L^p VERSIONS OF HARDY'S UNCERTAINTY PRINCIPLE ON HYPERBOLIC SPACES

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ABSTRACT. Hardy's uncertainty principle states that it is impossible for a function and its Fourier transform to be simultaneously very rapidly decreasing. In this paper we prove L^p versions of this principle for the Jacobi transform and for the Fourier transform on real hyperbolic spaces.

1. INTRODUCTION

The uncertainty principles roughly state that a non-zero function f and its Fourier transform $\widehat{f}$ cannot both be sharply localised. This is already evident in the Paley–Wiener theorem; the Fourier transform of a compactly supported smooth function extends to an entire function, hence it cannot have compact support. The Fourier transform of a rapidly decreasing function, i.e., a Schwartz function, is on the other hand again a rapidly decreasing function. Hardy's uncertainty principle (see [8]) tells us, however, that they cannot both be *very* rapidly decreasing. Generalisations to a L^p set-up have been studied and proved by, among others, Beurling ([12]) and Cowling–Price:

Theorem 1.1 (Cowling–Price [4]). *Let f be a measurable function on $\mathbb{R}$. If $e^{\alpha|\cdot|^2}f \in L^p(\mathbb{R})$ and $e^{\beta|\cdot|^2}\widehat{f} \in L^q(\mathbb{R})$, with $\min(p, q) < \infty$ and $\alpha\beta \geq \frac{1}{4}$, then $f = 0$.*

The case $p = q = \infty$, $\alpha\beta > \frac{1}{4}$ is Hardy's uncertainty principle.

Analogues of Hardy's uncertainty principle and its L^p versions for the Fourier transform on (semisimple) Lie groups have been the object of interest in several recent papers; see [5], [15] and the references therein. The Riemannian symmetric spaces of the non-compact type have also been studied; see [17] and [18].

The aim of this paper is to prove a version of Theorem 1.1 for the Jacobi transform and for the Fourier transform on the real hyperbolic spaces

$$SO_o(m, n)/SO_o(m-1, n), \quad m, n \in \mathbb{N}.$$

The proof of the latter case is based on the observation that the Fourier transform of functions of fixed K -type can be expressed in terms of modified Jacobi functions. This approach can be expanded to cover all hyperbolic spaces.

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2. JACOBI FUNCTIONS AND THE JACOBI TRANSFORM

Let $a, b, \lambda \in \mathbb{C}$ and $0 < t < \infty$. We consider the differential equation

$$(1) \quad (\Delta^{a,b}(t))^{-1} \frac{d}{dt} \left(\Delta^{a,b}(t) \frac{du(t)}{dt} \right) = -(\lambda^2 + \rho^2)u(t),$$

where $\rho = a + b + 1$ and $\Delta^{a,b}(t) = (2 \sinh t)^{2a+1} (2 \cosh t)^{2b+1}$. Using the substitution $x = -\sinh^2 t$, we can rewrite (1) as a hypergeometric differential equation with parameters $\frac{1}{2}(\rho + i\lambda)$, $\frac{1}{2}(\rho - i\lambda)$ and $a + 1$ (see [6, 2.1.1]). Let ${}_2F_1$ denote the Gauss hypergeometric function. The Jacobi function (of order (a, b)),

$$\varphi_\lambda^{a,b}(t) := {}_2F_1 \left(\frac{1}{2}(\rho + i\lambda), \frac{1}{2}(\rho - i\lambda), a + 1; -\sinh^2 t \right),$$

is for $a \notin -\mathbb{N}$ the unique solution to (1) satisfying $\varphi_\lambda^{a,b}(0) = 1$ and $\frac{d}{dt}|_{t=0} \varphi_\lambda^{a,b} = 0$. The Jacobi functions satisfy the following growth estimates:

Lemma 2.1. *There exists for each $a, b \in \mathbb{C}$ a constant $C > 0$ such that*

$$\left| \Gamma(a + 1)^{-1} \varphi_\lambda^{a,b}(t) \right| \leq C(1 + |\lambda|)^k (1 + t) e^{(|\Im \lambda| - \Re \rho)t},$$

for all $\lambda \in \mathbb{C}$ and all $t \geq 0$, where $k = 0$ if $\Re a > -\frac{1}{2}$ and $k = [\frac{1}{2} - \Re a]$ if $\Re a \leq -\frac{1}{2}$.

