IRREGULAR GABOR FRAMES AND THEIR STABILITY

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ABSTRACT. In this paper we give sufficient conditions for irregular Gabor systems to be frames. We show that for a large class of window functions, every relatively uniformly discrete sequence in $\mathbb{R}^2$ with sufficiently high density will generate a Gabor frame. Explicit frame bounds are given. We also study the stability of irregular Gabor frames and show that every Gabor frame with arbitrary time-frequency parameters is stable if the window function is nice enough. Explicit stability bounds are given.

1. INTRODUCTION

Fix some $g \in L^2(\mathbb{R})$; then the windowed Fourier transform is defined by

$$(F_g f)(t, \omega) = \int_{-\infty}^{+\infty} f(x) \overline{g(x-t)} e^{-ix\omega} dx, \quad \forall f \in L^2(\mathbb{R}),$$

where g is called a window function. If $\{e^{ib_n x} g(x - a_n) : n \in \mathbb{Z}\}$ forms a frame for $L^2(\mathbb{R})$, which is called a Gabor frame or a Weyl-Heisenberg frame, then every $f \in L^2(\mathbb{R})$ is uniquely determined by its discrete windowed Fourier transform $(F_g f)(a_n, b_n)$.

Since Gabor [19] proposed a signal representation with windowed Fourier transform, Gabor systems have had a fundamental impact on the development of modern time-frequency analysis and have been widely used in communication theory, quantum mechanics, and many other fields. For a collection of papers related to Gabor frames and their applications to signal and image processing, we refer to the monograph [18].

An important problem in practice is to determine conditions on the window function g and sampling points $\{(a_n, b_n)\}$ which imply that a Gabor system is a frame. For the regular case, i.e., $\{(a_n, b_n) : n \in \mathbb{Z}\} = \{(na, mb) : n, m \in \mathbb{Z}\}$, many results including necessary and sufficient conditions for $\{e^{imb_n x} g(x - na) : m, n \in \mathbb{Z}\}$ to be a frame are established. For details, see [4, 8, 9, 10, 11, 12, 17, 23, 27].

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For the irregular case, Christensen, Deng and Heil (see [9]) proved that for $\{e^{ib_n x} g(x - a_n) : n \in \mathbb{Z}\}$ to be a frame, it is necessary that $\{(a_n, b_n)\}$ is relatively uniformly discrete and is of lower Beurling density no less than $\frac{1}{2\pi}$. Also, Ramanathan and Steger [26] proved that the density must be exactly $\frac{1}{2\pi}$ in order to get an exact frame.

For sufficient conditions, however, very few results are known. Feichtinger and Gröchenig [16] proved that if $F_g g$ is integrable on $\mathbb{R}^2$, then there exists an open set $U \subset \mathbb{R}^2$ such that $\{e^{ib_n x} g(x - a_n) : n \in \mathbb{Z}\}$ is a frame for $L^2(\mathbb{R})$ for every separated set $\{(a_n, b_n) : n \in \mathbb{Z}\} \subset \mathbb{R}^2$ with $\bigcup_{n \in \mathbb{Z}} [(a_n, b_n) + U] = \mathbb{R}^2$. Unfortunately the Feichtinger-Gröchenig theory does not give much information on how to choose an appropriate set U . However, it has many attractive features, e.g., it actually delivers discrete expansions in a very large class of Banach spaces [5, 16]. Other results concerning specific window functions or sampling points can be found in [3, 24, 28].

For a band-limited window function g , Gröchenig [21] gave sufficient conditions which ensure $\{e^{i\lambda_m x} g(x - \mu_{m,n}) : m, n \in \mathbb{Z}\}$ to be a frame with explicit frame bounds. In this case, the sampling points $(\mu_{m,n}, \lambda_m)$ lie in parallel lines.

In [30], we proved that $\{e^{i\lambda_{m,n} x} g(x - \mu_{m,n}) : m, n \in \mathbb{Z}\}$ is a frame for certain g if $(\mu_{m,n}, \lambda_{m,n}) \in [na, (n+1)a] \times [mb, (m+1)b]$ and a, b are small enough. We also gave explicit frame bounds since they are necessary for implementing the frame algorithm.

In this paper, we show that if $g(x), xg(x) \in H^1$, then $\{e^{ib_n x} g(x - a_n) : n \in \mathbb{Z}\}$ is a frame for $L^2(\mathbb{R})$ for every relatively uniformly discrete and (a, b) -dense sequence $\{(a_n, b_n) : n \in \mathbb{Z}\}$ if a and b are small enough. All constants including frame bounds are determined explicitly.

