THE BANACH ALGEBRA INDUCED BY A DOUBLE CENTRALIZER

ETIENNE DESQUITH

(Communicated by David R. Larson)

ABSTRACT. Given a Banach algebra A , R. Larsen defined, in his book “*An introduction to the theory of multipliers*”, a Banach algebra A_T by means of a *multiplier* T on A , and essentially used it in the case of a *commutative semisimple* Banach algebra A to prove a result on multiplications which preserve regular maximal ideals. Here, we consider the analogue Banach algebra $\mathcal{A}_R$ induced by a bounded *double centralizer* $\langle L, R \rangle$ of a Banach algebra A . Then, our main concern is devoted to the relationships between L , R , and the algebras of bounded double centralizers $\mathcal{W}(A)$ and $\mathcal{W}(\mathcal{A}_R)$ of A and $\mathcal{A}_R$, respectively. By removing the assumption of semisimplicity, we generalize some results proven by Larsen.

1. PRELIMINARIES

Unless otherwise stated, we shall adopt throughout the sequel the following conventions: A will be an arbitrary Banach algebra, and the Banach algebra of all (resp. bounded) linear operators on it will be denoted by $\mathcal{L}(A)$ (resp. $B(A)$). The maps λ and μ , such that $a \mapsto \lambda_a$ and $a \mapsto \mu_a$ for $a \in A$, are the usual left and right regular representations of A . Composition of mappings will be denoted by simple juxtaposition.

Definition 1.1. A left (resp. right) *centralizer* of A is an element $L \in \mathcal{L}(A)$ (resp. $R \in \mathcal{L}(A)$), satisfying $L(ab) = L(a)b$ (resp. $R(ab) = aR(b)$), $\forall a, b \in A$.

A *double centralizer* of A is a pair $\langle L, R \rangle$ where L (resp. R) is a left (resp. right) centralizer, and which together satisfy the following *Double Centralizer Property (DC-Property)*:

$$(1) \quad aL(b) = R(a)b, \quad \forall a, b \in A.$$

The algebra $\mathcal{W}(A)$ of *bounded double centralizers* of A is the set of double centralizers with pointwise linear operations and with product and norm given by

$$(2) \quad \begin{aligned} \langle L, R \rangle \langle T, S \rangle &= \langle LT, SR \rangle, \quad \|\langle L, R \rangle\| = \max\{\|L\|, \|R\|\}, \\ \forall \langle L, R \rangle, \langle T, S \rangle &\in \mathcal{W}(A). \end{aligned}$$

Relation (2) above ensures the automatic continuity of the elements L and R of any double centralizer $\langle L, R \rangle$ of A . For the definitions of a *multiplier* and of the

Received by the editors November 28, 2001 and, in revised form, February 4, 2002.

2000 *Mathematics Subject Classification.* Primary 47D30.

Key words and phrases. Annihilator, Banach algebra, double centralizer, multiplier.

This work was supported by the Abdus Salam ICTP Associateship scheme (Trieste/Italy).

left annihilator $LAN(A)$, right annihilator $RAN(A)$ and annihilator $An(A)$ of A , the reader should refer to [3] and [4]. A is said to have no annihilators if $An(A) = LAN(A) = RAN(A) = \{0\}$.

2. THE BANACH ALGEBRA $\mathcal{A}_R$

Proposition 2.1. *Let $\langle L, R \rangle \in \mathcal{W}(A)$ such that $\|R\| \leq 1$, and define*

$$(3) \quad a \cdot b = R(a) b, \quad \forall a, b \in A.$$

- (i) *Relation (3) endows A with a Banach algebra structure denoted by $\mathcal{A}_R$.*
- (ii) *R (resp. L) is a homomorphism from $\mathcal{A}_R$ onto the image $Im(R)$ (resp. $Im(L)$) of R (resp. L).*
- (iii) *If A has no annihilator, or if A admits a two-sided approximate identity, then the kernels of L and R satisfy $Ker(R) = LAN(\mathcal{A}_R)$, $Ker(L) = RAN(\mathcal{A}_R)$, and hence $An(\mathcal{A}_R) = Ker(L) \cap Ker(R)$.*

Proof. (i) and (ii) are obtained by straightforward computations. It is also easy to show the relations $R[LAN(\mathcal{A}_R)] \subset LAN(A)$ and $L[RAN(\mathcal{A}_R)] \subset RAN(A)$, from which (iii) follows, since A is without annihilators. $\square$

Remark 2.2. It should be noticed that assuming $\|L\| \leq 1$, and defining the product in relation (3) as $a \cdot b = a L(b)$, $\forall a, b \in A$, gives the same algebra $\mathcal{A}_R$. Therefore, when dealing in the sequel with the algebra $\mathcal{A}_R$, we shall say that $\mathcal{A}_R$ is *induced* or *defined* by $\langle L, R \rangle$. Considering $\mathcal{A}_R$ as a *left Banach module* over itself, the map R satisfying relation (3) is an example of a bounded module map from the module into the algebra (cf. [1]).

As an immediate consequence of Proposition 2.1 above, we have:

Corollary 2.3. *Let A have no annihilators. Then, for any $\langle L, R \rangle \in \mathcal{W}(A)$, L (resp. R) is one-to-one implies that $An(\mathcal{A}_R) = \{0\}$. Moreover, $\mathcal{A}_R$ has no annihilators, if and only if L and R are both one-to-one.* $\square$

Remark 2.4. If A is commutative and without annihilators, then $\langle L, R \rangle$ is a multiplier $L = R = T$ and therefore $\mathcal{A}_R$ coincides with the algebra A_T of R. Larsen, and is also commutative.

