$; that is, when $y \in H$. Since H is not recursive, it follows that no algorithm can decide whether an existential formula in $\mathbf{C}$ is true or not. $\square$

2. PROOF OF THEOREM 1

By the Davis-Putnam-Robinson-Matijasevic theorem, there exists a polynomial $P(y, x_1, \dots, x_n)$ with integer coefficients such that the set

$$H = \{y \in \mathbf{Z} : \exists (k_1, \dots, k_n) \in \mathbf{Z}^n (P(y, k_1, \dots, k_n) = 0)\}$$

is not recursive. Let $Q = Q(y, x_0, x_1, \dots, x_n) \in \mathbf{Z}[y, x_0, x_1, \dots, x_n]$ be a polynomial such that Q is homogeneous in $x_0, x_1, \dots, x_n$ and satisfies

$$(1) \quad Q(y, 1, x_1, \dots, x_n) = P(y, x_1, \dots, x_n).$$

We shall denote by d the total degree of $x_0, \dots, x_n$ in Q . We fix a positive integer N such that

$$(2) \quad (\pi - 3)^N < \frac{1}{2 \cdot (n+1)},$$

and put

$$S = Q^2 + (x_0 - 3)^{2N} \quad \text{and} \quad \frac{\partial S}{\partial x_i} = h_i \quad (i = 0, 1, \dots, n).$$

Lemma 1. *For every $i = 0, 1, \dots, n$ there exists a $g_i \in \mathbf{Z}[y, x_0, \dots, x_n]$ such that $g_i \geq 1$ everywhere on $\mathbf{R}^{n+2}$, and for every $(y, x_0, \dots, x_n) \in \mathbf{R}^{n+2}$ and $(t_0, \dots, t_n) \in \mathbf{R}^{n+1}$ with $|t_i| \leq 2$ ($i = 0, \dots, n$) we have*

$$|h_i(y, x_0 + t_0, \dots, x_n + t_n)| \leq g_i(y, x_0, \dots, x_n) \quad (i = 0, \dots, n).$$

Proof. It is easy to check that if $h_i = \sum c \cdot y^\alpha x_0^{\beta_0} \dots x_n^{\beta_n}$, then

$$g_i = 1 + \sum |c| \cdot (y^2 + 1)^\alpha (x_0^2 + 3)^{\beta_0} \dots (x_n^2 + 3)^{\beta_n}$$

satisfies the requirements. $\square$

Now we define

$$F(y, x_0, \dots, x_n) = 4(n+1)^2 \left[S + \sum_{i=0}^n g_i^2 \cdot \sin^2 x_i \right].$$

Lemma 2. $\{y \in \mathbf{Z} : \exists (x_0, \dots, x_n) \in \mathbf{R}^{n+1} (F(y, x_0, \dots, x_n) < 1)\} = \{y \in \mathbf{Z} : \exists (x_0, \dots, x_n) \in \mathbf{R}^{n+1} (F(y, x_0, \dots, x_n) \leq 1)\} = H$.

Proof. Let $y \in H$, and let $(k_1, \dots, k_n) \in \mathbf{Z}^n$ be such that $P(y, k_1, \dots, k_n) = 0$. Then, putting $x_0 = \pi$ and $x_i = \pi k_i$ ($i = 1, \dots, n$) we have, by (1) and (2),

$$\begin{aligned} F(y, x_0, \dots, x_n) &= 4(n+1)^2 [Q^2(y, \pi, \pi k_1, \dots, \pi k_n) + (\pi - 3)^{2N}] \\ &= 4(n+1)^2 \cdot \pi^{2d} \cdot P^2(y, k_1, \dots, k_n) + 4(n+1)^2 (\pi - 3)^{2N} \\ &< 0 + 1 = 1. \end{aligned}$$

Next suppose $F(y, x_0, \dots, x_n) \leq 1$, where $y \in \mathbf{Z}$ and $(x_0, \dots, x_n) \in \mathbf{R}^{n+1}$. There are integers k_i such that $|\pi k_i - x_i| \leq \pi/2$ ($i = 0, \dots, n$). The proof will be completed if we show that $P(y, k_1, \dots, k_n) = 0$, as it implies $y \in H$.

Since $x/2 \leq 2x/\pi \leq \sin x$ for every $x \in [0, \pi/2]$, we obtain

$$(3) \quad \frac{1}{2} |\pi k_i - x_i| \leq \sin |\pi k_i - x_i| = |\sin x_i| \quad (i = 0, \dots, n).$$

It follows from the condition $F(y, x_0, \dots, x_n) \leq 1$ that

$$(4) \quad |\sin x_i| \cdot |g_i(y, x_0, \dots, x_n)| \leq \frac{1}{2(n+1)} \quad (i = 0, \dots, n).$$

Applying the mean value theorem, we find $c_0, \dots, c_n \in (-\pi/2, \pi/2)$ such that

$$\begin{aligned} (5) \quad & |S(y, \pi k_0, \dots, \pi k_n) - S(y, x_0, \dots, x_n)| \\ &= \left| \sum_{i=0}^n h_i(y, x_0 + c_0, \dots, x_n + c_n) \cdot (\pi k_i - x_i) \right| \\ &\leq \sum_{i=0}^n 2 \cdot g_i(y, x_0, \dots, x_n) \cdot |\sin x_i| \leq 1, \end{aligned}$$

where the inequalities follow from Lemma 1, (3), and (4). Since

$$S(y, x_0, \dots, x_n) \leq 1/(4(n+1)^2) < 1,$$

it follows from (5) that

$$(6) \quad |S(y, \pi k_0, \dots, \pi k_n)| < 2,$$

and thus

$$\begin{aligned} (7) \quad & (\pi k_0 - 3)^{2N} \leq Q^2(y, \pi k_0, \dots, \pi k_n) + (\pi k_0 - 3)^{2N} \\ &= S(y, \pi k_0, \dots, \pi k_n) < 2. \end{aligned}$$

