

A BEURLING-TYPE THEOREM FOR THE FOCK SPACE

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ABSTRACT. Let M be a finite codimensional quasi-invariant subspace of the Fock space $L_a^2(\mathbb{C})$. Then there exists a polynomial q such that $M = [q]$. We show that $[q] \ominus [zq]$ generates M if and only if $q = z^n$ for some $n \geq 0$.

INTRODUCTION

Let \mathbb{D} be the open unit disk in the complex plane \mathbb{C} , \mathbb{T} the unit circle, and $H^2(\mathbb{D})$ the Hardy space, consisting of all functions f holomorphic on \mathbb{D} satisfying

$$\|f\|_{H^2}^2 = \sup_{0 < r < 1} \int_{-\pi}^{\pi} |f(re^{i\theta})|^2 d\theta / (2\pi) < \infty.$$

We say that N is an invariant subspace if N is a (closed) subspace of $H^2(\mathbb{D})$ that is invariant for the multiplication operator M_z . In [Beu], A. Beurling proved that: If $N \neq 0$ is an invariant subspace of the Hardy space $H^2(\mathbb{D})$, then $N \ominus zN$ is a one dimensional subspace spanned by an inner function ϕ and

$$N = [\phi] = [N \ominus zN]$$

where $N \ominus zN = N \cap (zN)^\perp$ and $[\phi]$ denotes the smallest invariant subspace containing ϕ . Beurling's theorem has played an important role in operator theory, function theory and their intersection, function-theoretic operator theory. However, despite the great development in these fields over the past forty years, it is only recently that progress has been made in proving analogues for the other classical Hilbert spaces, the Dirichlet space and the Bergman space. In [Ric], Richter proved that the analogue of Beurling's theorem is true in the Dirichlet space. It is well known that the invariant subspace lattice of the Bergman space $L_a^2(\mathbb{D})$, defined to be the space of functions f analytic in \mathbb{D} for which

$$\|f\|_{L_a^2}^2 = \int_{|z| < 1} |f(z)|^2 \frac{dA(z)}{\pi} < \infty,$$

is very complicated. In fact the dimension of $N \ominus zN$ can be an arbitrary positive integer or ∞ [Hed]. However, a big breakthrough in the study of the analogue of Beurling's theorem on the Bergman space was made by A. Aleman, S. Richter and C. Sundberg [ARS]. They proved that any invariant subspace N of the Bergman space $L_a^2(\mathbb{D})$ also has the form $N = [N \ominus zN]$. H. Hedenmalm and K. Zhu showed

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that this wandering subspace property can fail in certain weighted Bergman spaces [HZ] and we thank the referee for calling our attention to this work. In this paper we will be concerned with the Fock space $L_a^2(\mathbb{C})$. The Fock space or the so-called Siegel-Bargmann space, defined to be the space of all μ -square-integrable entire functions on the complex plane \mathbb{C} , where

$$d\mu(z) = e^{-\frac{|z|^2}{2}} d\nu(z)(2\pi)^{-1}$$

is the Gaussian measure on \mathbb{C} ($d\nu$ is the ordinary Lebesgue measure). It is easy to see that $L_a^2(\mathbb{C})$ is a closed subspace of $L^2(\mathbb{C})$ with the reproducing kernel function $K_\lambda(z) = e^{\lambda z/2}$ and the normalized reproducing kernel function $k_\lambda(z) = e^{\lambda z/2 - |\lambda|^2/4}$. For general background on the Fock space one may consult [DG] and the references therein. As proved in [GZh], there exists no nontrivial invariant subspace for multiplication operator M_z in the Fock space. Thus, they introduced an substitute for invariant subspace, the so-called quasi-invariant subspace (see also [CGH]). Let $X = \{f \in L_a^2(\mathbb{C}) : zf \in L_a^2(\mathbb{C})\}$. Then X is a dense subspace of $L_a^2(\mathbb{C})$. Let M be a closed subspace of the Fock space $L_a^2(\mathbb{C})$, and let $X \cap M$ be dense in M . We say that M is quasi-invariant if $z(M \cap X) \subset M$. In this paper, we consider the analogue of Beurling's theorem for finite codimensional quasi-invariant subspaces of the Fock space. Our result shows that, unlike the cases of Hardy space, Dirichlet space and Bergman space, the analogue of Beurling's theorem is not true in the Fock space. Besides the Introduction, the paper has two sections. In Section 1, we review some basic terminologies and results concerning entire functions ([Con]) and the Fock space. The main result is proved in Section 2.