For $\langle L, R \rangle$ defining $\mathcal{A}_R$, direct computations show that the condition $\langle T, S \rangle \in \mathcal{W}(\mathcal{A}_R)$ implies $\langle LT, RS \rangle \in \mathcal{W}(A)$.

As noted by T. W. Palmer in [4], page 26, the set of left (resp. right) centralizers is contained in the commutant $(\mu_A)^c$ (resp. $(\lambda_A)^c$) of μ_A (resp. λ_A). Moreover, as pointed out in [2], page 775, if A has no annihilators, then given two arbitrary elements $\langle L, R \rangle$ and $\langle T, S \rangle$ of $\mathcal{W}(A)$, their components *cross-commute*, i.e. $TR = RT$ and $LS = SL$. As a consequence, any $\langle L, R \rangle \in \mathcal{W}(A)$ satisfies $LR = RL$.

For every $\langle L, R \rangle \in \mathcal{W}(A)$, let $Im(L)$ (resp. $Im(R)$) denote the image of L (resp. R), and define $I_{LR} = Im(L) \cap Im(R)$. Then we have:

Proposition 2.5. *Let A have no annihilators, and let $\langle L, R \rangle \in \mathcal{W}(A) \setminus \{0\}$. If $\overline{Im(L)} \neq A$ and $\overline{Im(R)} \neq A$, then each of L and R admits a nontrivial closed invariant subspace (the bar denotes the closure in the norm topology of A); moreover, $I_{LR} = \{0\}$ implies that L (resp. R) cannot be one-to-one.*

Proof. We shall write $L[Im(R)]$ and $R[Im(L)]$ to denote the restrictions of the maps L and R , respectively, on the subspaces $Im(R)$ and $Im(L)$ of A . According to the above comments, the components of $\langle L, R \rangle \in \mathcal{W}(A)$ commute. Hence the following hold: $L[Im(R)] \subset I_{LR}$ and $R[Im(L)] \subset I_{LR}$, from which we derive $L[\overline{Im(R)}] \subset \overline{Im(R)}$, $R[\overline{Im(L)}] \subset \overline{Im(L)}$. Then, if the images of L and R are not dense in A , these inclusions prove that each of L and R admits a nontrivial closed invariant subspace. L and R being nonzero, the same inclusions imply that L (resp. R) cannot be one-to-one if $I_{LR} = \{0\}$. $\square$

Theorem 2.6. *Let A and $\mathcal{A}_R$ be without annihilators and let $\langle L, R \rangle \in \mathcal{W}(A)$ be such that $Im(L) \subset R[Im(L)]$. Then, R is bijective, and R^{-1} is a right centralizer. The result remains clearly true if the roles of R and L are interchanged, and the right centralizer is replaced by the left centralizer.*

Proof. We have, according to Corollary 2.3 and the proof of Proposition 2.5, $R[Im(L)] \subset Im(L)$, which, together with the assumption on $Im(L)$ implies the equality $Im(L) = R[Im(L)]$. It then follows that R is onto A , according to the following result: $L(x_0) \in R[Im(L)] \implies x_0 \in Im(R)$. Indeed, let us assume by contradiction that $x_0 \notin Im(R)$. Then, L being one-to-one, we get $L(x_0) \neq LR(a)$, $\forall a \in A$. But since L commutes with R , it also follows that $L(x_0) \neq RL(a)$, $\forall a \in A$, which means that $L(x_0) \notin R[Im(L)]$. Hence, the desired result is established, and one easily checks that R^{-1} is a right centralizer. $\square$

Let us define

- (4) $\mathcal{C}_\ell(A) = \{L \in B(A) ; \exists R \in B(A) ; \langle L, R \rangle \in \mathcal{W}(A)\} ;$
- (5) $\mathcal{C}_r(A) = \{S \in B(A) ; \exists T \in B(A) ; \langle T, S \rangle \in \mathcal{W}(A)\} ;$
- (6) $\Phi(\langle L, R \rangle) = L \quad , \quad \Psi(\langle L, R \rangle) = R \quad , \quad \forall \langle L, R \rangle \in \mathcal{W}(A) .$

Then we get:

Theorem 2.7. *Let A be a Banach algebra with an approximate identity bounded by one. Then $\mathcal{C}_\ell(A)$ and $\mathcal{C}_r(A)$ are Banach subalgebras of $B(A)$, and Φ is a faithful representation of $\mathcal{W}(A)$ on A , with $\Phi(\mathcal{W}(A)) = \mathcal{C}_\ell(A)$ (resp. Ψ is a one-to-one anti-homomorphism of $\mathcal{W}(A)$ into $B(A)$, with $\Psi(\mathcal{W}(A)) = \mathcal{C}_r(A)$).*

Proof. $\mathcal{C}_\ell(A)$ is clearly a linear subspace of $B(A)$. Moreover, if $L, T \in \mathcal{C}_\ell(A)$, then by definition, there exist $R, S \in B(A)$ such that $\langle L, R \rangle \in \mathcal{W}(A)$, $\langle T, S \rangle \in \mathcal{W}(A)$, with $L = \Phi(\langle L, R \rangle)$ and $T = \Phi(\langle T, S \rangle)$. Hence, $\langle L, R \rangle \langle T, S \rangle = \langle L T, S R \rangle \in \mathcal{W}(A)$, and therefore $L T = \Phi(\langle L T, S R \rangle) \in \mathcal{C}_\ell(A)$. So $\mathcal{C}_\ell(A)$ is a subalgebra of $B(A)$ and more precisely, a Banach algebra. Indeed, if $(L_n)_{n \geq 1}$ is a sequence in $\mathcal{C}_\ell(A)$ converging to some $L \in B(A)$, then L is a left centralizer. Moreover, the corresponding sequence $(R_n)_{n \geq 1}$ in $B(A)$ such that $L_n = \Phi(\langle L_n, R_n \rangle)$, $\forall n \geq 1$, converges to some $R \in B(A)$. To see this, we proceed as follows: given $x \in A$, we let $x \mapsto \lambda_x$ be the left regular representation of A onto itself, and for fixed $n, m \in \mathbb{N}$ and $a \in A$, we consider the map

