PERIODIC SOLUTIONS FOR PLANAR 2N-BODY PROBLEMS

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ABSTRACT. In this paper we study some necessary conditions and sufficient conditions for the nested polygonal solutions of planar 2N-body problems.

1. MAIN RESULTS

This paper uses the same notations as those used in [6]. For $n \geq 2$ the equations of motion of the planar n -body problem ([1], [2], [3], [5], [6], [7]) can be written in the form

$$(1.1) \quad \ddot{z}_k = - \sum_{\substack{j=1 \\ j \neq k}}^n m_j \frac{z_k - z_j}{|z_k - z_j|^3},$$

where z_k is the complex coordinate of the k th mass m_k in an inertial coordinate system. Let ρ_k denote the N complex k th roots of unity, i.e.,

$$(1.2) \quad \rho_k = \exp(2\pi i k / N).$$

This equation will also serve to define ρ_k for any number k . We assume that the mass m_k ($k = 1, \dots, N$) is located at the vertex ρ_k of a regular polygon inscribed on the unit circle, and $\tilde{m}_k$ ($k = 1, \dots, N$) is located at

$$(1.3) \quad \tilde{\rho}_k = a \rho_k \cdot e^{i\theta},$$

where $a > 0$, $0 \leq \theta \leq 2\pi$, and $a \neq 1$ when $\theta = 0$ or 2π .

Then the center of masses $m_1, \dots, m_N; \tilde{m}_1, \dots, \tilde{m}_N$ is

$$(1.4) \quad z_0 = \sum_j (m_j \rho_j + \tilde{m}_j \tilde{\rho}_j) / M$$

where $M = \sum_j (m_j + \tilde{m}_j)$. In (1.4) and throughout this paper, unless specially restricted, all indices and summations will range from 1 to N . The functions describing their rotation about z_0 with angular velocity ω are then given by

$$(1.5) \quad z_k(t) = (\rho_k - z_0) \exp(i\omega t), \quad k = 1, \dots, N,$$

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and

$$(1.6) \quad \tilde{z}_k(t) = (a\rho_k e^{i\theta} - z_0) \exp(i\omega t), \quad k = 1, \dots, N.$$

Then the equations of motion of the planar 2N-body problem can be written in the following form:

$$(1.7) \quad \ddot{z}_k = \sum_{j \neq k} m_j \frac{z_j - z_k}{|z_j - z_k|^3} + \sum_j \tilde{m}_j \frac{\tilde{z}_j - z_k}{|\tilde{z}_j - z_k|^3}$$

and

$$(1.8) \quad \ddot{\tilde{z}}_k = \sum_j m_j \frac{z_j - \tilde{z}_k}{|z_j - \tilde{z}_k|^3} + \sum_{j \neq k} \tilde{m}_j \frac{\tilde{z}_j - \tilde{z}_k}{|\tilde{z}_j - \tilde{z}_k|^3}.$$

R. Moeckel and C. Simó ([5]) proved the following result:

Theorem (Moeckel-Simó). *If $\theta = 0$ and $m_1 = \dots = m_N$, $\tilde{m}_1 = \dots = \tilde{m}_N$, then for every mass ratio $b = \frac{\tilde{m}_1}{m_1}$, there are exactly two planar central configurations consisting of two nested regular N -gons. For one of these, the ratio of the sizes of the two polygons is less than 1, and for the other it is greater than 1.*

In this paper, we study the inverse problem of the theorem (Moeckel-Simó) and the following results are established.

Theorem 1. *If, for $N \geq 2$, the functions $z_k(t)$ and $\tilde{z}_k(t)$ given by (1.5) and (1.6) are solutions of the 2N-body problem (1.7) and (1.8), it follows that ω^2 satisfies*

$$(1.9) \quad \left[\frac{1}{4} \sum_{j \neq N} \csc\left(\frac{\pi j}{N}\right) - \frac{\omega^2}{M} N \right] \cdot \left[\frac{1}{4} a^{-2} \sum_{j \neq N} \csc\left(\frac{\pi j}{N}\right) - \frac{\omega^2}{M} a \cdot N \right] e^{i\theta} \\ = \left[\sum_j \frac{1 - a\rho_{j-1}e^{i\theta}}{|1 - a\rho_{j-1}e^{i\theta}|^3} - \frac{N\omega^2}{M} \right] \cdot \left[\sum_j \frac{ae^{i\theta} - \rho_{j-1}}{|ae^{i\theta} - \rho_{j-1}|^3} - \frac{\omega^2}{M} Nae^{i\theta} \right].$$