Now k_0 is an integer, therefore (7) can hold only if $k_0 = 1$. Consequently,

$$\pi^{2d} \cdot P^2(y, k_1, \dots, k_n) = \pi^{2d} Q^2(y, 1, k_1, \dots, k_n) = Q^2(y, \pi k_0, \dots, \pi k_n) < 2$$

by (1) and (6), and thus $P^2(y, k_1, \dots, k_n) < 1$. Since $P^2(y, k_1, \dots, k_n)$ is an integer, we have $P^2(y, k_1, \dots, k_n) = 0$. $\square$

Lemma 3. *For every $x_1, \dots, x_n \in \mathbf{R}$, $\delta > 0$ and $K > 0$ there exists a $t > K$ such that*

$$|x_i - t \cdot \sin t^{2i}| < \delta \quad (i = 1, \dots, n).$$

Proof. We prove by induction on n . The case $n = 1$ is clear. Suppose $n \geq 2$ and that the statement is true for $n - 1$. We may assume $0 < \delta < 1$ and $K > 1$. By the induction hypothesis there exists a $u > K + (\pi \cdot 2^{2n+1}/\delta) + |x_n|$ such that

$$|x_i - u \cdot \sin u^{2i}| < \delta/2 \quad (i = 1, \dots, n-1).$$

We put $v = u + \delta \cdot (8n \cdot (u+1)^{2n-2})^{-1}$. Then we have, for every $t \in [u, v]$ and $i = 1, \dots, n-1$,

$$\begin{aligned} |t \cdot \sin t^{2i} - u \cdot \sin u^{2i}| &\leq |t \cdot (\sin t^{2i} - \sin u^{2i})| + |t - u| \cdot |\sin u^{2i}| \\ &\leq (u+1) (v^{2i} - u^{2i}) + \frac{\delta}{8} \\ &\leq (u+1) \cdot 2i \cdot (u+1)^{2i-1} \cdot \frac{\delta}{8n(u+1)^{2n-2}} + \frac{\delta}{8} \\ &< \frac{\delta}{4} + \frac{\delta}{8} < \frac{\delta}{2}, \end{aligned}$$

and thus $|x_i - t \cdot \sin t^{2i}| < \delta$. Since $u > |x_n|$ and

$$v^{2n} \geq u^{2n} + 2n \cdot u^{2n-1} \cdot \frac{\delta}{8n(u+1)^{2n-2}} > u^{2n} + \frac{\delta}{4} \cdot \frac{u}{2^{2n-2}} > u^{2n} + 2\pi,$$

we can find a $t \in [u, v]$ such that $t \cdot \sin t^{2n} = x_n$. $\square$

Now we put

$$f(y, x) = 1 - F(y, x \cdot \sin x^2, x \cdot \sin x^4, \dots, x \cdot \sin x^{2n+2})$$

for every $y \in \mathbf{Z}$ and $x \in \mathbf{R}$. Then $f(y, x) \in \mathcal{S}_1$ for every $y \in \mathbf{Z}$. We prove that

$$(8) \quad \{y \in \mathbf{Z} : \exists x (f(y, x) > 0)\} = H$$

and

$$(9) \quad \{y \in \mathbf{Z} : \exists x (f(y, x) = 0)\} = H.$$

Indeed, (8) and the inclusion $\{y \in \mathbf{Z} : \exists x (f(y, x) = 0)\} \subset H$ are clear from Lemmas 2 and 3. Suppose $y \in H$. By (8), there is a $u \in \mathbf{R}$ such that $f(y, u) > 0$. Now we have $F \geq S \geq (x_0 - 3)^{2N}$ everywhere, and thus

$$f(y, x) \leq 1 - ((x \cdot \sin x^2) - 3)^{2N}$$

for every x . In particular, $f(y, 0) < 0$. Therefore, we can find an x with $f(y, x) = 0$, which proves (9). Since H is not recursive, this completes the proof of Theorem 1. $\square$

REFERENCES

- [1] B. F. Caviness, On canonical forms and simplification, *Journal of the Association for computing Machinery* **17** (2) (1970), 385-396. MR **43**:7104
- [2] L. Kuipers and H. Niederreiter, *Uniform distribution of sequences*. John Wiley & Sons, 1974. MR **54**:7415
- [3] A. J. Macintyre, Exponential algebra. *Logic and Algebra (Pontignano, 1994)*, 191-210. Lecture Notes in Pure and Applied Mathematics, No. 180. Dekker, 1996. MR **97h**:03064

- [4] D. Richardson, Some undecidable problems involving elementary functions of a real variable, *J. Symbolic Logic* **33** (1968), 514-520. MR **39**:1330
- [5] P. S. Wang, The undecidability of the existence of zeros of real elementary functions, *Journal of the Association for computing Machinery* **21** (4) (1974), 586-589. MR **51**:117

DEPARTMENT OF ANALYSIS, EÖTVÖS LORÁND UNIVERSITY, BUDAPEST, PÁZMÁNY PÉTER SÉTÁNY
1/C 1117, HUNGARY

E-mail address: `laczko@renyi.hu`