$$(7) \quad A \ni b \mapsto (R_n - R_m)(a) b = \lambda_{(R_n - R_m)(a)}(b) .$$

Then we get

$$(8) \quad \|(R_n - R_m)(a)\| = \|\lambda_{(R_n - R_m)(a)}\| = \sup \{ \|(R_n - R_m)(a)(b)\| ; \|b\| \leq 1 \},$$

so that the following holds:

$$(9) \quad \|R_n - R_m\| \leq \sup \left\{ \sup \{ \|(R_n - R_m)(a)(b)\| ; \|b\| \leq 1 \} ; \|a\| \leq 1 \right\}.$$

Since $\langle L_n, R_n \rangle, \langle L_m, R_m \rangle \in \mathcal{W}(A)$, we have, for each pair of elements $a, b \in A$,

$$(10) \quad R_n(a)b = aL_n(b) \quad \text{and} \quad R_m(a)b = aL_m(b),$$

and the equality (8) hence gives

$$\begin{aligned} \|R_n - R_m\| &\leq \sup \left\{ \|a\| \sup \{ \|L_n - L_m\| \|b\| ; \|b\| \leq 1 \} ; \|a\| \leq 1 \right\} \\ &\leq \|L_n - L_m\|. \end{aligned}$$

$(R_n)_{n \geq 1}$ is therefore a Cauchy sequence in $B(A)$, and hence converges to some $R \in B(A)$.

Now, for each $n \in \mathbb{N}$ and for all $a, b \in A$ the following relations are satisfied:

$$(11) \quad (i) \quad R_n(ab) = aR_n(b) \quad \text{and} \quad (ii) \quad R_n(a)b = aL_n(b),$$

and we get, taking the limits in equalities (i) and (ii) when n tends to infinity: $R(ab) = aR(b)$ and $R(a)b = aL(b)$, that is, R is a right centralizer and $\langle L, R \rangle \in \mathcal{W}(A)$. Therefore $L = \Phi(\langle L, R \rangle) \in \mathcal{C}_\ell(A)$, and $\mathcal{C}_\ell(A)$ is hence norm-closed in $B(A)$. That $\mathcal{C}_r(A)$ is a Banach subalgebra of $B(A)$ is obtained by quite similar arguments as in the case of $\mathcal{C}_\ell(A)$. That Φ is a faithful representation of $\mathcal{W}(A)$ on A , and Ψ a one-to-one anti-homomorphism of $\mathcal{W}(A)$ into A , are obtained by routine computations. $\square$

Given $R \in B(A)$, let $\{R\}^c = \{S \in B(A) ; RS = SR\}$ denote the *commutant* of $\{R\}$. According to the comments following Remark 2.4, if $\langle L, R \rangle \in \mathcal{W}(A)$ defines $\mathcal{A}_R$ and if $S \in \{R\}^c$ is such that $\langle T, S \rangle \in \mathcal{W}(\mathcal{A}_R)$, then $\langle LT, RS \rangle = \langle LT, SR \rangle \in \mathcal{W}(A)$, and it is “tempting” in this situation to write $\langle LT, SR \rangle = \langle L, R \rangle \langle T, S \rangle$. But, the product in the right-hand side of this equation would not make sense, unless $\mathcal{W}(A)$ and $\mathcal{W}(\mathcal{A}_R)$ are subalgebras of some larger Banach algebra. We construct in what follows such an algebra.

Consider the set $\mathcal{D}_{P_0}(A)$ of all pairs $\langle L, R \rangle$ with $L \in B(A)$, $R \in B(A)$, made into a linear algebra under pointwise linear operations and with the same product as in $\mathcal{W}(A)$. Then $\mathcal{D}_{P_0}(A)$ is a normed algebra when it is endowed with the same norm as in $\mathcal{W}(A)$. Now, since the elements of $\mathcal{D}_{P_0}(A)$ are not assumed to satisfy the *DC-Property*, $\mathcal{D}_{P_0}(A)$ may not be a Banach algebra. So we just have to consider the norm completion $\mathcal{D}_P(A)$ of $\mathcal{D}_{P_0}(A)$ to have our needed Banach algebra. The notation $\mathcal{D}_P(A)$, for the algebra just constructed, is motivated by the fact that we may think of its elements as bounded *Double Pre-centralizers* of A .

Proposition 2.8 below follows from the above comments and the previous results.

Proposition 2.8. *$\mathcal{W}(A)$ and $\mathcal{W}(\mathcal{A}_R)$ are Banach subalgebras of the Banach algebra $\mathcal{D}_P(A)$, with nonvoid intersection, since $\langle L, R \rangle \in \mathcal{W}(A) \cap \mathcal{W}(\mathcal{A}_R)$. Moreover, $\langle T, S \rangle \in \mathcal{W}(\mathcal{A}_R)$ if and only if $\langle LT, RS \rangle \in \mathcal{W}(A)$.* $\square$

N.B. Throughout the remainder of the sequel, and unless otherwise stated, A will always be a Banach algebra without annihilators and $\mathcal{A}_R$ will be the Banach algebra in Proposition 2.1, induced by some double centralizer $\langle L, R \rangle \in \mathcal{W}(A)$.