Theorem 2. *For $N \geq 2$, $m_k, \tilde{m}_k > 0$, if the functions $z_k(t)$ and $\tilde{z}_k(t)$ given by (1.5) and (1.6) with ω^2 given by (1.9) are solutions of the 2N-body problem (1.7) and (1.8), then $m_1 = m_2 = \dots = m_N$ and $\tilde{m}_1 = \tilde{m}_2 = \dots = \tilde{m}_N$.*

Conversely, if $m_1 = m_2 = \dots = m_N$ and $\tilde{m}_1 = \tilde{m}_2 = \dots = \tilde{m}_N$, let $b = \tilde{m}_1/m_1$. If b, a and ω^2 satisfy

$$(1.10) \quad \frac{N\omega^2}{M} = \frac{1}{1+b} \left[\sum_{j=1}^{N-1} \frac{1 - \rho_j}{|1 - \rho_j|^3} + \sum_{j=0}^{N-1} \frac{b(1 - a\rho_j e^{i\theta})}{|1 - a\rho_j e^{i\theta}|^3} \right],$$

$$(1.11) \quad \frac{N\omega^2}{M} = \frac{e^{-i\theta}}{a(1+b)} \left[\sum_j \frac{ae^{i\theta} - \rho_j}{|ae^{i\theta} - \rho_j|^3} + \sum_{j \neq N} \frac{b(a - a\rho_j)e^{i\theta}}{|a - a\rho_j|^3} \right],$$

then $z_k(t)$ and $\tilde{z}_k(t)$ are solutions of (1.7) and (1.8).

Remark. It is not difficult to prove that “ ω^2 ” and “ a ” are positive real numbers if $\theta = 0$ or $\frac{\pi}{N}$, and it seems that only $\theta = 0$ or $\frac{\pi}{N}$, “ ω^2 ” and “ a ” are positive real numbers, but the proof seems difficult.

Corollary 1 (MacMillan-Bartky [3]). *Under the above assumptions,*

(i) *if $N = 2, \theta = 0, a > 1$ and $z_k(t)$ and $\tilde{z}_k(t)$ given by (1.5) and (1.6) with ω^2 satisfying*

$$(1.12) \quad \begin{aligned} & \left[\frac{1}{4} - \frac{2\omega^2}{M} \right] \cdot \left[\frac{1}{4}a^{-2} - \frac{\omega^2}{M} \cdot 2a \right] \\ &= \left[\frac{2(a^2 + 1)}{(a^2 - 1)^2} - \frac{\omega^2}{M} \cdot 2a \right] \cdot \left[\frac{-4a}{(a^2 - 1)^2} - \frac{2\omega^2}{M} \right] \end{aligned}$$

are solutions of the 2×2 -body problems (1.7) and (1.8), then $m_1 = m_2, \tilde{m}_1 = \tilde{m}_2$. Conversely, if $\omega^2 = (m_1 + m_2 + \tilde{m}_1 + \tilde{m}_2)\gamma/N$ satisfying (1.12) and

$$(1.13) \quad \frac{2\omega^2}{M} = \frac{1}{1+b} \left[\frac{1}{4} - \frac{4ab}{(a^2 - 1)^2} \right]$$

where

$$(1.14) \quad b = \frac{a^7 - 2a^5 - 8a^4 + a^3 - 8a^2}{17a^4 - 2a^2 + 1},$$

then $z_k(t)$ and $\tilde{z}_k(t)$ given by (1.5) and (1.6) are solutions of (1.7) and (1.8).

