BEHAVIOR OF THE BERGMAN KERNEL AND METRIC NEAR CONVEX BOUNDARY POINTS

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ABSTRACT. The boundary behavior of the Bergman metric near a convex boundary point z_0 of a pseudoconvex domain $D \subset \mathbb{C}^n$ is studied. It turns out that the Bergman metric at points $z \in D$ in the direction of a fixed vector $X_0 \in \mathbb{C}^n$ tends to infinity, when z is approaching z_0 , if and only if the boundary of D does not contain any analytic disc through z_0 in the direction of X_0 .

For a domain $D \subset \mathbb{C}^n$ we denote by $L_h^2(D)$ the Hilbert space of all holomorphic functions f that are square-integrable and by $\|f\|_D$ the L_2 -norm of f . Let $K_D(z)$ be the restriction on the diagonal of the Bergman kernel function of D . It is well known (cf. [5]) that

$$K_D(z) = \sup\{|f(z)|^2 : f \in L_h^2(D), \|f\|_D \leq 1\}.$$

If $K_D(z) > 0$ for some point $z \in D$, then the Bergman metric $B_D(z; X)$, $X \in \mathbb{C}^n$, is well defined and can be given by the equality

$$B_D(z; X) = \frac{M_D(z; X)}{\sqrt{K_D(z)}},$$

where

$$M_D(z; X) = \sup\{|f'(z)X| : f \in L_h^2(D), \|f\|_D = 1, f(z) = 0\}.$$

We say that a boundary point z_0 of a domain $D \subset \mathbb{C}^n$ is *convex* if there is a neighborhood U of this point such that $D \cap U$ is convex.

In [4], Herbort proved the following

Theorem 1. *Let z_0 be a convex boundary point of a bounded pseudoconvex domain $D \subset \mathbb{C}^n$ whose boundary contains no nontrivial germ of an analytic curve near z_0 . Then*

$$\lim_{z \rightarrow z_0} B_D(z; X) = \infty$$

for any $X \in \mathbb{C}^n \setminus \{0\}$.

Herbort's proof is mainly based on Ohsawa's $\bar{\partial}$ -technique. The main purpose of this note is to generalize Theorem 1 using more elementary methods.

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For a convex boundary point z_0 of a domain $D \subset \mathbb{C}^n$ we denote by $L(z_0)$ the set of all $X \in \mathbb{C}^n$ for which there exists a number $\varepsilon_X > 0$ such that $z_0 + \lambda X \in \partial D$ for all complex numbers λ , $|\lambda| \leq \varepsilon_X$. Note that $L(z_0)$ is a complex linear space.

Then our result is the following one.

Theorem 2. *Let z_0 be a convex boundary point of a bounded pseudoconvex domain $D \subset \mathbb{C}^n$ and let $X \in \mathbb{C}^n$. Then*

- (a) $\liminf_{z \rightarrow z_0} K_D(z) \operatorname{dist}^2(z, \partial D) \in (0, \infty]$;
- (b) $\lim_{z \rightarrow z_0} B_D(z; X) = \infty$ if and only if $X \notin L(z_0)$. Moreover, this limit is locally uniform in $X \notin L(z_0)$;
- (c) if $L(z_0) = \{0\}$, then (a) and (b) are still true without the assumption that D is bounded.

Proof of Theorem 2. To prove (a) and (b) we will use the following localization theorem for the Bergman kernel and metric [2].

Theorem 3. *Let $D \subset \mathbb{C}^n$ be a bounded pseudoconvex domain and let $V \subset\subset U$ be open neighborhoods of a point $z_0 \in \partial D$. Then there exists a constant $\tilde{C} \geq 1$ such that*

$$\begin{aligned} \tilde{C}^{-1} K_{D \cap U}(z) &\leq K_D(z) \leq K_{D \cap U}(z), \\ \tilde{C}^{-1} B_{D \cap U}(z; X) &\leq B_D(z; X) \leq \tilde{C} B_{D \cap U}(z; X) \end{aligned}$$

for any $z \in D \cap V$ and any $X \in \mathbb{C}^n$. (Here $K_{D \cap U}(z)$ and $B_{D \cap U}(z; \cdot)$ denote the Bergman kernel and metric of the connected component of $D \cap U$ that contains z .)

So, we may assume that D is convex.