Given $a \in A$, let us set

$$(12) \quad \lambda_a^R(x) = R(a)x = \lambda_{R(a)}(x) = [(\lambda R)(a)](x), \quad \forall x \in A,$$

$$(13) \quad \mu_a^R(x) = R(x)a = xL(a) = [(\mu L)(a)](x), \quad \forall x \in A,$$

and let u_R be defined by $u_R(a) = \langle \lambda_a^R, \mu_a^R \rangle$, $\forall a \in \mathcal{A}_R$. Then we have:

Proposition 2.9. *Let $\langle L, R \rangle \in \mathcal{W}(A)$ define $\mathcal{W}(\mathcal{A}_R)$. Then*

(i) *u_R is the regular homomorphism of $\mathcal{A}_R$ into $\mathcal{W}(\mathcal{A}_R)$, and if there exists $z \in A$ such that $L(z) = R(z)$, we have in the algebra $\mathcal{D}_P(A)$:*

$$(14) \quad \left\{ \langle \lambda_z, \mu_z \rangle, \langle \lambda_z^R, \mu_z^R \rangle \right\} \subset \mathcal{W}(A) \cap \mathcal{W}(\mathcal{A}_R).$$

(ii) *If $\langle L, R \rangle$ is a multiplier ($T = L = R$), then $\mathcal{W}(A)$ is a Banach subalgebra of $\mathcal{W}(\mathcal{A}_R)$.*

Proof. (i) Short computations using the definition of the product in $\mathcal{W}(\mathcal{A}_R)$ show that $a \mapsto u_R(a) = \langle \lambda_a^R, \mu_a^R \rangle$ is a homomorphism into $\mathcal{W}(\mathcal{A}_R)$ and that λ_a^R and μ_a^R satisfy the DC-Property in $\mathcal{W}(\mathcal{A}_R)$. The regular homomorphism $u : a \mapsto \langle \lambda_a, \mu_a \rangle$, being from A into $\mathcal{W}(A)$, $\langle \lambda_z, \mu_z \rangle$ and $\langle \lambda_{R(z)}, \mu_{R(z)} \rangle$, belong to $\mathcal{W}(A)$, for each $z \in A$. Let us show that these two elements belong to $\mathcal{W}(\mathcal{A}_R)$, whenever $z \in A$ fulfills $L(z) = R(z)$. Indeed, for each $a \in A$, we have

$$(15) \quad x \cdot \lambda_a(y) = R(x)a y = xL(a)y, \quad \forall x, y \in A,$$

$$(16) \quad \mu_a(x) \cdot y = R(xa)y = xR(a)y, \quad \forall x, y \in A.$$

Hence, if $z \in A$ satisfies $L(z) = R(z)$, and since A is without annihilators, relations (15) and (16) above yield $\langle \lambda_z, \mu_z \rangle \in \mathcal{W}(\mathcal{A}_R)$. For z with the above property, we have

$$\begin{aligned} x \cdot \lambda_{R(z)}(y) &= R(x)R(z)y = xLR(z)y = xRL(z)y \quad (L \text{ and } R \text{ commute}) \\ &= R(xL(z))y = (xL(z)) \cdot y = (xR(z)) \cdot y \quad (L(z) = R(z)) \\ &= \mu_{R(z)}(x) \cdot y, \end{aligned}$$

that is, $\langle \lambda_{R(z)}, \mu_{R(z)} \rangle = \langle \lambda_z^R, \mu_z^R \rangle \in \mathcal{W}(\mathcal{A}_R)$, and the desired result, $\langle \lambda_z^R, \mu_z^R \rangle \in \mathcal{W}(\mathcal{A}_R) \cap \mathcal{W}(A)$, hence follows.

(ii) Let $\langle L, R \rangle \in \mathcal{W}(A)$ be such that $L = R$, so that R satisfies $aR(b) = R(a)b$ for all $a, b \in A$. Then given any $\langle T, S \rangle \in \mathcal{W}(A)$, we have

$$\begin{aligned} a \cdot T(b) &= R(a)T(b) = aR(T(b)) = R(aT(b)) \\ &= R(S(a)b) = R(S(a))b = S(a) \cdot b, \end{aligned}$$

which proves that $\langle T, S \rangle \in \mathcal{W}(\mathcal{A}_R)$. □

Remark 2.10. There exists a homomorphism φ from $\mathcal{W}(\mathcal{A}_R)$ into $\mathcal{W}(A)$ which makes the following diagram commutative:

$$\begin{array}{ccc} R : \mathcal{A}_R & \longrightarrow & \text{Im}(R) \subset A \\ u_R \downarrow & & \downarrow u \\ \varphi : \mathcal{W}(\mathcal{A}_R) & \longrightarrow & \mathcal{W}(A) \end{array}$$

where u and u_R denote respectively the regular homomorphisms of A into $\mathcal{W}(A)$ and of $\mathcal{A}_R$ into $\mathcal{W}(\mathcal{A}_R)$. Indeed, the homomorphisms uR and u_R are defined from $\mathcal{A}_R$, respectively into $\mathcal{W}(A)$ and $\mathcal{W}(\mathcal{A}_R)$, and satisfy $\text{Ker}(u_R) = \{0\} \subset \text{Ker}(uR)$. Then there exists a map $\varphi : \mathcal{W}(\mathcal{A}_R) \rightarrow \mathcal{W}(A)$ such that we have $\varphi u_R = uR$. Moreover, φ is clearly a homomorphism from $\mathcal{W}(\mathcal{A}_R)$ into $\mathcal{W}(A)$. $\square$

Under the hypotheses and notations of Remark 2.10, short computations lead to the following relations, for each $a \in A$, and each $\langle T, S \rangle \in \mathcal{W}(\mathcal{A}_R)$:

$$(17) \quad \langle T, S \rangle u_R(a) = \langle \lambda_{RT(a)}, \mu_{LT(a)} \rangle = \langle \lambda_{T(a)}^R, \mu_{T(a)}^R \rangle = u_R(T(a)),$$

$$(18) \quad u_R(a) \langle T, S \rangle = \langle \lambda_{RS(a)}, \mu_{LS(a)} \rangle = \langle \lambda_{S(a)}^R, \mu_{S(a)}^R \rangle = u_R(S(a)).$$