(ii) *if $N = 2, \theta = 0, 0 < a < 1$ and $z_k(t)$ and $\tilde{z}_k(t)$ given by (1.5) and (1.6) with ω^2 satisfying*

$$(1.15) \quad \begin{aligned} & \left[\frac{1}{4} - \frac{2\omega^2}{M} \right] \cdot \left[\frac{1}{4}a^{-2} - \frac{\omega^2}{M} \cdot 2a \right] \\ &= \left[\frac{-4a}{(1 - a^2)^2} - \frac{\omega^2}{M} \cdot 2a \right] \cdot \left[\frac{2(1 + a^2)}{(1 - a^2)^2} - \frac{2\omega^2}{M} \right] \end{aligned}$$

are solutions of (1.7) and (1.8), then $m_1 = m_2, \tilde{m}_1 = \tilde{m}_2$. Conversely, if $\omega^2 = M\gamma/N$ satisfying (1.15) and

$$(1.16) \quad \gamma = \frac{1}{1+b} \left[\frac{1}{4} + \frac{2b(a^2 + 1)}{(a^2 - 1)^2} \right]$$

where

$$(1.17) \quad b = \frac{a^7 - 2a^5 + 17a^3}{-8a^5 + a^4 - 8a^3 - 2a^2 + 1},$$

then $z_k(t)$ and $\tilde{z}_k(t)$ given by (1.5) and (1.6) are solutions of (1.7) and (1.8).

Corollary 2 (MacMillan-Bartky [3]). (i) *For $N = 2$ and $\theta = \frac{\pi}{2}$, then (1.10) and (1.11) imply*

$$(1.18) \quad b = [(1 + a^2)^{-3/2} - 2^{-3}] \cdot [(1 + a^2)^{-3/2} - (2a)^{-3}]^{-1}.$$

(ii) *For $N = 2$ and $\theta = \frac{\pi}{2}$, $b = 1$ if and only if $a = 1$. That is, $z_k, \tilde{z}_k$ are solutions of (1.7) and (1.8). If $m_1 = m_2, \tilde{m}_1 = \tilde{m}_2$, then $m_1 = m_2 = \tilde{m}_1 = \tilde{m}_2$ if and only if $m_1, m_2, \tilde{m}_1, \tilde{m}_2$ are at the vertices of a square.*

(iii) *For $N = 2$ and $\theta = \frac{\pi}{2}$, $m_1 = m_2, \tilde{m}_1 = \tilde{m}_2$ are at the vertices of a diamond. Then $m_1, m_2, \tilde{m}_1, \tilde{m}_2$ form a central configuration if and only if $\frac{\sqrt{3}}{3} < a < \sqrt{3}$, where a stands for the ratio of its diagonals and the ratio b of masses is determined uniquely by (1.18); especially considering the degenerate central configuration, when $b = \infty$, $a = \frac{\sqrt{3}}{3}$, and when $a = \sqrt{3}, b = 0$.*

Corollary 3 (Perko-Walter [6]). *For $N \geq 2, m_k, \tilde{m}_k > 0$, if $a = 1$ and $\theta = \frac{\pi}{N}$, then $z_k(t)$ and $\tilde{z}_k(t)$ given by (1.5) and (1.6) with $\omega^2 = M\gamma/N$ and ω^2 given by (1.9) are solutions of the $2N$ -body problem (1.7) and (1.8). Then $m_1 = m_2 = \dots = m_N = \tilde{m}_1 = \dots = \tilde{m}_N$.*

Corollary 4. *If $N = 3, \theta = 0$. If $z_k(t)$ and $\tilde{z}_k(t)$ given by (1.5) and (1.6) with ω^2 satisfying*

$$(1.19) \quad \omega^2 = \left[\frac{\sqrt{3}}{M} (a + a^{-2}) - \frac{3}{M} \left(\sum_{j=1}^3 \frac{a - \rho_{j-1}}{|a - \rho_{j-1}|^3} + a \sum_{j=1}^3 \frac{1 - a\rho_{j-1}}{|1 - a\rho_{j-1}|^3} \right) \right]^{-1} \\ \cdot \left[\frac{1}{3} a^{-2} - \left(\sum_{j=1}^3 \frac{a - \rho_{j-1}}{|a - \rho_{j-1}|^3} \right) \left(\sum_{j=1}^3 \frac{1 - a\rho_{j-1}}{|1 - a\rho_{j-1}|^3} \right) \right]$$

are the solution of the nested 2×3 -body problems (1.7) and (1.8), then $m_1 = m_2 = m_3, \tilde{m}_1 = \tilde{m}_2 = \tilde{m}_3$.