To prove part (a) of Theorem 2, for any $z \in D$ we choose a point $\tilde{z} \in \partial D$ such that $\|z - \tilde{z}\| = \operatorname{dist}(z, \partial D)$. We denote by l the complex line through z and $\tilde{z}$. By the Oshawa-Takegoshi extension theorem for L^2 -holomorphic functions [7], it follows that there exists a constant $C_1 > 0$ only depending on the diameter of D (not on l) such that

$$(1) \quad K_D(z) \geq C_1 K_{D \cap l}(z).$$

Since $D \cap l$ is convex, it is contained in an open half-plane Π of the l -plane with $\tilde{z} \in \partial \Pi$. Then

$$(2) \quad K_{D \cap l}(z) \geq K_\Pi(z) = \frac{1}{4\pi \operatorname{dist}^2(z, \partial \Pi)}.$$

Now, part (a) of Theorem 2 follows from the inequalities (1), (2) and the fact that $\operatorname{dist}(z, \partial \Pi) \leq \|z - \tilde{z}\| = \operatorname{dist}(z, \partial D)$.

To prove part (b) of Theorem 2, we denote by $N(z_0)$ the complex affine space through z_0 that is orthogonal to $L(z_0)$. Set $E(z_0) = D \cap N(z_0)$. Note that $E(z_0)$ is a nonempty convex set. So, part (b) of Theorem 2 will be a consequence of the following:

Theorem 4. *Let z_0 be a boundary point of a bounded convex domain $D \subset \mathbb{C}^n$. Then:*

- (i) $\lim_{z \rightarrow z_0} B_D(z; X) = \infty$ locally uniformly in $X \notin L(z_0)$;

(ii) for any compact set $K \subset\subset E(z_0)$ there exists a constant $C > 0$ such that

$$B_D(z; X) \leq C\|X\|, \quad z \in K^0, \quad X \in L(z_0),$$

where $K^0 := \{z_0 + tz : z \in K, 0 < t \leq 1\}$ is the cone generated by K .

Proof of Theorem 4. To prove (i) we will use the well-known fact that the Carathéodory metric $C_D(z; X)$ of D does not exceed $B_D(z; X)$. On the other hand, we have the following simple geometric inequality [1]:

$$C_D(z; X) \geq \frac{1}{2d(z; X)},$$

where $d(z; X)$ denotes the distance from z to the boundary of D in the X -direction, i.e., $d(z; X) := \sup\{r : z + \lambda X \in D, \lambda \in \mathbb{C}, |\lambda| < r\}$. So, if we assume that (i) does not hold, then we may find a number $a > 0$ and sequences $D \supset (z_j)_j$, $z_j \rightarrow z_0$, $\mathbb{C}^n \supset (X_j)_j$, $X_j \rightarrow X \notin L(z_0)$, such that $B_D(z_j; X_j) \leq \frac{1}{2a}$. Hence $d(z_j; X_j) \geq a$ which implies that for $|\lambda| \leq a$ the points $z_0 + \lambda X$ belong to $\bar{D}$ and, in view of convexity, they belong to ∂D . This means that $X \in L(z_0)$, a contradiction.

To prove part (ii) of Theorem 4, we may assume that $z_0 = 0$ and $L := L(0) = \{z \in \mathbb{C}^n : z_1 = \dots = z_k = 0\}$ for some $k < n$. Then $N := N(0) = \{z \in \mathbb{C}^n : z_{k+1} = \dots = z_n = 0\}$. From now on we will write any point $z \in \mathbb{C}^n$ in the form $z = (z', z'')$, $z' \in \mathbb{C}^k$, $z'' \in \mathbb{C}^{n-k}$. Note that $L \in \partial D$ near 0, i.e., there exists a $c > 0$ such that

$$(3) \quad \{0'\} \times \Delta_c'' \in \partial D,$$

where $\Delta_c'' \subset \mathbb{C}^{n-k}$ is the polydisc with center at the origin and radius c . Since $K \subset\subset E := E(0)$ and since E is convex, there exists an $\alpha > 1$ such that $K \subset\subset E_\alpha$, where $E_\alpha := \{z : \alpha z \in E\}$. Note that $K^0 \subset E_\alpha$. Using (3), the equality

$$(z', z'') = \frac{1}{\alpha}(\alpha z', 0'') + (1 - \frac{1}{\alpha})(0', (1 - \frac{1}{\alpha})^{-1}z''),$$

and the convexity of D , it follows that

$$(4) \quad F_\alpha \times \Delta_\varepsilon'' \subset D,$$

where $\varepsilon := c(1 - \frac{1}{\alpha})$ and where F_α is the projection of E_α in $\mathbb{C}^k$ (we can identify E_α with F_α). For $\delta := c(\alpha - 1)$ we obtain in the same way that

$$(5) \quad \tilde{D} := D \cap (\mathbb{C}^k \times \Delta_\delta'') \subset F_{\frac{1}{\alpha}} \times \Delta_\delta''.$$