Proof. See [13, Lemma 2.3]. $\square$

Here $[\cdot]$ denotes the integer part. We note that $\Gamma(a + 1)^{-1} \varphi_\lambda^{a,b}(t)$ is an entire function in the variables a, b and $\lambda \in \mathbb{C}$ (also for $a \in -\mathbb{N}$). The Jacobi transform (of order (a, b)) is defined by

$$\widehat{f}^{a,b}(\lambda) := \int_{\mathbb{R}_+} f(t) \varphi_\lambda^{a,b}(t) \Delta^{a,b}(t) dt,$$

for all even functions f and all complex numbers λ for which the right-hand side is well-defined. The Paley–Wiener theorem for the Jacobi transform, [13, Theorem 3.4], states that the (normalised) application $f \mapsto \Gamma(a + 1)^{-1} \widehat{f}^{a,b}$ is a bijection from $C_c^\infty(\mathbb{R})_{\text{even}}$ onto $\mathcal{H}(\mathbb{C})_{\text{even}}$, the space of even entire rapidly decreasing functions of exponential type, for all $a, b \in \mathbb{C}$.

The Jacobi functions of the second kind

$$\phi_\lambda^{a,b}(t) = (2 \cosh t)^{i\lambda - \rho} {}_2F_1 \left(\frac{1}{2}(\rho - i\lambda), \frac{1}{2}(a - b + 1 - i\lambda), 1 - i\lambda; \cosh^{-2} t \right),$$

defines for $\lambda \notin -i\mathbb{N}$ another solution of (1), characterised by the property that $\phi_\lambda^{a,b}(t) \sim e^{i(\lambda - \rho)t}$ for $t \rightarrow \infty$. We remark that $\phi_\lambda^{a,b}$ is singular if and only if $\lambda \in -i\mathbb{N}$, with simple poles. Define the meromorphic Jacobi c -functions as

$$(2) \quad c^{a,b}(\lambda) := 2^{\rho - i\lambda} \frac{\Gamma(a + 1)\Gamma(i\lambda)}{\Gamma(\frac{1}{2}(i\lambda + \rho))\Gamma(\frac{1}{2}(i\lambda + a - b + 1))}.$$

Then

$$(3) \quad \varphi_\lambda^{a,b} = c^{a,b}(\lambda) \phi_\lambda^{a,b} + c^{a,b}(-\lambda) \phi_{-\lambda}^{a,b},$$

as a meromorphic identity; see [14, (2.15–18)]. The inversion formula for the Jacobi transform can be written as (with $\mu \geq 0$, $\mu > -\Re(a \pm b + 1)$)

$$(4) \quad f(t) = \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{f}^{a,b}(\lambda + i\mu) \phi_{\lambda + i\mu}^{a,b}(t) \frac{d\lambda}{c^{a,b}(-\lambda - i\mu)} \quad (t > 0)$$

for $f \in C_c^\infty(\mathbb{R})_{\text{even}}$; see [14, Theorem 2.2]. Using residual calculus we can rewrite (4) as follows:

Theorem 2.2. *Assume that $a \notin -\mathbb{N}$. Let $D_{a,b}$ denote the finite set of zeroes for $c^{a,b}(-\lambda)$ with $\Im \lambda \geq 0$. Let $\eta = 0$ if $D_{a,b} \cap \mathbb{R} = \{\emptyset\}$ and otherwise choose $0 < \eta < \frac{1}{2}$ such that $c^{a,b}(\pm\lambda) \neq 0$ for $\Im \lambda \in [-\eta, \eta] \setminus \{0\}$. Then*