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1. PRELIMINARIES

Let M be a finite codimensional subspace of the Fock space $L_a^2(\mathbb{C})$. We begin with the special case (in the case of the complex plane) of the algebraic reduction theorem ([GZh], Theorem 5.5) for such subspaces:

Lemma 1.1 (Theorem 5.5 [GZh]). *Let M be a quasi-invariant subspace of finite codimension. Then $\mathcal{C} \cap M$ is an ideal in the polynomial ring \mathcal{C} and $\mathcal{C} \cap M$ is dense in M . Conversely, if I is an ideal in \mathcal{C} of finite codimension, then $[I]$ is quasi-invariant subspaces of the same codimension and $[I] \cap \mathcal{C} = I$.*

By Lemma 1.1, each finite codimensional quasi-invariant subspace M has the form

$$M = [I],$$

where I is a finite codimensional ideal with the same codimension as M . Note that on the complex plane \mathbb{C} , every nonzero ideal I is principle, that is, there is a polynomial p such that $I = p\mathcal{C}$. Therefore, on the Fock space $L_a^2(\mathbb{C})$, finite codimensional quasi-invariant subspaces are exactly $[p]$, where p range over all non-zero polynomials. Let $M = [p]$. It is easy to check that

$$\text{codim } M = \dim L_a^2(\mathbb{C})/M = \deg p,$$

where $\text{codim } M$ denotes the codimension of M and $\deg p$ denotes the degree of p . Thus $\text{codim } [p] = \deg p$ and $\text{codim } [zp] = \deg p + 1$. Using the fact $[zp] \subset [p]$, we have

$$\dim[p]/[zp] = 1.$$

Therefore, every finite codimensional quasi-invariant subspace has codimension one property.

We also need some conceptions and results concerning entire functions [Con]. Let us recall the Weierstrass Factorization Theorem for entire functions ([Con], VII. 5.14). Let f be an entire function with a zero of multiplicity $m \geq 0$ at $z = 0$; let $\{a_n\}$ be the zeros of f , $a_n \neq 0$, arranged so that a zero of multiplicity k is repeated in this sequence k times. Also assume that $|a_1| \leq |a_2| \leq \dots$. If $\{l_n\}$ is a sequence of integers such that $\sum_{n=1}^{\infty} (\frac{r}{|a_n|})^{l_n+1} < \infty$ for every $r > 0$, then

$$P(z) = \prod_{n=1}^{\infty} E_{l_n}(z/a_n)$$

converges uniformly on compact subsets of the plane, where

$$E_l(z) = (1 - z) \exp(z + \frac{z^2}{2} + \dots + \frac{z^l}{l})$$

for $l \geq 1$ and $E_0(z) = 1 - z$. Consequently, the Weierstrass Factorization Theorem says that $f(z) = z^m e^{g(z)} P(z)$, where g is an entire function.

Let f be an entire function with zeros $\{a_1, a_2, \dots\}$, repeated according to multiplicity and arranged such that $|a_1| \leq |a_2| \leq \dots$. We say that f is of *finite rank* if there is an integer k such that $\sum_{n=1}^{\infty} |a_n|^{-(k+1)} < \infty$. If k is the smallest integer such that this occurs, then f is said to be of rank k ; a function with only a finite number of zeros has rank 0. An entire function f has *finite genus* if f has finite rank and if

$$f(z) = z^m e^{g(z)} P(z),$$

where $P(z)$ is as above, and g is a polynomial. If k is the rank of f and j is the degree of the polynomial g , then $\mu = \max(k, j)$ is called the *genus* of f . An entire function f is of *finite order* if there is a positive constant a and an $r_0 > 0$ such that $|f(z)| < \exp(|z|^a)$ for $|z| > r_0$. If f is of finite order, then the number $\lambda = \inf\{a : |f(z)| < \exp(|z|^a) \text{ for } z \text{ sufficiently large}\}$ is called the *order* of f . We also recall that the Hadamard's Factorization Theorem says that if f is an entire function of finite order λ , then f has finite genus $\mu \leq \lambda$ ([Con], p. 289).

2. THE MAIN RESULT AND ITS PROOF

Let M be a finite codimensional quasi-invariant subspace of the Fock space $L_a^2(\mathbb{C})$. The following lemma characterizes the structure of M^\perp , which will be used to prove our main result.

Lemma 2.1. *Suppose that $p = z^{i_0}(z - \lambda_1)^{i_1}(z - \lambda_2)^{i_2} \dots (z - \lambda_m)^{i_m}$ and $M = [p]$. Then*

$$M^\perp = \text{span}\{1, z, \dots, z^{i_0-1}, e^{\frac{\bar{\lambda}_1 z}{2}}, \dots, z^{i_1-1} e^{\frac{\bar{\lambda}_1 z}{2}}, \dots, e^{\frac{\bar{\lambda}_m z}{2}}, \dots, z^{i_m-1} e^{\frac{\bar{\lambda}_m z}{2}}\}.$$

Proof. Since $\dim M^\perp = \deg p$ and the elements in the following set are linearly independent, we only need to show that

$$\{1, z, \dots, z^{i_0-1}, e^{\frac{\bar{\lambda}_1 z}{2}}, z e^{\frac{\bar{\lambda}_1 z}{2}}, \dots, z^{i_1-1} e^{\frac{\bar{\lambda}_1 z}{2}}, \dots, e^{\frac{\bar{\lambda}_m z}{2}}, \dots, z^{i_m-1} e^{\frac{\bar{\lambda}_m z}{2}}\} \subseteq M^\perp.$$