Now, let $(z, X) \in K^0 \times L$. Note that $z = (z', 0'')$ and $X = (0', X'')$. Then, using (4) and the product properties of the Bergman kernel and metric, we have

$$(6) \quad \begin{aligned} M_D(z; X) &\leq M_{F_\alpha \times \Delta_\varepsilon''}(z; X) \\ &= M_{\Delta_\varepsilon''}(0''; X'') \sqrt{K_{F_\alpha}(z')} \leq C_1 \|X\| \sqrt{K_{F_\alpha}(z')} \end{aligned}$$

for some constant $C_1 > 0$. On the other hand, since $K^0 \subset\subset \mathbb{C}^k \times \Delta_\delta''$, by virtue of Theorem 3 there exists a constant $\tilde{C} \geq 1$ such that

$$K_D(z) \geq \tilde{C}^{-1} K_{\tilde{D}}(z).$$

Moreover, in view of (5), we have

$$K_{\tilde{D}}(z) \geq K_{F_{\frac{1}{\alpha}}}(z') K_{\Delta_\delta''}(0'')$$

and hence

$$(7) \quad K_D(z) \geq (C_2)^2 K_{F_{\frac{1}{\alpha}}}(z')$$

for some constant $C_2 > 0$. Now, by (6) and (7), it follows that

$$(8) \quad B_D(z; X) = \frac{M_D(z; X)}{\sqrt{K_D(z)}} \leq \frac{C_1}{C_2} \|X\| \sqrt{\frac{K_{F_\alpha}(z')}{K_{F_{\frac{1}{\alpha}}}(z')}}.$$

Note that $z' \rightarrow \alpha^{-2}z'$ is a biholomorphic mapping from $F_{\frac{1}{\alpha}}$ onto F_α and, therefore,

$$(9) \quad K_{F_{\frac{1}{\alpha}}}(z') = \alpha^{-4k} K_{F_\alpha}(\alpha^{-2}z').$$

In view of (8) and (9), in order to finish (ii) we have to find a constant $C_3 > 0$ such that

$$(10) \quad K_{F_\alpha}(z') \leq C_3 K_{F_\alpha}(\alpha^{-2}z')$$

for any $z' \in H^0$ with $H^0 := \{tz' : z' \in H, 0 < t \leq 1\}$, where H is the projection of K into $\mathbb{C}^k$ (we can identify K with H).

To do this, note first that $\gamma := \text{dist}(H, \partial F_\alpha) > 0$ since $K \subset\subset E_\alpha$. Fix $\tau \in (0, 1]$ and $z' \in H^0$, and denote by $T_{\tau, z'}$ the translation that maps the origin in the point $\tau z'$. It is easy to check that

$$(11) \quad T_{\tau, z'}(\bar{F}_\alpha \cap B_\gamma) \subset F_\alpha,$$

where B_γ is the ball in $\mathbb{C}^k$ with center at the origin and radius γ . To prove (10), we will consider the following two cases:

Case I. $z' \in H^0 \setminus B_{\frac{\gamma}{2}} \subset\subset F_\alpha$: Then

$$(12) \quad K_{F_\alpha}(z') \leq \frac{m_1}{m_2} K_{F_\alpha}(\alpha^{-2}z'),$$

where $m_1 := \sup_{H^0 \setminus B_{\frac{\gamma}{2}}} K_{F_\alpha}$ and $m_2 := \inf K_{F_\alpha}$.