The stability of frames is needed in practice. Given a frame $\{e^{ib_n x} g(x - a_n) : n \in \mathbb{Z}\}$, we hope that it remains a frame when a_n or b_n has some small perturbation. For a regular Gabor frame $\{e^{imb_n x} g(x - na) : m, n \in \mathbb{Z}\}$, it was shown in [6, 7, 15, 29, 30] that for certain window functions g , if $|\mu_{m,n} - na|$ and $|\lambda_{m,n} - mb|$ are small enough, then $\{e^{i\lambda_{m,n} x} g(x - \mu_{m,n}) : m, n \in \mathbb{Z}\}$ is also a frame.

In this paper, we study the stability of Gabor frames with arbitrary time-frequency parameters. We show that for any g with $g(x), xg(x), x^2g(x) \in H^2$ and any sequence $\{(a_n, b_n)\}$, if $\{e^{ib_n x} g(x - a_n) : n \in \mathbb{Z}\}$ is a frame, so is $\{e^{ib'_n x} g(x - a'_n) : n \in \mathbb{Z}\}$ provided $|a'_n - a_n|$ and $|b'_n - b_n|$ are small enough. The stability bounds are determined explicitly.

Notation and definition. The Fourier transform of $f \in L^2(\mathbb{R})$ is defined by $\hat{f}(\omega) = \int_{\mathbb{R}} f(x) e^{-ix\omega} dx$.

$H^s = \{f : \int_{-\infty}^{+\infty} (1 + |\omega|^2)^s |\hat{f}(\omega)|^2 < +\infty\}$ is the Sobolev space.

Let $\Gamma = \{(a_n, b_n) : n \in \mathbb{Z}\} \subset \mathbb{R}^2$. Γ is called (p, q) -uniformly discrete if $|a_n - a_m| \geq p$ or $|b_n - b_m| \geq q$ for any $m \neq n$. Γ is called relatively uniformly discrete if it is a finite union of uniformly discrete sequences. Γ is called (a, b) -dense if $\bigcup_{n \in \mathbb{Z}} [a_n - \frac{a}{2}, a_n + \frac{a}{2}] \times [b_n - \frac{b}{2}, b_n + \frac{b}{2}] = \mathbb{R}^2$.

We say a rectangle $E = [t_1, t_2] \times [\omega_1, \omega_2] \subset \mathbb{R}^2$ is of size (a, b) if $t_2 - t_1 = a$ and $\omega_2 - \omega_1 = b$. $|E|$ denotes the Lebesgue measure of E . The norms of both $L^2(\mathbb{R})$ and $L^2(\mathbb{R}^2)$ are denoted by $\|\cdot\|$. The exact meaning can be seen by the context.

A family of functions $\{f_j : j \in J\}$ belonging to a separable Hilbert space $\mathcal{H}$ is said to be a frame if there exist positive constants A and B such that $A\|f\|^2 \leq$

$\sum_{j \in J} |\langle f, f_j \rangle|^2 \leq B \|f\|^2$ for every $f \in \mathcal{H}$. The numbers A and B are called the lower and upper *frame bounds*, respectively. If only the right-hand inequality holds, $\{f_j : j \in J\}$ is called a *Bessel sequence*.

2. IRREGULAR GABOR FRAMES

In this section, we study the construction of Gabor frames with arbitrary time-frequency parameters.

Lemma 2.1. (i) For any integer $n \geq 1$, $f \in H^n$ if and only if f is $n - 1$ times continuously differentiable, $f^{(n-1)}$ is locally absolutely continuous and $f, f', \dots, f^{(n)} \in L^2(\mathbb{R})$.

(ii) If $g(x), xg(x), \dots, x^n g(x) \in L^2(\mathbb{R})$, then $\hat{g}$ is $n - 1$ times continuously differentiable, $\hat{g}^{(n-1)}$ is locally absolutely continuous and $\hat{g}, \hat{g}', \dots, \hat{g}^{(n)} \in L^2(\mathbb{R})$.

Proof. (i) This is a consequence of [3, Theorem 5.2].

(ii) Let $f(t) = \frac{1}{2\pi} \hat{g}(-t)$. Then $\hat{f}(x) = g(x)$ and so $f \in H^n$. Now the conclusion follows from (i).

Remark. Although Sobolev spaces are involved in this paper, the differentiation is in the classical sense.

Lemma 2.2. Suppose that $f, g \in L^2(\mathbb{R})$. Then

- (i) $\|(F_g f)(t, \omega)\|^2 = 2\pi \|g(x)\|^2 \cdot \|f(x)\|^2$.
- (ii) $(F_g f)(t, \omega) = \frac{1}{2\pi} e^{-i\omega t} (F_{\hat{g}} \hat{f})(\omega, -t)$.
- (iii) $\frac{\partial}{\partial t} (F_g f)(t, \omega) = -(F_{g'} f)(t, \omega)$ if g is differentiable and $g' \in L^2(\mathbb{R})$.