Relations (17) and (18) show that for each $a \in A$, φ is still defined on $\mathcal{W}(\mathcal{A}_R) u_R(a)$ and on $u_R(a) \mathcal{W}(\mathcal{A}_R)$ as follows:

$$(19) \quad \varphi(\langle T, S \rangle u_R(a)) = \varphi(u_R(T(a))) = u[RT(a)] = \langle \lambda_{RT(a)}, \mu_{RT(a)} \rangle,$$

$$(20) \quad \varphi(u_R(a) \langle T, S \rangle) = \varphi(u_R(S(a))) = u[RS(a)] = \langle \lambda_{RS(a)}, \mu_{RS(a)} \rangle.$$

The next result gives the conditions under which φ extends to the whole space $\mathcal{W}(\mathcal{A}_R)$.

Theorem 2.11. *If for each $\langle T, S \rangle \in \mathcal{W}(\mathcal{A}_R)$ there exists $\langle P, Q \rangle \in \mathcal{W}(A)$ such that the system below is satisfied:*

$$(*) \quad \begin{cases} P R = R T, \\ Q R = R S. \end{cases}$$

Then the map $\Theta : \mathcal{W}(\mathcal{A}_R) \rightarrow \mathcal{W}(A)$, defined by $\Theta(\langle T, S \rangle) = \langle P, Q \rangle$, extends the homomorphism $\varphi : \mathcal{W}(\mathcal{A}_R) \rightarrow \mathcal{W}(A)$. Furthermore, Θ is an isomorphism, provided R is one-to-one.

Proof. If φ extends to the whole of $\mathcal{W}(\mathcal{A}_R)$, then for each $a \in A$ and each $\langle T, S \rangle \in \mathcal{W}(\mathcal{A}_R)$, the element $\langle P, Q \rangle = \varphi(\langle T, S \rangle)$ would satisfy, according to (19) and (20),

$$\begin{aligned} \varphi(\langle T, S \rangle u_R(a)) &= \langle P, Q \rangle \varphi(u_R(a)) = \langle P, Q \rangle \langle \lambda_{R(a)}, \mu_{R(a)} \rangle \\ &= \langle \lambda_{RT(a)}, \mu_{RT(a)} \rangle, \\ \varphi(u_R(a) \langle T, S \rangle) &= \varphi(u_R(a)) \langle P, Q \rangle = \langle \lambda_{R(a)}, \mu_{R(a)} \rangle \langle P, Q \rangle \\ &= \langle \lambda_{RS(a)}, \mu_{RS(a)} \rangle. \end{aligned}$$

But we have

$$(21) \quad \langle P, Q \rangle \langle \lambda_{R(a)}, \mu_{R(a)} \rangle = \langle P \lambda_{R(a)}, \mu_{R(a)} Q \rangle = \langle \lambda_{PR(a)}, \mu_{PR(a)} \rangle,$$

$$(22) \quad \langle \lambda_{R(a)}, \mu_{R(a)} \rangle \langle P, Q \rangle = \langle \lambda_{R(a)} P, Q \mu_{R(a)} \rangle = \langle \lambda_{QR(a)}, \mu_{QR(a)} \rangle.$$

Hence, if φ extends to $\mathcal{W}(\mathcal{A}_R)$, then we must have

$$\begin{cases} \langle \lambda_{RT(a)}, \mu_{RT(a)} \rangle = \langle \lambda_{PR(a)}, \mu_{PR(a)} \rangle, \\ \langle \lambda_{RS(a)}, \mu_{RS(a)} \rangle = \langle \lambda_{QR(a)}, \mu_{QR(a)} \rangle, \end{cases}$$

from which the system $(*)$ follows. So, if for each $\langle T, S \rangle \in \mathcal{W}(\mathcal{A}_R)$ there exists $\langle P, Q \rangle \in \mathcal{W}(A)$ such that the system $(*)$ is satisfied, we define the map $\Theta : \mathcal{W}(\mathcal{A}_R) \longrightarrow \mathcal{W}(A)$ by

$$(23) \quad \Theta(\langle T, S \rangle) = \langle P, Q \rangle.$$

The restriction of Θ coincides with φ on $u_R(\mathcal{A}_R)$ since for each $u_R(a) = \langle \lambda_{R(a)}, \mu_{R(a)} \rangle$, the element $\langle P, Q \rangle = \langle \lambda_{R(a)}, \mu_{R(a)} \rangle \in \mathcal{W}(A)$ clearly fulfills the system $(*)$.

We need to show that Θ as defined above is a homomorphism. So let $\langle T, S \rangle$ and $\langle T', S' \rangle$ belong to $\mathcal{W}(\mathcal{A}_R)$, and let $\langle P, Q \rangle$ and $\langle P', Q' \rangle$ be elements of $\mathcal{W}(A)$ satisfying

$$(24) \quad \Theta(\langle T, S \rangle) = \langle P, Q \rangle \quad \text{and} \quad \Theta(\langle T', S' \rangle) = \langle P', Q' \rangle.$$

Then $\langle T T', S' S \rangle \in \mathcal{W}(\mathcal{A}_R)$, $\langle P P', Q' Q \rangle \in \mathcal{W}(A)$, and we have

$$\begin{aligned} (P P') R &= P (P' R) = P (R T') = (P R) T' = R (T T') , \\ (Q' Q) R &= Q' (Q R) = Q' (R S) = (Q' R) S = R (S' S) . \end{aligned}$$

So, $\langle P P', Q' Q \rangle \in \mathcal{W}(A)$ fulfills the system $(*)$. Therefore, by definition of Θ , we get

$$\begin{aligned} \Theta(\langle T T', S' S \rangle) &= \langle P P', Q' Q \rangle = \langle P, Q \rangle \langle P', Q' \rangle \\ &= \Theta(\langle T, S \rangle) \Theta(\langle T', S' \rangle) \end{aligned}$$

and Θ is hence a homomorphism from $\mathcal{W}(\mathcal{A}_R)$ into $\mathcal{W}(A)$, as desired.