$$f(t) = \frac{1}{4\pi} \int_{\mathbb{R}} \frac{\widehat{f}^{a,b}(\lambda + i\eta) \varphi_{\lambda+i\eta}^{a,b}(t)}{c^{a,b}(-\lambda - i\eta) c^{a,b}(\lambda + i\eta)} d\lambda - \sum_{\nu \in D_{a,b}} ik_\nu \text{Res}_{\lambda=\nu} \left\{ \frac{\widehat{f}^{a,b}(\lambda) \varphi_\lambda^{a,b}(t)}{c^{a,b}(-\lambda) c^{a,b}(\lambda)} \right\}$$

for $f \in C_c^\infty(\mathbb{R})_{\text{even}}$, where $k_\nu := 1/2$ if $\nu \in i\mathbb{N} \cup \mathbb{R}$ and $k_\nu := 1$ otherwise.

Proof. The set $D_{a,b}$ is determined by the poles of the Γ -functions of (2). It follows that $D_{a,b}$ consists of those elements $\nu \neq 0$, with $\Im \nu \geq 0$, which are of the form $\nu = i(\pm b - a - 1 - 2m)$, $m \in \mathbb{N} \cup \{0\}$.

Let $\nu \in D_{a,b}$, that is, $c^{a,b}(-\nu) = 0$. Assume first that $\nu \notin i\mathbb{N}$; then $c^{a,b}(\nu) \neq 0$ by the condition $a \notin -\mathbb{N}$, and

$$\begin{aligned} \text{Res}_{\lambda=\nu} \left\{ \frac{\widehat{f}^{a,b}(\lambda) \phi_\lambda^{a,b}(t)}{c^{a,b}(-\lambda)} \right\} &= \text{Res}_{\lambda=\nu} \left\{ \widehat{f}^{a,b}(\lambda) \left(\frac{\phi_\lambda^{a,b}(t)}{c^{a,b}(-\lambda)} + \frac{\phi_{-\lambda}^{a,b}(t)}{c^{a,b}(\lambda)} \right) \right\} \\ &= \text{Res}_{\lambda=\nu} \left\{ \frac{\widehat{f}^{a,b}(\lambda) \varphi_\lambda^{a,b}(t)}{c^{a,b}(-\lambda) c^{a,b}(\lambda)} \right\}, \end{aligned}$$

by (3), since $\phi_{-\lambda}^{a,b}(t)/c^{a,b}(\lambda)$ is regular at $\lambda = \nu$.

Now assume $\nu \in i\mathbb{N} \cap D_{a,b}$. Then ν is a zero for $c^{a,b}(-\lambda)$ of order 1; a double pole in the denominator of $c^{a,b}(-\lambda)$ at $\nu \in i\mathbb{N}$ would imply $a \in -\mathbb{N}$, which we have excluded. The c -function $c^{a,b}(\lambda)$ is regular and non-zero at $\lambda = \nu$, as the poles arising from the Γ -functions in (2) cancel each other (we have excluded the cases with double poles in the denominator). We also note that $\phi_\lambda^{a,b}(t)$ is regular at $\lambda = \nu$.

Write $\nu = i(\pm b - a - 1 - 2m)$. Fix a and m , and define, for λ in some small neighbourhood of ν , a continuous function $b(\lambda)$ by the condition

$$\lambda = i(\pm b(\lambda) - a - 1 - 2m).$$

It follows that $c^{a,b(\lambda)}(-\lambda) = 0$ and $\varphi_\lambda^{a,b(\lambda)}(t) = c^{a,b(\lambda)}(\lambda) \phi_\lambda^{a,b(\lambda)}(t)$, for $\lambda \neq \nu$, by (3), and $b(\nu) = b$. Since $\lim_{\lambda \rightarrow \nu} \frac{\Gamma((\lambda - \nu)/2)}{\Gamma(\lambda)} = 2$ for $\nu \in -\mathbb{N} \cup \{0\}$, it can be seen from (2) that $\lim_{\lambda \rightarrow \nu} c^{a,b(\lambda)}(\lambda) = 2 \lim_{\lambda \rightarrow \nu} c^{a,b}(\lambda)$, and thus, by continuity of the Jacobi functions in all the variables,