Proof. The first two equalities are consequences of Parseval's identity. For the third one, it suffices to prove that

$$\lim_{\Delta x \rightarrow 0} \left\| \frac{g(x + \Delta x) - g(x)}{\Delta x} - g'(x) \right\|^2 = \lim_{\Delta x \rightarrow 0} \frac{1}{2\pi} \left\| \left(\frac{e^{i\Delta x \xi} - 1}{\Delta x} - i\xi \right) \hat{g}(\xi) \right\|^2 = 0.$$

Since $\left| \frac{e^{i\Delta x \xi} - 1}{\Delta x} \right| = \left| \frac{2 \sin \frac{\Delta x \xi}{2}}{\Delta x} \right| \leq |\xi|$ and $\xi \hat{g}(\xi) = -i \widehat{g'}(\xi) \in L^2(\mathbb{R})$, the conclusion follows from the dominated convergence theorem.

Proposition 2.3 (Wirtinger's inequality [22]). If f is differentiable on $[a, b]$, $f, f' \in L^2[a, b]$ and $f(a)f(b) = 0$, then

$$\int_a^b |f(x)|^2 dx \leq \frac{4}{\pi^2} (b-a)^2 \int_a^b |f'(x)|^2 dx.$$

The following lemma is an immediate consequence.

Lemma 2.4. If f is differentiable on $[a, b]$, $f, f' \in L^2[a, b]$ and there is some $c \in [a, b]$ such that $f(c) = 0$, then

$$\int_a^b |f(x)|^2 dx \leq \frac{4\delta^2}{\pi^2} \int_a^b |f'(x)|^2 dx,$$

where $\delta = \max\{c - a, b - c\}$.

Lemma 2.5. *Let $\{E_n : n \in \mathbb{Z}\}$ be a sequence of measurable sets in $\mathbb{R}^2$ such that $|E_n \cap E_m| = 0$, for $n \neq m$. Suppose that $u_n(t, \omega), v_n(t, \omega) \in L^2(E_n)$ and that $\sum_{n \in \mathbb{Z}} \|u_n(t, \omega)\|_{L^2(E_n)}^2$ and $\sum_{n \in \mathbb{Z}} \|v_n(t, \omega)\|_{L^2(E_n)}^2$ are finite. Then*

$$\begin{aligned} & \left| \left(\sum_{n \in \mathbb{Z}} \|u_n(t, \omega)\|_{L^2(E_n)}^2 \right)^{1/2} - \left(\sum_{n \in \mathbb{Z}} \|v_n(t, \omega)\|_{L^2(E_n)}^2 \right)^{1/2} \right| \\ & \leq \left(\sum_{n \in \mathbb{Z}} \|u_n(t, \omega) - v_n(t, \omega)\|_{L^2(E_n)}^2 \right)^{1/2} \\ & \leq \left(\sum_{n \in \mathbb{Z}} \|u_n(t, \omega)\|_{L^2(E_n)}^2 \right)^{1/2} + \left(\sum_{n \in \mathbb{Z}} \|v_n(t, \omega)\|_{L^2(E_n)}^2 \right)^{1/2}. \end{aligned}$$

Proof. By the triangle inequalities in $L^2(E_n)$ and $\ell^2(\mathbb{Z})$, we have

$$\begin{aligned} & \left(\sum_{n \in \mathbb{Z}} \|u_n(t, \omega) - v_n(t, \omega)\|_{L^2(E_n)}^2 \right)^{1/2} \\ & \geq \left(\sum_{n \in \mathbb{Z}} \left| \|u_n(t, \omega)\|_{L^2(E_n)} - \|v_n(t, \omega)\|_{L^2(E_n)} \right|^2 \right)^{1/2} \\ & \geq \left| \left(\sum_{n \in \mathbb{Z}} \|u_n(t, \omega)\|_{L^2(E_n)}^2 \right)^{1/2} - \left(\sum_{n \in \mathbb{Z}} \|v_n(t, \omega)\|_{L^2(E_n)}^2 \right)^{1/2} \right|. \end{aligned}$$

Similarly we can prove the second inequality.