Now, if $\langle T, S \rangle \in \mathcal{W}(\mathcal{A}_R)$ is such that $\langle P, Q \rangle = \Theta(\langle T, S \rangle) = 0$, then by virtue of the system $(*)$, the following holds: $\text{Im}(T) \cup \text{Im}(S) \subset \text{Ker}(R)$, and if R is a one-to-one map, then T and S must be identically null. $\square$

We next deal with the uniqueness of the above extension of the homomorphism φ .

Theorem 2.12. *Let us assume that there exists another extension Σ of the homomorphism φ to the entire $\mathcal{W}(\mathcal{A}_R)$. Then, Σ “fulfills” the system $(*)$, and in the particular case where R is bijective, there exists a unique way of extending φ to $\mathcal{W}(\mathcal{A}_R)$.*

Proof. Σ must coincide with φ on the image $u_R(\mathcal{A}_R)$ of the regular homomorphism from $\mathcal{A}_R$ into $\mathcal{W}(\mathcal{A}_R)$, that is,

$$(25) \quad \Sigma(u_R(a)) = \Sigma(\langle \lambda_{R(a)}, \mu_{L(a)} \rangle) = \langle \lambda_{R(a)}, \mu_{R(a)} \rangle,$$

and since $u_R(A)$ is a two-sided ideal in $\mathcal{W}(\mathcal{A}_R)$, Σ must also satisfy

$$(26) \quad \Sigma(\langle T, S \rangle u_R(a)) = \Sigma(\langle \lambda_{RT(a)}, \mu_{LT(a)} \rangle) = \langle \lambda_{RT(a)}, \mu_{RT(a)} \rangle,$$

$$(27) \quad \Sigma(u_R(a) \langle T, S \rangle) = \Sigma(\langle \lambda_{RS(a)}, \mu_{LS(a)} \rangle) = \langle \lambda_{RS(a)}, \mu_{RS(a)} \rangle.$$

So, if $\Sigma(\langle T, S \rangle) = \langle U, V \rangle$, then we must have

$$\begin{aligned} \Sigma(\langle T, S \rangle u_R(a)) &= \langle U, V \rangle \langle \lambda_{R(a)}, \mu_{R(a)} \rangle = \langle \lambda_{UR(a)}, \mu_{UR(a)} \rangle \\ &= \langle \lambda_{RT(a)}, \mu_{RT(a)} \rangle, \\ \Sigma(u_R(a) \langle T, S \rangle) &= \langle \lambda_{R(a)}, \mu_{R(a)} \rangle \langle U, V \rangle = \langle \lambda_{VR(a)}, \mu_{VR(a)} \rangle \\ &= \langle \lambda_{RS(a)}, \mu_{RS(a)} \rangle. \end{aligned}$$

The above relations show that $\Sigma(\langle T, S \rangle) = \langle U, V \rangle$ satisfies the system $(*)$ as well, and therefore the following system $(**)$ is also fulfilled by $\langle P, Q \rangle$ and $\langle U, V \rangle$:

$$(**) \quad \begin{cases} P R = U R, \\ Q R = V R. \end{cases}$$

But the conditions in $(**)$ above may also be interpreted in the following way, in terms of restrictions of maps: $P|_{Im(R)} = U|_{Im(R)}$ and $Q|_{Im(R)} = V|_{Im(R)}$, which, in the case where R is bijective (and hence onto A), gives

$$(28) \quad \Theta(\langle T, S \rangle) = \langle P, Q \rangle = \langle U, V \rangle = \Sigma(\langle T, S \rangle),$$

that is, $\Theta = \Sigma$, which proves the uniqueness of the extension of φ . In such a case where the map R is bijective, R admits a well-defined inverse $R^{-1} \in B(A)$ which is a right centralizer, and therefore, the system $(*)$ gives, for $\langle P, Q \rangle = \Theta(\langle T, S \rangle)$,

$$(29) \quad P = R T R^{-1} \quad \text{and} \quad Q = R S R^{-1},$$

which completes the proof. $\square$

Let us denote by $W^R(A)$, the normed algebra of bounded double centralizers of $Im(R) \subseteq A$. Then it is clear that $\mathcal{W}(A) \subseteq W^R(A)$, and that equality holds whenever R is onto A . The next result links $W^R(A)$ with the space of elements $P, Q \in B(A)$ which satisfy the system $(*)$ with the elements of $\mathcal{W}(\mathcal{A}_R)$.

Proposition 2.13. *Let $P, Q \in B(A)$ satisfy the system $(*)$ with some $\langle T, S \rangle \in \mathcal{W}(\mathcal{A}_R)$. Then $\langle P, Q \rangle \in W^R(A)$.*

Proof. For all $a, b \in Im(R)$, we have $\alpha, \beta \in A$, such that $a = R(\alpha)$; $b = R(\beta)$, and then, for each $\langle P, Q \rangle \in B(A)$ satisfying the system $(*)$ with some $\langle T, S \rangle \in \mathcal{W}(\mathcal{A}_R)$, the following holds:

$$\begin{aligned} a P(b) &= R(\alpha) (P R)(\beta) = R(\alpha) (R T)(\beta) \quad (\text{since } (*) \text{ holds}) \\ &= R(R(\alpha) T(\beta)) \\ &= R(\alpha \cdot T(\beta)) = R(S(\alpha) \cdot \beta) \quad (\text{since } \langle T, S \rangle \in \mathcal{W}(\mathcal{A}_R)) \\ &= R(R S(\alpha) \beta) \\ &= R S(\alpha) R(\beta) = Q R(\alpha) R(\beta) \quad (\text{since } (*) \text{ holds}) \\ &= Q(a) b, \end{aligned}$$

which is the *DC-Property* in $W^R(A)$, and the proof is complete. $\square$

As is well known, the theory of double centralizers is a helpful device in the study of extensions of algebras. Let us recall the following definition (cf. [4], pp. 33 - 34).