For each $f \in M$, we can write $f = (z - \lambda_k)^{i_k} f_k$ where $k = 0, 1, \dots, m$ and $\lambda_0 = 0$. Note that in the Fock space

$$f(z) = \int f(w) e^{\frac{\bar{w}z}{2}} d\mu(w)$$

because $e^{\frac{\bar{z}w}{2}}$ is the reproducing kernel. Thus we have

$$\begin{aligned} \frac{d}{dz} f(z) &= \frac{d}{dz} \int f(w) e^{\frac{\bar{w}z}{2}} d\mu(w) \\ &= \int f(w) \frac{\bar{w}}{2} e^{\frac{\bar{w}z}{2}} d\mu(w) \\ &= \frac{1}{2} \langle f(w), w e^{\frac{\bar{z}w}{2}} \rangle. \end{aligned}$$

Similarly, one can show that

$$\frac{d^j}{dz^j} f(z) = \frac{1}{2^j} \langle f(w), w^j e^{\frac{\bar{z}w}{2}} \rangle \text{ for each } j > 0.$$

Thus for each non-negative positive integer j ($0 \leq j < i_k$), we have

$$\begin{aligned} \langle f(z), z^j e^{\frac{\bar{\lambda}_k z}{2}} \rangle &= \langle (z - \lambda_k)^{i_k} f_k(z), z^j e^{\frac{\bar{\lambda}_k z}{2}} \rangle \\ &= 2^j \frac{1}{2^j} \langle (z - \lambda_k)^{i_k} f_k(z), z^j e^{\frac{\bar{w}z}{2}} \rangle|_{w=\lambda_k} \\ &= 2^j \frac{d^j}{dw^j} ((w - \lambda_k)^{i_k} f_k(w))|_{w=\lambda_k} \\ &= 0. \end{aligned}$$

This completes the proof. \square

Now we are ready to prove our main result.

Theorem 2.1. $[q] \ominus [zq]$ generates $[q]$ if and only if $q = z^n$ for some $n \geq 0$.

Proof. It is obvious that $[q] \ominus [zq]$ generates $[q]$ when $q = z^n$.

Suppose that there exists a polynomial

$$q = z^{i_0-1} (z - \lambda_1)^{i_1} (z - \lambda_2)^{i_2} \cdots (z - \lambda_m)^{i_m} \text{ with } \prod_{k=1}^m i_k \neq 0$$

such that $[q] \ominus [zq]$ generates $[q]$. Since $\dim [q] \ominus [zq] = 1$, there is a $\phi \in [q] \ominus [zq]$ such that ϕ generates $[q]$. Using Lemma 2.1, we can write ϕ as

$$(1) \quad \phi = p_0(z) + p_1(z) e^{\bar{\lambda}_1 z} + \cdots + p_m(z) e^{\bar{\lambda}_m z}.$$

Since $\deg p_0 \leq \deg q - 1$, $p_0(z)$ does not generate $[q]$. Thus there exists $1 \leq i \leq m$ such that $p_i(z) \neq 0$. Similarly, one can show that if ϕ generates $[q]$, then there exist at least two polynomials $p_i \neq 0$ and $p_j \neq 0$ in (1). Thus we may assume that each $p_j \neq 0$ in (1).

It is easy to see that ϕ has finitely many zeros because M has finite codimension. By the Weierstrass Factorization Theorem we have

$$\phi = p(z) e^{g(z)}$$

where $p(z)$ is a polynomial. On the other hand, for each $\lambda > 1$ there exists $M_\lambda > 0$ such that

$$|\phi(z)| = |p_0(z) + p_1(z) e^{\bar{\lambda}_1 z} + \cdots + p_m(z) e^{\bar{\lambda}_m z}| < e^{|z|^\lambda} \text{ for all } |z| > M_\lambda.$$

So, by definition, the order of ϕ is less than or equal to 1. It is obvious that ϕ and e^g have the same order. By the Hadamard's Factorization Theorem, g is a polynomial of degree ≤ 1 . However, by using the assumption that $p_j \neq 0$ in (1), the order of ϕ is nonzero. This, together with the fact that the order of e^g is equal to the degree of g , lets us write $g = az + b$. Without loss of generality, we assume that

$$\phi(z) = q(z)e^{az}.$$

Thus we have

$$(2) \quad p_0(z) + p_1(z)e^{\frac{\bar{\lambda}_1 z}{2}} + \cdots + p_m(z)e^{\frac{\bar{\lambda}_m z}{2}} = q(z)e^{az}.$$

The fact that (2) does not hold in the case that $p_j \neq 0$ is elementary. We refer the interested reader to any elementary ordinary differential equations book (see, e.g., [BD]). Thus (1) does not hold. Therefore ϕ does not generate $[q]$. This completes the proof. \square

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