Conversly, if $m_1 = m_2 = m_3, \tilde{m}_1 = \tilde{m}_2 = \tilde{m}_3$, let $b = \frac{\tilde{m}_1}{m_1}$. If ω^2 satisfies (1.19) and a and b have the relation

$$(1.20) \quad b = \left(\frac{\sqrt{3}}{3} a^{-3} - \sum_{j=1}^3 \frac{1 - a\rho_{j-1}}{|1 - a\rho_{j-1}|^3} \right)^{-1} \left(\frac{\sqrt{3}}{3} - a^{-1} \sum_{j=1}^3 \frac{a - \rho_{j-1}}{|a - \rho_{j-1}|^3} \right),$$

then $z_k(t)$ and $\tilde{z}_k(t)$ defined by (1.5) and (1.6) are solutions of (1.7) and (1.8).

Corollary 5. *Assume $N = 3, \theta = \frac{\pi}{3}$. If $z_k(t)$ and $\tilde{z}_k(t)$ given by (1.5) and (1.6) with ω^2 satisfying*

$$(1.21) \quad \omega^2 = \frac{M}{3} \left[\frac{\sqrt{3}}{3} (a + a^{-2}) - \left(\sum_{j=1}^3 \frac{a - e^{-\frac{\pi}{3}\sqrt{-1}}\rho_{j-1}}{|a - e^{-\frac{\pi}{3}\sqrt{-1}}\rho_{j-1}|^3} + a \sum_{j=1}^3 \frac{1 - ae^{\frac{\pi}{3}\sqrt{-1}}\rho_{j-1}}{|1 - ae^{\frac{\pi}{3}\sqrt{-1}}\rho_{j-1}|^3} \right) \right]^{-1} \\ \cdot \left[\frac{1}{3} a^{-2} e^{\frac{\pi}{3}\sqrt{-1}} - \left(\sum_{j=1}^3 \frac{a - e^{-\frac{\pi}{3}\sqrt{-1}}\rho_{j-1}}{|a - e^{-\frac{\pi}{3}\sqrt{-1}}\rho_{j-1}|^3} \right) \left(\sum_{j=1}^3 \frac{1 - ae^{\frac{\pi}{3}\sqrt{-1}}\rho_{j-1}}{|1 - ae^{\frac{\pi}{3}\sqrt{-1}}\rho_{j-1}|^3} \right) \right]$$

are the solution of the twisted 2×3 -body problems (1.7) and (1.8), then $m_1 = m_2 = m_3, \tilde{m}_1 = \tilde{m}_2 = \tilde{m}_3$.

Conversly, if $m_1 = m_2 = m_3, \tilde{m}_1 = \tilde{m}_2 = \tilde{m}_3$, let $b = \frac{\tilde{m}_1}{m_1}$. If ω^2 satisfies (1.21) and a and b have the following relation:

$$(1.22) \quad b = \left(\frac{\sqrt{3}}{3} a^{-2} - \sum_{j=1}^3 \frac{a(1 - ae^{\frac{\pi}{3}\sqrt{-1}}\rho_{j-1})}{|1 - ae^{\frac{\pi}{3}\sqrt{-1}}\rho_{j-1}|^3} \right)^{-1} \left(\frac{\sqrt{3}}{3} a - \sum_{j=1}^3 \frac{a - e^{-\frac{\pi}{3}\sqrt{-1}}\rho_{j-1}}{|a - e^{-\frac{\pi}{3}\sqrt{-1}}\rho_{j-1}|^3} \right),$$

then $z_k(t)$ and $\tilde{z}_k(t)$ defined by (1.5) and (1.6) are solutions of (1.7) and (1.8).

2. EIGENVALUES AND EIGENVECTORS FOR CIRCULANT MATRICES

Definition 2.1 ([4]). *If $N \times N$ matrix $A = (a_{i,j})$ satisfies*

$$(2.1) \quad a_{i,j} = a_{i-1,j-1}, \quad 1 \leq i, j \leq N,$$

where we assume $a_{i,0} = a_{i,N}$ and $a_{0,j} = a_{N,j}$. Then we call A a circulant matrix.