$$\text{Res}_{\lambda=\nu} \left\{ \frac{\widehat{f}^{a,b}(\lambda) \phi_\lambda^{a,b}(t)}{c^{a,b}(-\lambda)} \right\} = \frac{1}{2} \text{Res}_{\lambda=\nu} \left\{ \frac{\widehat{f}^{a,b}(\lambda) \varphi_\lambda^{a,b}(t)}{c^{a,b}(-\lambda) c^{a,b}(\lambda)} \right\},$$

since $2\phi_\nu^{a,b}(t) = 2 \lim_{\lambda \rightarrow \nu} \phi_\lambda^{a,b(\lambda)}(t) = 2 \lim_{\lambda \rightarrow \nu} \frac{\varphi_\lambda^{a,b(\lambda)}(t)}{c^{a,b(\lambda)}(\lambda)} = \lim_{\lambda \rightarrow \nu} \frac{\varphi_\lambda^{a,b(\lambda)}(t)}{c^{a,b}(\lambda)} = \frac{\varphi_\nu^{a,b}(t)}{c^{a,b}(\nu)}$.

Let $0 \leq \eta < \frac{1}{2}$ and let $\delta > 0$. We can extend the estimates in [13, Lemma 2.1] (for $\Im \lambda \geq 0$) as follows: There exists a positive constant C such that

$$|\phi_\lambda^{a,b}(t)| \leq Ce^{-(\Im \lambda + \Re \rho)t}$$

for all $t \geq \delta$ and all $\lambda \in \mathbb{C}$ with $\Im \lambda \geq -\eta$; see [3] for details. The polynomial estimates on $|c^{a,b}(-\lambda)^{-1}|$ away from the poles given by [13, Lemma 2.2] can also be extended to $\Im \lambda \geq -\eta$.

Now choose η as in the theorem. By the above estimates and since $\widehat{f}^{a,b}$ satisfies the usual Paley–Wiener growth estimates, we can shift the contour toward the real axis, and (4) becomes

$$\begin{aligned} f(t) &= \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{f}^{a,b}(\lambda + i\eta) \phi_{\lambda+i\eta}^{a,b}(t) \frac{d\lambda}{c^{a,b}(-\lambda - i\eta)} + \text{Residual terms} \\ &= \frac{1}{4\pi} \int_{\mathbb{R}} \widehat{f}^{a,b}(\lambda + i\eta) \phi_{\lambda+i\eta}^{a,b}(t) \frac{d\lambda}{c^{a,b}(-\lambda - i\eta)} \\ &\quad + \frac{1}{4\pi} \int_{\mathbb{R}} \widehat{f}^{a,b}(-\lambda - i\eta) \phi_{-\lambda-i\eta}^{a,b}(t) \frac{d\lambda}{c^{a,b}(\lambda + i\eta)} + \text{Residual terms}, \end{aligned}$$

where we have moved half the integral across the real axis if $D_{a,b} \cap \mathbb{R} \neq \{\emptyset\}$ and made a sign change $\lambda \mapsto -\lambda$ in the integral over the line $\Im \lambda = -\eta$. Since $\widehat{f}^{a,b}$ is even, we get our inversion formula from the identity (3). $\square$

Remark 2.3. Theorem 2.2 and its proof were communicated to us by H. Schlichtkrull. For $a > -1$, $b \in \mathbb{R}$ (which implies $\eta = 0$), it is due to [7, Appendix 1] (a minor error has been corrected with the introduction of the constant k_ν).

Our proof of the L^p -version of Hardy’s uncertainty principle for the Jacobi transform is inspired by the proof of the semisimple Lie group case; see [15]. The following lemma is crucial:

Lemma 2.4. *Let $1 \leq q < \infty$. Assume that h is an entire function on $\mathbb{C}$ such that*

$$|h(\lambda)| \leq C e^{\gamma |\Re \lambda|^2} (1 + |\Im \lambda|)^N, \lambda \in \mathbb{C} \quad \text{and} \quad \int_{\mathbb{R}} |h(\lambda)|^q d\lambda < \infty$$

for positive constants γ , N and C . Then $h = 0$.