Lemma 2.6. *Suppose that $g(x), xg(x) \in H^1$. Let $a, b > 0$ be such that*

$$\Delta := \frac{2a}{\pi} \|g'(x)\| + \frac{2b}{\pi} \|xg(x)\| + \frac{4ab}{\pi^2} \|xg'(x)\| < \|g(x)\|.$$

Suppose that E_n are rectangles with size no greater than (a, b) such that $\bigcup_{n \in \mathbb{Z}} E_n = \mathbb{R}^2$ and $|E_m \cap E_n| = 0, n \neq m$. Then for any $(a_n, b_n) \in E_n$, $\{|E_n|^{1/2} e^{ib_n x} g(x - a_n) : n \in \mathbb{Z}\}$ is a frame for $L^2(\mathbb{R})$ with bounds $2\pi(\|g(x)\| - \Delta)^2$ and $2\pi(\|g(x)\| + \Delta)^2$.

Proof. By Lemma 2.1, $g(x), g'(x), xg(x), xg'(x) \in L^2(\mathbb{R})$.

Put $E_n = [t_{n,1}, t_{n,2}] \times [\omega_{n,1}, \omega_{n,2}]$. Then $t_{n,2} - t_{n,1} \leq a$ and $\omega_{n,2} - \omega_{n,1} \leq b$. For any $f \in L^2(\mathbb{R})$, we derive from Lemma 2.2 that

$$\begin{aligned} (2.1) \quad & \sum_{n \in \mathbb{Z}} \int_{E_n} |(F_g f)(t, \omega) e^{i\omega t} - (F_g f)(t, b_n) e^{ib_n t}|^2 d\omega dt \\ & = \sum_{n \in \mathbb{Z}} \int_{t_{n,1}}^{t_{n,2}} dt \int_{\omega_{n,1}}^{\omega_{n,2}} \frac{1}{4\pi^2} |(F_{\hat{g}} \hat{f})(\omega, -t) - (F_{\hat{g}} \hat{f})(b_n, -t)|^2 d\omega \end{aligned}$$

$$\begin{aligned}
&\leq \sum_{n \in \mathbb{Z}} \int_{t_{n,1}}^{t_{n,2}} dt \cdot \frac{4b^2}{\pi^2} \int_{\omega_{n,1}}^{\omega_{n,2}} \frac{1}{4\pi^2} |(F_{\tilde{g}'} \hat{f})(\omega, -t)|^2 d\omega \quad (\text{Lemma 2.4}) \\
&= \sum_{n \in \mathbb{Z}} \frac{4b^2}{\pi^2} \iint_{E_n} |(F_{\tilde{g}} f)(t, \omega)|^2 d\omega dt \quad (\tilde{g}(x) = -ixg(x)) \\
&= \frac{4b^2}{\pi^2} \|(F_{\tilde{g}} f)(t, \omega)\|^2 \\
&= \frac{4b^2}{\pi^2} \cdot 2\pi \|xg(x)\|^2 \|f(x)\|^2.
\end{aligned}$$

Noting that

$$(2.2) \quad \sum_{n \in \mathbb{Z}} \iint_{E_n} |(F_g f)(t, \omega) e^{i\omega t}|^2 d\omega dt = \|(F_g f)(t, \omega)\|^2 = 2\pi \|g(x)\|^2 \|f(x)\|^2,$$

we see from Lemma 2.5 that

$$\begin{aligned}
(2.3) \quad &\sum_{n \in \mathbb{Z}} \iint_{E_n} |(F_g f)(t, b_n)|^2 d\omega dt \\
&= \sum_{n \in \mathbb{Z}} \iint_{E_n} |(F_g f)(t, \omega) e^{i\omega t} - ((F_g f)(t, \omega) e^{i\omega t} - (F_g f)(t, b_n) e^{ib_n t})|^2 d\omega dt \\
&\leq 2\pi \left(\|g(x)\| + \frac{2b}{\pi} \|xg(x)\| \right)^2 \|f(x)\|^2.
\end{aligned}$$

Hence

$$\begin{aligned}
(2.4) \quad &\sum_{n \in \mathbb{Z}} \iint_{E_n} |(F_g f)(t, b_n) e^{ib_n t} - (F_g f)(a_n, b_n) e^{ib_n t}|^2 d\omega dt \\
&= \sum_{n \in \mathbb{Z}} \int_{\omega_{n,1}}^{\omega_{n,2}} d\omega \int_{t_{n,1}}^{t_{n,2}} |(F_g f)(t, b_n) - (F_g f)(a_n, b_n)|^2 dt \\
&\leq \sum_{n \in \mathbb{Z}} \frac{4a^2}{\pi^2} \iint_{E_n} |(F_{g'} f)(t, b_n)|^2 d\omega dt \quad (\text{Lemma 2.4}) \\
&\leq \frac{4a^2}{\pi^2} 2\pi \left(\|g'(x)\| + \frac{2b}{\pi} \|xg'(x)\| \right)^2 \|f(x)\|^2,
\end{aligned}$$

where (2.3) is used in the last step.