Lemma 2.1 ([4]). *If A and B are $N \times N$ circulant matrices, then $A+B$, $A-B$, $A \cdot B$ are also circulant matrices and $AB = BA$.*

Lemma 2.2 ([4]). *Let $A = (a_{i,j})$ be an $N \times N$ circulant matrix. The eigenvalues λ_k and the eigenvectors $\vec{v}_k$ of A are*

$$(2.2) \quad \lambda_k(A) = \sum_j a_{1,j} \rho_{k-1}^{j-1}$$

and

$$(2.3) \quad \vec{v}_k = (\rho_{k-1}, \rho_{k-1}^2, \dots, \rho_{k-1}^N)^T.$$

Lemma 2.3 ([4]). *Let A, B be circulant matrices, where $\lambda_k(A)$, $\lambda_k(B)$ are eigenvalues of A, B . Then the eigenvalues of $A+B$, $A-B$, $A \cdot B$ are $\lambda_k(A) + \lambda_k(B)$, $\lambda_k(A) - \lambda_k(B)$, $\lambda_k(A) \cdot \lambda_k(B)$.*

The following useful lemma can be simply proved using the properties of circulant matrices.

Lemma 2.4. *If $A = (a_{i,j})$ is an $N \times N$ circulant matrix, and $A \cdot X = 0$, where $X = (x_1, \dots, x_N)^T$, $\sum_i x_i \neq 0$, then*

$$(2.4) \quad \begin{aligned} a_{1,j} + \dots + a_{N,j} &= 0, \quad 1 \leq j \leq N, \\ a_{i,1} + \dots + a_{i,N} &= 0, \quad 1 \leq i \leq N. \end{aligned}$$

3. PROOF OF THE MAIN RESULTS

For two nested regular polygons, we define

$$(3.1) \quad \rho_k = \exp(2\pi i k / N),$$

$$(3.2) \quad \tilde{\rho}_k = a \exp(2\pi i k / N) e^{i\theta},$$

$$(3.3) \quad z_0 = \sum_j (m_j \rho_j + \tilde{m}_j \tilde{\rho}_j) / M,$$

where

$$(3.4) \quad M = \sum_j (m_j + \tilde{m}_j),$$

$$(3.5) \quad z_k(t) = (\rho_k - z_0) \exp(i\omega t), \quad k = 1, \dots, N,$$

and

$$(3.6) \quad \tilde{z}_k(t) = (a\rho_k e^{i\theta} - z_0) \exp(i\omega t), \quad k = 1, \dots, N.$$

Proof of Theorem 1. (3.1)–(3.6) imply that the $z_k(t)$ and $\tilde{z}_k(t)$ are the solutions of (1.7) and (1.8) if and only if

$$(3.7) \quad (\rho_k - z_0)\omega^2 \exp(i\omega t) = \left(\sum_{j \neq k} m_j \frac{\rho_k - \rho_j}{|\rho_k - \rho_j|^3} + \sum_j \tilde{m}_j \frac{\rho_k - \tilde{\rho}_j}{|\rho_k - \tilde{\rho}_j|^3} \right) \exp(i\omega t)$$

and

(3.8)

$$(\tilde{\rho}_k - z_0)\omega^2 \exp(i\omega t) = \left(\sum_j m_j \frac{\tilde{\rho}_k - \rho_j}{|\tilde{\rho}_k - \rho_j|^3} + \sum_{j \neq k} \tilde{m}_j \frac{\tilde{\rho}_k - \tilde{\rho}_j}{|\tilde{\rho}_k - \tilde{\rho}_j|^3} \right) \exp(i\omega t)$$

or if and only if

(3.9)

$$\sum_{j \neq k} m_j \left(\frac{1}{|\rho_k - \rho_j|^3} - \frac{\omega^2}{M} \right) (\rho_k - \rho_j) + \sum_j \tilde{m}_j \left(\frac{1}{|\rho_k - \tilde{\rho}_j|^3} - \frac{\omega^2}{M} \right) (\rho_k - \tilde{\rho}_j) = 0$$

and

(3.10)

$$\sum_j m_j \left(\frac{1}{|\tilde{\rho}_k - \rho_j|^3} - \frac{\omega^2}{M} \right) (\tilde{\rho}_k - \rho_j) + \sum_{j \neq k} \tilde{m}_j \left(\frac{1}{|\tilde{\rho}_k - \tilde{\rho}_j|^3} - \frac{\omega^2}{M} \right) (\tilde{\rho}_k - \tilde{\rho}_j) = 0.$$