Proof. See [15, Lemma 2.3]. $\square$

Theorem 2.5. *Let $a, b \in \mathbb{C}$, $a \notin -\mathbb{N}$, and suppose that $1 \leq p \leq \infty$, $1 \leq q < \infty$. Assume that f is an even measurable function on $\mathbb{R}$ satisfying*

$$e^{\alpha|\cdot|^2} e^{(1-\frac{2}{p})\Re \rho|\cdot|} f \in L^p(\mathbb{R}, |\Delta^{a,b}(t)|) \quad \text{and} \quad e^{\beta|\cdot|^2} \widehat{f}^{a,b} \in L^q(\mathbb{R}),$$

for positive constants α, β such that $\alpha\beta \geq \frac{1}{4}$. Then $f = 0$ almost everywhere.

Proof. Let f be an even measurable function satisfying the above growth conditions. The very rapid decay implies that $f \in L^1(\mathbb{R}_+, |\Delta^{a,b}(t)| dt) \cap L^2(\mathbb{R}_+, |\Delta^{a,b}(t)| dt)$ and that $\widehat{f}^{a,b}(\lambda)$ defines an analytic function in $\lambda \in \mathbb{C}$ for all $a, b \in \mathbb{C}$. We note that it suffices to prove the theorem for $\alpha\beta = \frac{1}{4}$. First let $p < \infty$. Using Lemma 2.1, we

get the following estimates on $\widehat{f}^{a,b}(\lambda)$ (for different constants $C, N > 0$):

$$\begin{aligned}
 |\widehat{f}^{a,b}(\lambda)| &\leq C \int_{\mathbb{R}_+} |f(t)|(1+|\lambda|)^k(1+t)e^{(|\Im \lambda|-\Re \rho)t} |\Delta^{a,b}(t)| dt \\
 &\leq C(1+|\lambda|)^k \int_{\mathbb{R}_+} |f(t)|e^{\alpha t^2} e^{(1-\frac{2}{p})\Re \rho t} (1+t)e^{|\Im \lambda|t} e^{-\alpha t^2} e^{(\frac{2}{p}-2)\Re \rho t} |\Delta^{a,b}(t)| dt \\
 &\leq C(1+|\lambda|)^k \left(\int_{\mathbb{R}_+} \left((1+t)e^{|\Im \lambda|t} e^{-\alpha t^2} e^{-\frac{2}{p'}\Re \rho t} \right)^{p'} |\Delta^{a,b}(t)| dt \right)^{\frac{1}{p'}} \\
 &\leq C(1+|\lambda|)^k \left(\int_{\mathbb{R}_+} (1+t)^{p'} e^{p'|\Im \lambda|t} e^{-p'\alpha t^2} dt \right)^{\frac{1}{p'}} \\
 &= C(1+|\lambda|)^k e^{|\Im \lambda|^2/4\alpha} \left(\int_{\mathbb{R}_+} (1+t)^{p'} e^{-p'\alpha(t-|\Im \lambda|/2\alpha)^2} dt \right)^{\frac{1}{p'}} \\
 &= C(1+|\lambda|)^k e^{|\Im \lambda|^2/4\alpha} \left(\int_{-|\Im \lambda|/2\alpha}^{\infty} (1+t+|\Im \lambda|/2\alpha)^{p'} e^{-p'\alpha t^2} dt \right)^{\frac{1}{p'}} \\
 &\leq C(1+|\lambda|)^N e^{|\Im \lambda|^2/4\alpha}
 \end{aligned}$$

for $\lambda \in \mathbb{C}$, using translation invariance of dt , the Hölder inequality (with $\frac{1}{p} + \frac{1}{p'} = 1$) and the inequality $|\Im \lambda| \leq |\lambda|$. For $p = \infty$, we have $|f(t)| \leq C e^{-\alpha|t|^2} e^{-\Re \rho|t|}$ for some positive constant C , and we get