Putting (2.1) and (2.4) together, we see from Lemma 2.5 that

$$\begin{aligned}
(2.5) \quad &\sum_{n \in \mathbb{Z}} \iint_{E_n} |(F_g f)(t, \omega) e^{i\omega t} - (F_g f)(a_n, b_n) e^{ib_n t}|^2 d\omega dt \\
&\leq 2\pi \left(\frac{2a}{\pi} \|g'(x)\| + \frac{2b}{\pi} \|xg(x)\| + \frac{4ab}{\pi^2} \|xg'(x)\| \right)^2 \|f(x)\|^2 \\
&= 2\pi \Delta^2 \|f(x)\|^2.
\end{aligned}$$

Using (2.2) and Lemma 2.5 again, we get

$$\begin{aligned} & \sum_{n \in \mathbb{Z}} |E_n| \cdot |\langle f(x), e^{ib_n x} g(x - a_n) \rangle|^2 = \sum_{n \in \mathbb{Z}} \iint_{E_n} |(F_g f)(a_n, b_n)|^2 d\omega dt \\ &= \sum_{n \in \mathbb{Z}} \left\| (F_g f)(t, \omega) e^{i\omega t} - ((F_g f)(t, \omega) e^{i\omega t} - (F_g f)(a_n, b_n) e^{ib_n t}) \right\|_{L^2(E_n)}^2 \\ &\geq 2\pi (\|g(x)\| - \Delta)^2 \|f(x)\|^2. \end{aligned}$$

Similarly we can prove that

$$\sum_{n \in \mathbb{Z}} |E_n| \cdot |\langle f(x), e^{ib_n x} g(x - a_n) \rangle|^2 \leq 2\pi (\|g(x)\| + \Delta)^2 \|f(x)\|^2.$$

Now the conclusion follows.

Lemma 2.7. *Suppose that $g(x), xg(x) \in H^1$, p and q are positive constants. Let*

$$M(g(x); p, q) = \|g(x)\| + \frac{p}{\pi} \|g'(x)\| + \frac{q}{\pi} \|xg(x)\| + \frac{pq}{\pi^2} \|xg'(x)\|.$$

Then $\{e^{ib_n x} g(x - a_n) : n \in \mathbb{Z}\}$ is a Bessel sequence in $L^2(\mathbb{R})$ with the bound $\frac{2\pi}{pq} M(g(x); p, q)^2$ for any (p, q) -uniformly discrete sequence $\{(a_n, b_n) : n \in \mathbb{Z}\}$.

Proof. Let $E_n = [a_n - \frac{p}{2}, a_n + \frac{p}{2}] \times [b_n - \frac{q}{2}, b_n + \frac{q}{2}]$. Then $|E_n \cap E_m| = 0, m \neq n$. Similarly to Lemma 2.6 we can prove that

$$\begin{aligned} & \sum_{n \in \mathbb{Z}} \iint_{E_n} |(F_g f)(t, \omega) e^{i\omega t} - (F_g f)(a_n, b_n) e^{ib_n t}|^2 d\omega dt \\ &\leq 2\pi \left(\frac{p}{\pi} \|g'(x)\| + \frac{q}{\pi} \|xg(x)\| + \frac{pq}{\pi^2} \|xg'(x)\| \right)^2 \|f(x)\|^2. \end{aligned}$$

Hence

$$\sum_{n \in \mathbb{Z}} |E_n| \cdot |\langle f(x), e^{ib_n x} g(x - a_n) \rangle|^2 \leq 2\pi M(g(x); p, q)^2 \|f(x)\|^2.$$

Since $|E_n| = pq$, the conclusion follows.

We are now ready to state the main results of this paper.

Theorem 2.8. *Suppose that $g(x), xg(x) \in H^1$. Let $a, b > 0$ be such that*

$$(2.6) \quad \Delta := \frac{2a}{\pi} \|g'(x)\| + \frac{b}{\pi} \|xg(x)\| + \frac{2ab}{\pi^2} \|xg'(x)\| < \|g(x)\|.$$

Then $\{e^{ib_n x} g(x - a_n) : n \in \mathbb{Z}\}$ is a frame for $L^2(\mathbb{R})$ for any relatively uniformly discrete and (a, b) -dense sequence $\{(a_n, b_n) : n \in \mathbb{Z}\}$.