Multiplying both sides by ρ_{N-k} and noting that $|\rho_k - \rho_j| = |\rho_k| |1 - \rho_{j-k}| = |1 - \rho_{j-k}|$ and using $\tilde{\rho}_k = a\rho_k e^{i\theta}$,

$$\begin{aligned} & \sum_{j \neq k} m_j \left(\frac{1}{|1 - \rho_{j-k}|^3} - \frac{\omega^2}{M} \right) (1 - \rho_{j-k}) \\ & + \sum_j \tilde{m}_j \left(\frac{1}{|1 - a\rho_{j-k}e^{i\theta}|^3} - \frac{\omega^2}{M} \right) (1 - a\rho_{j-k}e^{i\theta}) = 0 \end{aligned} \quad (3.11)$$

and

$$\begin{aligned} & \sum_j m_j \left(\frac{1}{|ae^{i\theta} - \rho_{j-k}|^3} - \frac{\omega^2}{M} \right) (ae^{i\theta} - \rho_{j-k}) \\ & + \sum_{j \neq k} \tilde{m}_j \left(\frac{1}{|a - a\rho_{j-k}|^3} - \frac{\omega^2}{M} \right) (a - a\rho_{j-k})e^{i\theta} = 0. \end{aligned} \quad (3.12)$$

Notice that every step from (3.7) to (3.12) can be conversed, respectively. Now we define the $N \times N$ circulant matrices $C = [c_{k,j}]$, $A = [a_{k,j}]$, $B = [b_{k,j}]$, $D = [d_{k,j}]$ as follows:

$$\begin{aligned} c_{k,j} &= 0, & \text{for } k &= j, \\ c_{k,j} &= \left(\frac{1}{|1 - \rho_{j-k}|^3} - \frac{\omega^2}{M} \right) (1 - \rho_{j-k}), & \text{for } k &\neq j, \end{aligned} \quad (3.13)$$

$$a_{k,j} = \left(\frac{1}{|1 - a\rho_{j-k}e^{i\theta}|^3} - \frac{\omega^2}{M} \right) (1 - a\rho_{j-k}e^{i\theta}), \quad (3.14)$$

$$b_{k,j} = \left(\frac{1}{|ae^{i\theta} - \rho_{j-k}|^3} - \frac{\omega^2}{M} \right) (ae^{i\theta} - \rho_{j-k}), \quad (3.15)$$

$$\begin{aligned} d_{k,j} &= 0, & \text{for } k &= j, \\ d_{k,j} &= \left(\frac{1}{|a - a\rho_{j-k}|^3} - \frac{\omega^2}{M} \right) (a - a\rho_{j-k})e^{i\theta} & \text{for } k &\neq j. \end{aligned} \quad (3.16)$$

Then (3.11) and (3.12) hold if and only if the matrix equation

$$(3.17) \quad \begin{pmatrix} C & A \\ B & D \end{pmatrix} \begin{pmatrix} m_1 \\ \vdots \\ m_N \\ \tilde{m}_1 \\ \vdots \\ \tilde{m}_N \end{pmatrix} = 0$$

has a positive solution.

Let

$$(3.18) \quad m = (m_1, \dots, m_N)^T, \tilde{m} = (\tilde{m}_1, \dots, \tilde{m}_N)^T.$$

Then (3.17) is equivalent to

$$(3.19) \quad C \cdot m + A \cdot \tilde{m} = 0,$$

$$(3.20) \quad B \cdot m + D \cdot \tilde{m} = 0.$$

By (3.19) and (3.20) we have

$$(3.21) \quad (CD - BA) \cdot m = 0,$$

$$(3.22) \quad (AB - CD) \cdot \tilde{m} = 0.$$

We notice that (3.21) and (3.22) have a solution is equivalent to that $AB - CD$ has a positive real eigenvector corresponding to eigenvalue 0. By Lemma 2.3 we have

$$(3.23) \quad \begin{aligned} & \lambda_k(AB - CD) \\ &= \lambda_k(AB) - \lambda_k(CD) \\ &= \lambda_k(A)\lambda_k(B) - \lambda_k(C)\lambda_k(D). \end{aligned}$$