$$\begin{aligned}
 |\widehat{f}^{a,b}(\lambda)| &\leq C \int_{\mathbb{R}_+} e^{-\alpha t^2} e^{-\Re \rho|t|} (1+|\lambda|)^k(1+t)e^{(|\Im \lambda|-\Re \rho)t} |\Delta^{a,b}(t)| dt \\
 &\leq C(1+|\lambda|)^k e^{|\Im \lambda|^2/4\alpha} \int_{\mathbb{R}_+} (1+t)e^{-\alpha(t-|\Im \lambda|/2\alpha)^2} dt \\
 &\leq C(1+|\lambda|)^k e^{|\Im \lambda|^2/4\alpha} \int_{-|\Im \lambda|/2\alpha}^{\infty} (1+t+|\Im \lambda|/2\alpha)e^{-\alpha t^2} dt \\
 &\leq C(1+|\lambda|)^N e^{|\Im \lambda|^2/4\alpha}
 \end{aligned}$$

for $\lambda \in \mathbb{C}$.

Define $g(\lambda) := \widehat{f}^{a,b}(\lambda)e^{\langle \lambda, \lambda \rangle / 4\alpha}$. Then

$$|g(\lambda)| \leq C(1+|\lambda|)^N e^{|\Re \lambda|^2/4\alpha} \leq C(1+|\Im \lambda|)^N e^{|\Re \lambda|^2/4\alpha'},$$

for some $0 < \alpha' < \alpha$. Furthermore we see that

$$\int_{\mathbb{R}} |g(\lambda)|^q d\lambda = \int_{\mathbb{R}} \left(|\widehat{f}^{a,b}(\lambda)| e^{\beta|\lambda|^2} \right)^q d\lambda < \infty,$$

so Lemma 2.4 implies that g , and hence also $\widehat{f}^{a,b}$, is identically zero on $\mathbb{C}$.

Using (the proof of) Theorem 2.2, we see that

$$\begin{aligned}
 \int_{\mathbb{R}_+} f(t)h(t)\Delta^{a,b}(t)dt &= \frac{1}{2\pi} \int_{\mathbb{R}_+} \int_{\mathbb{R}} f(t)\widehat{h}^{a,b}(\lambda+i\mu)\phi_{\lambda+i\mu}^{a,b}(t)\frac{d\lambda\Delta^{a,b}(t)dt}{c^{a,b}(-\lambda-i\mu)} \\
 &= \frac{1}{4\pi} \int_{\mathbb{R}} \int_{\mathbb{R}_+} \frac{f(t)\varphi_{\lambda+i\eta}^{a,b}(t)\widehat{g}^{a,b}(\lambda+i\eta)}{c^{a,b}(-\lambda-i\eta)c^{a,b}(\lambda+i\eta)}\Delta^{a,b}(t)dtd\lambda \\
 &\quad - \sum_{\nu \in D_{a,b}} ik_{\nu}\text{Res}_{\lambda=\nu} \left\{ \frac{\int_{\mathbb{R}_+} f(t)\varphi_{\lambda}^{a,b}(t)\widehat{h}^{a,b}(\lambda)\Delta^{a,b}(t)dt}{c^{a,b}(-\lambda)c^{a,b}(\lambda)} \right\} \\
 &= \frac{1}{4\pi} \int_{\mathbb{R}} \frac{\widehat{f}^{a,b}(\lambda+i\eta)\widehat{h}^{a,b}(\lambda+i\eta)}{c^{a,b}(-\lambda-i\eta)c^{a,b}(\lambda+i\eta)}d\lambda \\
 &\quad - \sum_{\nu \in D_{a,b}} ik_{\nu}\text{Res}_{\lambda=\nu} \left\{ \frac{\widehat{f}^{a,b}(\lambda)\widehat{h}^{a,b}(\lambda)}{c^{a,b}(-\lambda)c^{a,b}(\lambda)} \right\}
 \end{aligned}$$

is identically zero for any $h \in C_c^\infty(\mathbb{R})_{\text{even}}$, and we conclude that f is zero almost everywhere. $\square$