Proof. Put $\Gamma = \{(a_n, b_n) : n \in \mathbb{Z}\}$. We conclude that

$$(2.7) \quad \Gamma \cap [t, t+a) \times [\omega, \omega+b) \neq \emptyset, \quad \forall (t, \omega) \in \mathbb{R}^2.$$

Otherwise, there is some $(t_0, \omega_0) \in \mathbb{R}^2$ such that $\Gamma \cap [t_0, t_0+a) \times [\omega_0, \omega_0+b) = \emptyset$. Since Γ is relatively uniformly discrete, there is some $\varepsilon > 0$ such that $\Gamma \cap [t_0 - \varepsilon, t_0 + a) \times [\omega_0 - \varepsilon, \omega_0 + b) = \emptyset$. Hence $(t_0 + \frac{a-\varepsilon}{2}, \omega_0 + \frac{b-\varepsilon}{2}) \notin \bigcup_{n \in \mathbb{Z}} [a_n - \frac{a}{2}, a_n + \frac{a}{2}] \times [b_n - \frac{b}{2}, b_n + \frac{b}{2}] = \mathbb{R}^2$, which is impossible.

For any $m \in \mathbb{Z}$, let $\Gamma_m = \{(a_n, b_n) : ma \leq a_n < (m+1)a\}$. By (2.7), Γ_m has infinitely many points and so we can write $\Gamma_m = \{(a_{n_{m,k}}, b_{n_{m,k}}) : k \in \mathbb{Z}\}$. Without

loss of generality, we can assume that $b_{n_m,k} \leq b_{n_m,k+1}$. Using (2.7) again, we have $\lim_{k \rightarrow \pm\infty} b_{n_m,k} = \pm\infty$ and $b_{n_m,k+1} - b_{n_m,k} \leq b$. Let

$$E_{n_m,k}^1 = [ma, (m+1)a] \times \left[\frac{b_{n_m,k-1} + b_{n_m,k}}{2}, b_{n_m,k} \right]$$

and

$$E_{n_m,k}^2 = [ma, (m+1)a] \times \left[b_{n_m,k}, \frac{b_{n_m,k} + b_{n_m,k+1}}{2} \right].$$

Then $\{E_{n_m,k}^j : j = 1 \text{ or } 2, m, k \in \mathbb{Z}\}$ meets Lemma 2.6 and $(a_{n_m,k}, b_{n_m,k}) \in E_{n_m,k}^1 \cap E_{n_m,k}^2$. By substituting $\frac{b}{2}$ for b in Lemma 2.6, we have

$$\begin{aligned} 2\pi(\|g(x)\| - \Delta)^2 \|f(x)\|^2 &\leq \sum_{m,k \in \mathbb{Z}} \left(|E_{n_m,k}^1| + |E_{n_m,k}^2| \right) |(F_g f)(a_{n_m,k}, b_{n_m,k})|^2 \\ &\leq \sum_{m,k \in \mathbb{Z}} ab |\langle f(x), e^{ib_{n_m,k}x} g(x - a_{n_m,k}) \rangle|^2 = ab \sum_{n \in \mathbb{Z}} |\langle f(x), e^{ib_n x} g(x - a_n) \rangle|^2. \end{aligned}$$

On the other hand, since Γ is relatively uniformly discrete, we can write $\Gamma = \bigcup_{k=1}^r F_k$, where F_k is a (p_k, q_k) -uniformly discrete sequence, $p_k, q_k > 0, 1 \leq k \leq r$, and r is a positive integer. By Lemma 2.7,

$$\begin{aligned} \sum_{n \in \mathbb{Z}} |\langle f(x), e^{ib_n x} g(x - a_n) \rangle|^2 &= \sum_{k=1}^r \sum_{(a_n, b_n) \in F_k} |\langle f(x), e^{ib_n x} g(x - a_n) \rangle|^2 \\ &\leq \left(\sum_{k=1}^r \frac{2\pi}{p_k q_k} M(g(x), p_k, q_k)^2 \right) \|f(x)\|^2. \end{aligned}$$

This completes the proof.

Remark. The condition (2.6) can be replaced by

$$\Delta := \frac{a}{\pi} \|g'(x)\| + \frac{2b}{\pi} \|xg(x)\| + \frac{2ab}{\pi^2} \|xg'(x)\| < \|g(x)\|$$

and the proof is similar to the one above.

3. STABILITY OF GABOR FRAMES

In this section, we study the stability of Gabor frames.

Noting that $(F_g f)(a_n + t_0, b_n + \omega_0) = e^{-i(b_n + \omega_0)t_0} \langle f(x + t_0) e^{-i\omega_0 x}, e^{ib_n x} g(x - a_n) \rangle$, we have the following result.