Hence

$$(3.24) \quad \lambda_k(AB - CD) = 0$$

for some $1 \leq k \leq N$ if and only if

$$(3.25) \quad \lambda_k(A)\lambda_k(B) = \lambda_k(C)\lambda_k(D).$$

Also, we notice by Lemma 2.2 and Lemma 2.4 $\lambda_1(AB - CD) = 0$. Hence by Lemma 2.2 we have

$$(3.26) \quad \begin{aligned} & \left[\sum_{j \neq 1} \left(\frac{1}{|1 - \rho_{j-1}|^3} - \frac{\omega^2}{M} \right) (1 - \rho_{j-1}) \right] \cdot \left[\sum_{j \neq 1} \left(\frac{1}{|a - a\rho_{j-1}|^3} - \frac{\omega^2}{M} \right) (a - a\rho_{j-1}) e^{i\theta} \right] \\ &= \left[\sum_j \left(\frac{1}{|1 - a\rho_{j-1}e^{i\theta}|^3} - \frac{\omega^2}{M} \right) (1 - a\rho_{j-1}e^{i\theta}) \right] \\ & \quad \cdot \left[\sum_j \left(\frac{1}{|ae^{i\theta} - \rho_{j-1}|^3} - \frac{\omega^2}{M} \right) (ae^{i\theta} - \rho_{j-1}) \right]. \end{aligned}$$

We notice that

$$\begin{aligned}
 (3.27) \quad & \left[\sum_{j \neq 1} \frac{1 - \rho_{j-1}}{|1 - \rho_{j-1}|^3} - \frac{\omega^2}{M} \sum_j (1 - \rho_{j-1}) \right] \\
 & \cdot \left[a^{-2} \sum_{j \neq 1} \frac{1 - \rho_{j-1}}{|1 - \rho_{j-1}|^3} - \frac{\omega^2}{M} a \cdot \sum_j (1 - \rho_{j-1}) \right] \cdot e^{i\theta} \\
 & = \left[\frac{1}{4} \sum_{j \neq N} \csc \left(\frac{\pi j}{N} \right) - \frac{\omega^2}{M} \cdot N \right] \cdot \left[\frac{1}{4} a^{-2} \sum_{j \neq N} \csc \left(\frac{\pi j}{N} \right) - \frac{\omega^2}{M} \cdot a \cdot N \right] \cdot e^{i\theta},
 \end{aligned}$$

$$\begin{aligned}
 (3.28) \quad & \sum_j \left[\frac{1 - ae^{i\theta} \rho_{j-1}}{|1 - ae^{i\theta} \rho_{j-1}|^3} - \frac{\omega^2}{M} \cdot (1 - ae^{i\theta} \rho_{j-1}) \right] \\
 & \cdot \sum_j \left[\frac{ae^{i\theta} - \rho_{j-1}}{|ae^{i\theta} - \rho_{j-1}|^3} - \frac{\omega^2}{M} (ae^{i\theta} - \rho_{j-1}) \right] \\
 & = \left[\sum_j \frac{1 - ae^{i\theta} \rho_{j-1}}{|1 - ae^{i\theta} \rho_{j-1}|^3} - \frac{\omega^2 \cdot N}{M} \right] \cdot \left[\sum_j \frac{ae^{i\theta} - \rho_{j-1}}{|ae^{i\theta} - \rho_{j-1}|^3} - \frac{\omega^2}{M} \cdot Nae^{i\theta} \right].
 \end{aligned}$$

Proof of Theorem 2. $\lambda_1(AB - CD) = 0$ is simple so $\vec{v}_1 = (1, 1, \dots, 1)^T$ is the only positive real eigenvector for $\lambda_1 = 0$.

Assume $m_1 = m_2 = \dots = m_N > 0$ and $\tilde{m}_1 = m_2 = \dots = \tilde{m}_N > 0$, ω^2 is determined by (1.9) and a, b is determined by (1.10) and (1.11). Then $(m_1, \dots, m_1)^T$ is a solution of (3.11) and (3.12) or (1.7) and (1.8), since

$$\begin{aligned}
 (3.29) \quad & \sum_{j \neq k} m_j \left(\frac{1}{|1 - \rho_{j-k}|^3} - \frac{\omega^2}{M} \right) (1 - \rho_{j-k}) \\
 & + \sum_j b m_j \left(\frac{1}{|1 - a \rho_{j-k} e^{i\theta}|^3} - \frac{\omega^2}{M} \right) (1 - a \rho_{j-k} e^{i\theta})
 \end{aligned}$$

$$\begin{aligned}
 (3.30) \quad & = m_1 \left[\sum_{j \neq k} \left(\frac{1}{|1 - \rho_{j-k}|^3} - \frac{\omega^2}{M} \right) (1 - \rho_{j-k}) \right. \\
 & \left. + \sum_j b \left(\frac{1}{|1 - a \rho_{j-k} e^{i\theta}|^3} - \frac{\omega^2}{M} \right) (1 - a \rho_{j-k} e^{i\theta}) \right]
 \end{aligned}$$