Case II. $z' \in H^0 \cap B_{\frac{\gamma}{2}}$: By Theorem 3 there exists a constant $\tilde{C}_3 \geq 1$ such that $\tilde{C}_3 K_{F_\alpha} \geq K_{F_\alpha \cap B_\gamma}$ on $F_\alpha \cap B_{\frac{\gamma}{2}}$. In particular,

$$(13) \quad \tilde{C}_3 K_{F_\alpha}(\alpha^{-2}z') \geq K_{F_\alpha \cap B_\gamma}(\alpha^{-2}z').$$

On the other hand, by (11) with data $T := T_{1-\alpha^{-2}, z'}$ it follows that

$$(14) \quad K_{F_\alpha \cap B_\gamma}(\alpha^{-2}z') = K_{T(F_\alpha \cap B_\gamma)}(z') \geq K_{F_\alpha}(z').$$

Now, (12), (13), and (14) imply that (10) holds for $C_3 := \max\{\frac{m_1}{m_2}, \tilde{C}_3\}$. This completes the proofs of Theorem 4 and part (b) of Theorem 2. $\square$

Remark. The approximation (5) of the domain $D \cap (\mathbb{C}^k \times \Delta_\delta'')$ by the domain $E_{\frac{1}{\alpha}} \times \Delta_\delta''$ can be replaced by using the Oshawa-Takegoshi theorem [7] with the data D and N .

Finally, part (c) of Theorem 2 will be a consequence of the following two theorems.

Theorem 5 ([6]). *Let $D \subset \mathbb{C}^n$ be a pseudoconvex domain and let U be an open neighborhood of a local (holomorphic) peak point $z_0 \in \partial D$. Then*

$$\lim_{z \rightarrow z_0} \frac{K_D(z)}{K_{D \cap U}(z)} = 1$$

and

$$\lim_{z \rightarrow z_0} \frac{B_D(z; X)}{B_{D \cap U}(z; X)} = 1$$

locally uniformly in $X \in \mathbb{C}^n \setminus \{0\}$.

Theorem 6. *Let z_0 be a boundary point of a bounded convex domain $D \subset \mathbb{C}^n$. Then the following conditions are equivalent:*

- (1) z_0 is a (holomorphic) peak point;
- (2) z_0 is the unique analytic curve in $\bar{D}$ containing z_0 ;
- (3) $L(z_0) = \{0\}$.

Note that the only nontrivial implication is (3) $\implies$ (1). It is contained in [8]. Now, part (c) of Theorem 2 is a consequence of this implication, Theorem 5, and part (b) of Theorem 2.

Proof of Theorem 6. The implication (2) $\implies$ (3) is trivial.

The implication (1) $\implies$ (2) easily follows by the maximum principle and the fact that there are a neighborhood U of z_0 and a vector $X \in \mathbb{C}^n$ such that $(\bar{D} \cap U) + (0, 1]X \subset D$ (cf. (11)).

Denote by $A^0(D)$ the algebra of holomorphic functions on D which are continuous on $\bar{D}$. Now, following [8] we shall prove the implication (3) $\implies$ (1); namely, (3) implies that z_0 is a peak point with respect to $A^0(D)$. This is equivalent to the fact (cf. [3]) that the point mass at z_0 is the unique element of the set $A(z_0)$ of all representing measures for z_0 with respect to $A^0(D)$, i.e. $\text{supp } \mu = \{z_0\}$ for any $\mu \in A(z_0)$.

Let $\mu \in A(z_0)$. Since D is convex, we may assume that $z_0 = 0$ and $D \subset \{z \in \mathbb{C}^n : \text{Re}(z_1) < 0\}$. Note that if a is a positive number such that $a \inf_{z \in D} \text{Re}(z_1) > -1$ (D is bounded), then the function $f_1(z) = \exp(z_1 + az_1^2)$ belongs to $A^0(D)$ and $|f_1(z)| < 1$ for $z \in \bar{D} \setminus \{z_1 = 0\}$. This easily implies (cf. [3]) that $\text{supp } \mu \subset D_1 := \partial D \cap \{z_1 = 0\}$. Since $L(0) = 0$, the origin is a boundary point of the compact convex set D_1 . As above, we may assume that $D_1 \subset \{z \in \mathbb{C}^n : \text{Re}(z_2) \leq 0\}$ (z_2 is independent of z_1) and then construct a function $f_2 \in A^0(D)$ such that $|f_2(z)| < 1$ for $z \in D_1 \setminus \{z_2 = 0\}$. This implies that $\text{supp } \mu \subset D_1 \cap \{z_2 = 0\}$. Repeating this argument we conclude that $\text{supp } \mu = \{0\}$, which completes the proofs of Theorems 6 and 2. $\square$

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