Definition 2.14. Let A and B be Banach algebras. An *extension of A by C* is a short exact sequence $0 \longrightarrow A \xrightarrow{\rho} B \xrightarrow{\psi} C \longrightarrow 0$ of Banach algebras. The extension is called a *semidirect product* if there is a continuous homomorphism $\chi : C \longrightarrow B$, such that $\psi \chi$ is the identity map on C . The short exact sequence is then said to *split*, with splitting homomorphism χ .

Theorem 2.15. Let A be a Banach algebra with a left approximate identity bounded by one. Let $\mathcal{A}_R$ be induced by $\langle L, R \rangle \in \mathcal{W}(A)$ with R bijective. Then there exists an extension of A by $\mathcal{W}(\mathcal{A}_R)$:

$$(30) \quad 0 \longrightarrow A \xrightarrow{\rho} B \xrightarrow{\psi} \mathcal{W}(\mathcal{A}_R) \longrightarrow 0,$$

and a continuous homomorphism $\Theta : B \longrightarrow \mathcal{W}(A)$, such that the following diagram commutes:

$$\begin{array}{ccccccc} 0 & \longrightarrow & A & \xrightarrow{u} & \mathcal{W}(A) & \xrightarrow{\pi} & \mathcal{W}(A)/A \longrightarrow 0 \\ & & \uparrow I_A & & \uparrow \Theta & & \uparrow \tau \\ 0 & \longrightarrow & A & \xrightarrow{\rho} & B & \xrightarrow{\psi} & \mathcal{W}(\mathcal{A}_R) \longrightarrow 0. \end{array}$$

This extension, which is unique up to equivalence, is moreover a semidirect product.

Proof. Identifying A with $u(A)$ under the regular homomorphism u of A into $\mathcal{W}(A)$, consider the short exact sequence $0 \longrightarrow A \xrightarrow{u} \mathcal{W}(A) \xrightarrow{\pi} C \longrightarrow 0$, where π is the natural map. Since R is bijective, Theorem 2.12 asserts that Θ is a homeomorphic isomorphism of $\mathcal{W}(\mathcal{A}_R)$ onto $\mathcal{W}(A)$. The map $\tau : \mathcal{W}(\mathcal{A}_R) \longrightarrow \mathcal{W}(A)/A$, defined by

$$(31) \quad \tau(\langle T, S \rangle) = \Theta(\langle T, S \rangle) + A, \quad \forall \langle T, S \rangle \in \mathcal{W}(\mathcal{A}_R),$$

is hence a continuous homomorphism satisfying: $\tau = \pi \Theta$. Then, the hypotheses of Theorem 1.2.10 in [4], page 34, are satisfied, and its conclusion provides the existence of the extension of A given by relation (30), and unique up to equivalence. Furthermore, in virtue of Theorem 1.2.11 in [4], page 35, the existence of τ satisfying $\tau = \pi \Theta$ shows that the extension is a semidirect product. $\square$

Now, let us assume that A is *commutative*, and let us denote by $\mathcal{M}(A)$ the algebra of multipliers of A , which necessarily coincides with $\mathcal{W}(A)$. In [3], Corollary 1.3.1, page 23, R. Larsen proved that for a *commutative semisimple* Banach algebra A , the following equality holds: $\mathcal{M}(A) = \mathcal{M}(\mathcal{A}_R)$. In what follows, we still obtain the same result when the assumption of semisimplicity on A is removed. We shall need the following:

Lemma 2.16. Let A be a commutative Banach algebra and let $R \in \mathcal{M}(A)$. Then the following conditions are equivalent:

- (i) $\mathcal{A}_R$ has no annihilators;
- (ii) A has no annihilators and R is one-to-one.

Proof. Assume that (i) is fulfilled, and let $a \in A$ be given. Then, A being commutative and R being a multiplier of A , we get for all $x \in A$:

$$(32) \quad R(xa) = R(ax) = R(a)x = a \cdot x = x \cdot a \quad (\mathcal{A}_R \text{ is commutative}),$$

which, since $\mathcal{A}_R$ is without annihilators, implies that A has no annihilators. Moreover, due to Corollary 2.3, R is one-to-one, if $\mathcal{A}_R$ has no annihilators. Hence (i) implies (ii).

Conversely if (ii) is satisfied, then Corollary 2.3 again implies that $\mathcal{A}_R$ has no annihilators, and the equivalence of (i) and (ii) is therefore established. $\square$

We can now prove:

Theorem 2.17. *Let A be a commutative Banach algebra and let $R \in \mathcal{M}(A)$. Then $\mathcal{M}(A) = \mathcal{M}(\mathcal{A}_R)$ if and only if one of the equivalent conditions in Lemma 2.16 is satisfied.*

Proof. We assume that condition (i) of Lemma 2.16 holds. Let $T \in \mathcal{M}(A)$; then for all $a, b \in A$, we get $T(a \cdot b) = T[R(a) b] = R(a) T(b) = a \cdot T(b)$ and

$$\begin{aligned} T(a \cdot b) &= R(a) T(b) = T(b) R(a) = R[T(b) a] = R[b T(a)] \\ &= R(T(a) b) = R[T(a)] b = T(a) \cdot b, \end{aligned}$$

which means that $T \in \mathcal{M}(A)$, and consequently, $\mathcal{M}(A) \subseteq \mathcal{M}(\mathcal{A}_R)$.