(3.31)

$$= m_1 \left[\sum_{j \neq N} \left(\frac{1}{|1 - \rho_j|^3} - \frac{\omega^2}{M} \right) (1 - \rho_j) + \sum_j b \left(\frac{1}{|1 - a\rho_j e^{i\theta}|^3} - \frac{\omega^2}{M} \right) (1 - a\rho_j e^{i\theta}) \right] = 0,$$

and the same results for (3.12) are easily obtained. Thus the proof is completed.

Corollary 1 is obvious from Theorem 2.

Proof of Corollary 2. (i) For $N = 2$ and $\theta = \frac{\pi}{2}$, b and a have the following relationship:

$$(3.32) \quad b = \frac{\left(\sum_{j=0}^{N-1} \frac{ae^{i\theta} - \rho_j}{|ae^{i\theta} - \rho_j|^3} - \sum_{j=1}^{N-1} \frac{ae^{i\theta}(1 - \rho_j)}{|1 - \rho_j|^3} \right)}{\left(\sum_{j=0}^{N-1} \frac{ae^{i\theta}(1 - a\rho_j e^{i\theta})}{|1 - a\rho_j e^{i\theta}|^3} - \sum_{j=1}^{N-1} \frac{ae^{i\theta}(1 - \rho_j)}{a^3|1 - \rho_j|^3} \right)}.$$

Then

$$(3.33) \quad \begin{aligned} \sum_{j=0}^{N-1} \frac{ae^{i\theta} - \rho_j}{|ae^{i\theta} - \rho_j|^3} - \sum_{j=1}^{N-1} \frac{ae^{i\theta}(1 - \rho_j)}{|1 - \rho_j|^3} &= \frac{ai - 1}{|ai - 1|^3} + \frac{ai + 1}{|ai + 1|^3} - \frac{ai(1 + 1)}{|2|^3} \\ &= \frac{ai - 1}{(a^2 + 1)^{3/2}} + \frac{ai + 1}{(a^2 + 1)^{3/2}} - \frac{2ai}{8} \\ &= \frac{2ai}{(a^2 + 1)^{3/2}} - \frac{2ai}{8} = 2ai \left(\frac{1}{(a^2 + 1)^{3/2}} - \frac{1}{8} \right) \end{aligned}$$

and

$$(3.34) \quad \begin{aligned} \sum_{j=0}^{N-1} \frac{ae^{i\theta}(1 - a\rho_j e^{i\theta})}{|1 - a\rho_j e^{i\theta}|^3} - \sum_{j=1}^{N-1} \frac{ae^{i\theta}(1 - \rho_j)}{a^3|1 - \rho_j|^3} \\ = \frac{ai(1 - ai)}{|1 - ai|^3} + \frac{ai(1 - a(-1)i)}{|1 - a(-1)i|^3} - \frac{ai(1 + 1)}{|2a|^3} \\ = 2ai \left(\frac{1}{(a^2 + 1)^{3/2}} - \frac{1}{8a^3} \right), \end{aligned}$$

so

$$(3.35) \quad b = [(1 + a^2)^{-3/2} - 2^{-3}] \cdot [(1 + a^2)^{-3/2} - (2a)^{-3}]^{-1}.$$

(ii) That $b = 1$ if and only if $a = 1$ can be obtained directly from (1.18).

(iii) Since (1.11) holds and b must be greater than zero, we have

$$(3.36) \quad \begin{cases} (1 + a^2)^{-3/2} - 2^{-3} & > 0, \\ (1 + a^2)^{-3/2} - (2a)^{-3} & > 0, \end{cases}$$

or

$$(3.37) \quad \begin{cases} (1 + a^2)^{-3/2} - 2^{-3} & < 0, \\ (1 + a^2)^{-3/2} - (2a)^{-3} & < 0. \end{cases}$$

Then the result of (iii) is easily obtained.

Proof of Corollary 3. Under the assumptions of Corollary 3, we have $b = 1$.

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