Lemma 3.1. *For any $g \in L^2(\mathbb{R})$, if $\{e^{ib_n x} g(x - a_n) : n \in \mathbb{Z}\}$ is of a lower (upper) frame bound A (B), so is $\{e^{i(b_n + \omega_0)x} g(x - a_n - t_0) : n \in \mathbb{Z}\}$ for the same bound for any $t_0, \omega_0 \in \mathbb{R}$.*

Theorem 3.2. *Suppose that $g(x), xg(x), x^2g(x) \in H^2$. Let $\{(a_n, b_n)\}$ be a (p, q) -uniformly discrete sequence and $\{e^{ib_n x} g(x - a_n) : n \in \mathbb{Z}\}$ be a frame for $L^2(\mathbb{R})$ with bounds A and B . Let $\delta, \eta > 0$ be such that*

$$\Delta^2 = \frac{2\pi}{pq} \left(\frac{4\delta}{\pi} M(xg(x); p, q) + \frac{4\eta}{\pi} M(g'(x); p, q) + \frac{16\delta\eta}{\pi^2} M(xg'(x); p, q) \right)^2 < A.$$

Then for any (a'_n, b'_n) satisfying

$$|a_n - a'_n| \leq \eta \quad \text{and} \quad |b_n - b'_n| \leq \delta,$$

$\{e^{ib'_n x} g(x - a'_n) : n \in \mathbb{Z}\}$ is a frame for $L^2(\mathbb{R})$ with bounds $(A^{1/2} - \Delta)^2$ and $(B^{1/2} + \Delta)^2$.

Proof. It is easy to check that all of $M(xg(x); p, q)$, $M(g'(x); p, q)$ and $M(xg'(x); p, q)$ are finite numbers, thanks to Lemma 2.1.

For any $f \in L^2(\mathbb{R})$, by Lemma 2.2 and Lemma 2.4, we have

$$\begin{aligned}
 (3.1) \quad & \sum_{n \in \mathbb{Z}} \int_{-\eta}^{\eta} dt \int_{-\delta}^{\delta} \left| (F_g f)(a_n + t, b_n + \omega) e^{i(b_n + \omega)(a_n + t)} \right. \\
 & \quad \left. - (F_g f)(a_n + t, b'_n) e^{ib'_n(a_n + t)} \right|^2 d\omega \\
 &= \sum_{n \in \mathbb{Z}} \int_{-\eta}^{\eta} dt \int_{-\delta}^{\delta} \frac{1}{4\pi^2} \left| (F_{\hat{g}} \hat{f})(b_n + \omega, -a_n - t) - (F_{\hat{g}} \hat{f})(b'_n, -a_n - t) \right|^2 d\omega \\
 &\leq \sum_{n \in \mathbb{Z}} \frac{16\delta^2}{\pi^2} \int_{-\eta}^{\eta} dt \int_{-\delta}^{\delta} \frac{1}{4\pi^2} \left| (F_{\hat{g}'} \hat{f})(b_n + \omega, -a_n - t) \right|^2 d\omega \\
 &= \sum_{n \in \mathbb{Z}} \frac{16\delta^2}{\pi^2} \int_{-\eta}^{\eta} dt \int_{-\delta}^{\delta} |(F_{\tilde{g}} f)(a_n + t, b_n + \omega)|^2 d\omega \quad (\tilde{g}(x) = -ixg(x)) \\
 &\leq \frac{16\delta^2}{\pi^2} \cdot 4\delta\eta \cdot \frac{2\pi}{pq} M(xg(x); p, q)^2 \|f(x)\|^2,
 \end{aligned}$$

where Lemma 2.7 and Lemma 3.1 are used in the last step. Using Lemma 2.7 and Lemma 3.1 again, we get

$$\sum_{n \in \mathbb{Z}} \int_{-\delta}^{\delta} d\omega \int_{-\eta}^{\eta} |(F_g f)(a_n + t, b_n + \omega)|^2 dt \leq 4\delta\eta \cdot \frac{2\pi}{pq} M(g(x); p, q)^2 \|f(x)\|^2.$$

It follows from Lemma 2.5 that

$$\begin{aligned}
 & \sum_{n \in \mathbb{Z}} \int_{-\delta}^{\delta} d\omega \int_{-\eta}^{\eta} |(F_g f)(a_n + t, b'_n)|^2 dt \\
 & \leq 4\delta\eta \cdot \frac{2\pi}{pq} \left(M(g(x); p, q) + \frac{4\delta}{\pi} M(xg(x); p, q) \right)^2 \|f(x)\|^2.
 \end{aligned}$$

