ALGEBRAIC STRUCTURES DETERMINED BY 3 BY 3 MATRIX GEOMETRY

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(Communicated by David R. Larson)

Dedicated to Masamichi and Kyoko Takesaki and the memory of Yuki

ABSTRACT. Using a “3 by 3 matrix trick” we show that multiplication (an algebraic structure) in a C^* -algebra $\mathcal{A}$ is determined by the geometry of the C^* -algebra of the 3 by 3 matrices with entries from $\mathcal{A}$, $M_3(\mathcal{A})$. This is an example of an algebra-geometry duality which, we claim, has applications.

INTRODUCTION

The motivation for this research was the question, posed about 25 years ago in [10], to find a “natural” generalization of the van Kampen–Pontriagin duality theorem to any locally compact group G , based on the order structure of the semi-group of continuous positive definite functions on G , $P(G)$. The nonabelian group duality we seek follows in part from a more general duality between the algebraic structure of a C^* -algebra and its intrinsic matrix geometry. Since the proof of this algebra-geometry duality for C^* -algebras is relatively simple and of independent interest, we present the result in this note.

KEY LEMMA

In [8] J. L. Smul’jan proved a version of our key lemma which was subsequently translated from Russian to English (and also into contemporary operator-theoretic language) by the late Domingo Herrero. A slight modification of our key lemma, more appropriate to numerical linear algebra because it does not involve taking the square root of operators, can be found in [1]. The version of our key lemma which we state below can be found in [2].

In the following lemma $\mathcal{H}_1, \mathcal{H}_2$ are Hilbert spaces, $\mathcal{B}(\mathcal{H}_i, \mathcal{H}_j)$ is the set of all bounded linear operators from $\mathcal{H}_i$ to $\mathcal{H}_j$, $i = 1, 2$, $j = 1, 2$, and $A \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_1)$, $B \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_1)$, $B^* \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$, $D \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_2)$. Double vertical bars, $\|\cdot\|$, denote the usual uniform (sup) operator norm, e.g., $\|D\| = \sup\{\|D\zeta\|_{\mathcal{H}_2} \mid \|\zeta\|_{\mathcal{H}_2} \leq 1\}$.

Key lemma. *Let $\tilde{A} = \begin{bmatrix} A & B \\ B^* & D \end{bmatrix}$. If $\tilde{A}$ is a positive operator in $\mathcal{H}_1 \oplus \mathcal{H}_2$, then there exists a bounded operator $X \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_1)$ satisfying the following conditions.*

Received by the editors July 15, 2001 and, in revised form, February 10, 2002.

2000 *Mathematics Subject Classification.* Primary 46L89, 43A35; Secondary 43A40, 43A30.

Key words and phrases. C^* -algebra, convolution, completely bounded, duality, Fourier-Stieltjes algebra, locally compact group, positive definite function, matrix entry, unitary representation.

- (1) $B = A^{1/2}X$,
 (2) $\text{Range}(X) \subseteq \overline{\text{Range}(A)}$.

On the other hand, if B is of the above form in (1) for some operator X , then $\tilde{A} \geq 0$ if and only if $D \geq X^*X$.

THE DUALITY BETWEEN THE PRODUCT OPERATION
 IN C^* -ALGEBRA $\mathcal{A}$ AND THE GEOMETRY IN $M_3(\mathcal{A})$

In the following $M_3(\mathcal{A})$ is the set of 3 by 3 matrices with entries from C^* -algebra $\mathcal{A}$. With the usual operations of matrix addition, multiplication and taking the adjoint, $M_3(\mathcal{A})$ is a C^* -algebra; cf. [9], p. 192. The definition of “the geometry of $M_3(\mathcal{A})$ ” will be evident in the subsequent discussion.

Theorem 1. *If $a, b \in \mathcal{A}$, a C^* -algebra with unit e , then the product ab in $\mathcal{A}$ is determined by the geometry of $M_3(\mathcal{A})$, most notably the order structure of $M_3(\mathcal{A})$.*

The proof of Theorem 1 depends on the following lemma.