Conversely, let $T \in \mathcal{M}(\mathcal{A}_R)$. Then, for all $a, b, x \in A$, we get on one hand

$$\begin{aligned} R(x) T(a b) &= x \cdot T(a b) = T(x) \cdot a b = a b T(x) = R(ab) T(x) \\ &= [R(a) b] T(x) = R(a) T(x) b = [a \cdot T(x)] b = [T(a) \cdot x] b \\ &= (R[T(a)] x) = T(a) R(x) b = R(x) [T(a) b], \end{aligned}$$

which implies, since $\mathcal{A}_R$ is without annihilators, $T(a b) = T(a) b$.

On the other hand, let $a, b, x \in A$; then

$$R(x) T(a b) = R(a b) T(x) = a [b \cdot T(x)] = a [T(b) R(x)] = R(x) [a T(b)],$$

and once again, the fact that $\mathcal{A}_R$ is without annihilators, leads to $T(a b) = a T(b)$. Therefore, $T \in \mathcal{M}(A)$, and this provides the reverse inclusion $\mathcal{M}(\mathcal{A}_R) \subseteq \mathcal{M}(A)$, and hence the desired equality $\mathcal{M}(\mathcal{A}_R) = \mathcal{M}(A)$. $\square$

The following result also appears in [3], Corollary 1.3.2, page 23:

Proposition 2.18. *Let A be a commutative semisimple Banach algebra which admits factorization, i.e.: For every $a \in A$, there exist $x, y \in A$, such that $a = xy$. Then the following conditions are equivalent:*

- (i) A_T admits factorization;
- (ii) T is invertible.

Dropping the assumption of semisimplicity on A , we can prove the following:

Theorem 2.19. *Let A be a commutative Banach algebra which admits factorization, and let $R \in \mathcal{M}(A)$ be such that $\mathcal{A}_R$ is without annihilators. Then $\mathcal{A}_R$ admits factorization, if and only if R is invertible.*

Proof. Assume that $\mathcal{A}_R$ admits factorization. Then, since $\mathcal{A}_R$ is without annihilators, R is one-to-one, according to Lemma 2.16. Moreover, since $\mathcal{A}_R$ admits factorization, for each $a \in \mathcal{A}_R$ there exist $\alpha, \beta \in \mathcal{A}_R$, such that $a = \alpha \cdot \beta = R(\alpha) \beta = R(\alpha \beta)$, which means that R is onto A , whence R is bijective and R^{-1} exists.

Conversely, let R be invertible; then R is one-to-one, and since it is also onto A , we get for each $a \in A$ some $\alpha \in \mathcal{A}_R$ satisfying $a = R(\alpha)$. But, A admits factorization, and therefore α has the following decomposition: $\alpha = u v$, $u, v \in A$.

Hence we get $a = R(\alpha) = R(uv) = R(u)v = u \cdot v$, so that $\mathcal{A}_R$ also admits factorization, and the proof is complete. $\square$

Coming back to the general case where A is neither commutative nor semisimple, we give the analogous version of Theorem 2.17, when the multiplier algebra is replaced by the double centralizer algebra, as follows:

Theorem 2.20. *Let A be without annihilators, and let $\langle L, R \rangle \in \mathcal{W}(A)$ define $\mathcal{A}_R$ and be such that R is bijective. Then $\mathcal{W}(\mathcal{A}_R)$ and $\mathcal{W}(A)$ are homeomorphically isomorphic.*

Proof. According to Theorems 2.11 and 2.12, there exists a homomorphism Θ which extends in a unique way the homomorphism φ constructed in Remark 2.10 to the algebra $\mathcal{W}(\mathcal{A}_R)$. Moreover, according to Theorem 2.11, each element $\langle T, S \rangle \in \text{Ker}(\Theta)$ satisfies $\text{Im}(T) \cup \text{Im}(S) \subset \text{Ker}(R)$. So, if R is bijective and hence one-to-one, $\text{Ker}(R) = \{0\}$, and Θ is also one-to-one. Therefore, Θ is a bijection of $\mathcal{W}(\mathcal{A}_R)$ onto $\mathcal{W}(A)$. But, according to relation (29), the positive constant $K_R = \|R^{-1}\|\|R\|$ (depending only on R) satisfies $\|\Theta(\langle T, S \rangle)\| \leq K_R \|\langle T, S \rangle\|$; $\forall \langle T, S \rangle \in \mathcal{W}(\mathcal{A}_R)$, so that Θ is a bounded linear map from $\mathcal{W}(\mathcal{A}_R)$ onto the Banach space $\mathcal{W}(A)$, that is, a homeomorphism, and the proof is complete. $\square$

REFERENCES

1. Desquith, E.: Banach algebras associated to bounded module maps; Preprint ICTP N^o IC 98/194; (1998).
2. Doran, R.S. & Fell, J.M.G.: Representations of $*$ -algebras, locally compact groups, and Banach $*$ -algebraic Bundles, Vol. 2; Academic Press (1988). MR **90c**:46002
3. Larsen, R.: Introduction to the theory of multipliers; Springer-Verlag (1971). MR **55**:8695
4. Palmer, T.W.: Banach algebras and the general theory of Banach $*$ -algebras, Vol. I: Algebras and Banach algebras; Cambridge University Press (1994). MR **95c**:46002

INSTITUT DE RECHERCHES MATHÉMATIQUES (IRMA), 08 BP 2030 ABIDJAN 08, IVORY COAST
E-mail address: desquith@hotmail.com