Hence

$$\begin{aligned}
 (3.2) \quad & \sum_{n \in \mathbb{Z}} \int_{-\delta}^{\delta} d\omega \int_{-\eta}^{\eta} \left| (F_g f)(a_n + t, b'_n) e^{ib'_n(a_n + t)} - (F_g f)(a'_n, b'_n) e^{ib'_n(a_n + t)} \right|^2 dt \\
 &= \sum_{n \in \mathbb{Z}} \int_{-\delta}^{\delta} d\omega \int_{-\eta}^{\eta} |(F_g f)(a_n + t, b'_n) - (F_g f)(a'_n, b'_n)|^2 dt \\
 &\leq \sum_{n \in \mathbb{Z}} \frac{16\eta^2}{\pi^2} \int_{-\delta}^{\delta} d\omega \int_{-\eta}^{\eta} |(F_{g'} f)(a_n + t, b'_n)|^2 dt \\
 &\leq \frac{16\eta^2}{\pi^2} \cdot 4\delta\eta \cdot \frac{2\pi}{pq} \left(M(g'(x); p, q) + \frac{4\delta}{\pi} M(xg'(x); p, q) \right)^2 \|f(x)\|^2.
 \end{aligned}$$

By (3.1) and Lemma 2.5, we have

$$\sum_{n \in \mathbb{Z}} \int_{-\delta}^{\delta} d\omega \int_{-\eta}^{\eta} \left| (F_g f)(a_n + t, b_n + \omega) e^{i(b_n + \omega)(a_n + t)} - (F_g f)(a'_n, b'_n) e^{ib'_n(a_n + t)} \right|^2 dt \leq 4\delta\eta\Delta^2 \|f(x)\|^2.$$

Using Lemma 3.1 again, we get

$$\begin{aligned} 4\delta\eta A \|f(x)\|^2 &\leq \sum_{n \in \mathbb{Z}} \int_{-\delta}^{\delta} d\omega \int_{-\eta}^{\eta} |(F_g f)(a_n + t, b_n + \omega)|^2 dt \\ &\leq 4\delta\eta B \|f(x)\|^2. \end{aligned}$$

By Lemma 2.5, we have

$$\begin{aligned} (A^{1/2} - \Delta)^2 \|f(x)\|^2 &\leq \sum_{n \in \mathbb{Z}} |(F_g f)(a'_n, b'_n)|^2 \\ &\leq (B^{1/2} + \Delta)^2 \|f(x)\|^2. \end{aligned}$$

This completes the proof.

Since $\{(a_n, b_n) : n \in \mathbb{Z}\}$ is the finite union of uniformly discrete sequences whenever $\{e^{ib_n x} g(x - a_n) : n \in \mathbb{Z}\}$ is a frame [9], the following theorem can be proved similarly.

Theorem 3.3. *Let g be defined as in Theorem 3.2. Suppose $\{e^{ib_n x} g(x - a_n) : n \in \mathbb{Z}\}$ is a frame for $L^2(\mathbb{R})$. Then there exist two positive constants δ and η such that $\{e^{ib'_n x} g(x - a'_n) : n \in \mathbb{Z}\}$ is a frame whenever $|a_n - a'_n| \leq \eta$ and $|b_n - b'_n| \leq \delta$.*

Remark. 1. The stability bounds δ and η can be determined explicitly as in Theorem 3.2. By [9, Theorem 3.1], there exist some $r \geq 1$ and $p_k, q_k > 0, 1 \leq k \leq r$, such that $\{(a_n, b_n) : n \in \mathbb{Z}\} = \bigcup_{k=1}^r \{(a_{k,n}, b_{k,n}) : n \in \Lambda_k\}$, $\Lambda_k \subset \mathbb{Z}$ and $\{(a_{k,n}, b_{k,n}) : n \in \Lambda_k\}$ is (p_k, q_k) -uniformly discrete. Let

$$\begin{aligned} \Delta'^2 &= \sum_{k=1}^r \frac{2\pi}{p_k q_k} \left(\frac{4\delta}{\pi} M(xg(x); p_k, q_k) \right. \\ &\quad \left. + \frac{4\eta}{\pi} M(g'(x); p_k, q_k) + \frac{16\delta\eta}{\pi^2} M(xg'(x); p_k, q_k) \right)^2. \end{aligned}$$

Then $\Delta' < A^{1/2}$ meets the requirement of Theorem 3.3 and the frame bounds are $(A^{1/2} - \Delta')^2$ and $(B^{1/2} + \Delta')^2$.

2. It is easy to see that g meets Theorem 3.2 and 3.3 if g or $\hat{g}$ is 2 times continuously differentiable and compactly supported.

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