Lemma 1. *Let $\mathcal{A}$ be a C^* -algebra with unit e . Let a, b and c be unitary operators in $\mathcal{A}$, e.g., $a^*a = e = aa^*$. We have*

$$\begin{pmatrix} e & a & b \\ a^* & e & c \\ b^* & c^* & e \end{pmatrix} \geq 0 \quad \text{in } M_3(\mathcal{A})$$

if and only if $c = a^{-1}b$ if and only if $b = ac$.

Proof of Lemma 1. In the notation of our key lemma above, $\tilde{A}$ is the 3 by 3 matrix given in the statement of Lemma 1, and $A = \begin{pmatrix} e & a \\ a^* & e \end{pmatrix}$. Since a is unitary, $A^{1/2} = \frac{1}{\sqrt{2}}A$. We assume $\tilde{A}$ is a positive operator in the C^* -algebra $M_3(\mathcal{A})$, hence it will be a positive operator in any realization of $M_3(\mathcal{A})$ on Hilbert space.

Thus by (1) of our key lemma, there will be bounded operators x_1, x_2 satisfying the following pair of equations:

$$\begin{aligned} ex_1 + ax_2 &= \sqrt{2}b, \\ a^*x_1 + ex_2 &= \sqrt{2}c. \end{aligned}$$

Multiplying the first equation by $a^* = a^{-1}$ immediately yields $c = a^{-1}b$.

The converse is elementary, since $\begin{pmatrix} e & a & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}^* \begin{pmatrix} e & a & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \geq 0$; cf. [9], p. 193. □

Proof of Theorem 1. We consider the order structure on $M_3(\mathcal{A})$ to be part of “geometry”. Furthermore, note that $\frac{1}{2}A = \frac{1}{2}\begin{pmatrix} e & a \\ a^* & e \end{pmatrix} \in M_2(\mathcal{A})$ is a projection in C^* -algebra $M_2(\mathcal{A})$, i.e., $(\frac{1}{2}A)^* = \frac{1}{2}A = (\frac{1}{2}A)^2$, if and only if a is a unitary in C^* -algebra $\mathcal{A}$. Projections in a C^* -algebra are characterized geometrically as the extreme points of the convex intersection of the unit ball and the positive cone of the C^* -algebra; cf. [7], p. 484. Similarly projections $\frac{1}{2}\begin{pmatrix} e & b \\ b^* & e \end{pmatrix}$ and $\frac{1}{2}\begin{pmatrix} e & c \\ c^* & e \end{pmatrix}$ determine (and are determined by) unitaries b and c , respectively. Since every element in a C^* -algebra with unit is a linear combination of unitaries (cf. [4]), we have determined the product in $\mathcal{A}$ by various aspects of the geometry of $\mathcal{A}$, $M_2(\mathcal{A})$ and $M_3(\mathcal{A})$, and all three of these C^* -algebras can be considered subalgebras of $M_3(\mathcal{A})$. □

POSSIBLE APPLICATIONS

In [12], which is in preparation, we apply Theorem 1 above to construct the duality for locally compact groups mentioned in the Introduction as our motivation for searching for Theorem 1. It is our as yet unrealized goal to make a significant contribution to (via a simplification of) the existing classification of finite groups using our duality theory.

In the “standard model” which appears in theoretical physics, 3 by 3 matrices play a role. This was brought to my attention by a lecture of Alain Connes. Whether the structures explored in this paper might contribute any insights in physics, however minor, remain to be discovered.

ACKNOWLEDGMENTS

I thank Professor Paul Muhly at the University of Iowa not only for connecting me with the work of Smul’jan and Curto but also for financially supporting a full week visit to the University of Iowa where key points of this work were accomplished.

I would also like to mention that I have profitably discussed this work and/or possible applications at the Great Plains Operator Theory Symposium with Professors R. Curto, B. Blackadar, A. Kumjian, V. Paulsen, J. Quigg and J. Spielberg. A hearty thanks also goes to the referee who pointed out misprints and a missing hypothesis in the statement of Lemma 1 which is indeed used in its proof.

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