3. THE FOURIER TRANSFORM ON REAL HYPERBOLIC SPACES

Let $m \geq 1$ and $n \geq 2$ be two integers and consider the bilinear form $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^{m+n}$ given by

$$\langle x, y \rangle = x_1y_1 + \cdots + x_my_m - x_{m+1}y_{m+1} - \cdots - x_{m+n}y_{m+n}, \quad x, y \in \mathbb{R}^{m+n}.$$

Let $G = SO_o(m, n)$ denote the connected group of $(m+n) \times (m+n)$ matrices preserving $\langle \cdot, \cdot \rangle$ and let $H = SO_o(m-1, n) \subset G$ denote the isotropy subgroup of the point $(1, 0, \dots, 0) \in \mathbb{R}^{m+n}$. Let $K = SO(m) \times SO(n) \subset G$ be the (maximal compact) subgroup of elements fixed by the classical Cartan involution on G : $\theta(g) = (g^*)^{-1}$.

The space $\mathbb{X} := G/H$ is a semisimple symmetric space (an involution τ of G fixing H is given by $\tau(g) = JgJ$, where J is the diagonal matrix with entries $(1, -1, \dots, -1)$). The map $g \mapsto g \cdot (1, 0, \dots, 0)$ induces an embedding of $\mathbb{X}$ in $\mathbb{R}^{m+n}$ as the hypersurface (with $x_1 > 0$ if $m = 1$)

$$\mathbb{X} = \{x \in \mathbb{R}^{m+n} | \langle x, x \rangle = 1\}.$$

Let $\mathbb{Y} := \mathbb{S}^{m-1} \times \mathbb{S}^{n-1}$. We introduce spherical coordinates on $\mathbb{X}$ as

$$x(t, y) = (v \cosh t, w \sinh t), \quad t \in \mathbb{R}_+, \quad y = (v, w) \in \mathbb{Y}.$$

The map is injective, continuous and maps onto a dense subset of $\mathbb{X}$. The (K -invariant) metric distance from $x \in \mathbb{X}$ to the origin is given by $|x| = |x(t, y)| = |t|$.

The unique (up to a constant) G -invariant measure on $\mathbb{X}$ is in spherical coordinates given by

$$\int_{\mathbb{X}} f(x)dx = \int_{\mathbb{R}_+ \times \mathbb{Y}} f(x(t, y))J(t)dtdy$$

(see [9, Part II, Example 2.3]), where $J(t) = \cosh^{m-1} t \sinh^{n-1} t$ is the Jacobian, dt the Lebesgue measure on $\mathbb{R}$ and dy an invariant measure on $\mathbb{Y}$, normalised such that $\int_{\mathbb{Y}} 1dy = 1$.

The action of $SO(m)$ on $C^\infty(\mathbb{S}^{m-1})$ decomposes into irreducible representations $\mathcal{H}^r$ of spherical harmonics of degree $|r|$ (see [10, Introduction]) characterised as

the eigenfunctions of the Laplace–Beltrami operator Δ_m on $\mathbb{S}^{m-1}$ with eigenvalue $-r(r+m-2)$. Here $r=0$ if $m=1$, $r \in \mathbb{Z}$ for $m=2$ and $r \in \mathbb{N} \cup \{0\}$ for $m>2$.

Let $\mathcal{H}^{r,s} = \mathcal{H}^r \otimes \mathcal{H}^s$ and denote the representation of K on $\mathcal{H}^{r,s}$ by $\delta_{r,s}$. Let $d_{r,s} = \dim \mathcal{H}^{r,s}$ and $\chi_{r,s}$ denote the dimension and the character of $\delta_{r,s}$. A function in $L^2(\mathbb{X})$ is said to be of K -type (r,s) if its translates under the left regular action of K span a vector space which is equivalent to $\mathcal{H}^{r,s}$ as a K -module. We write $L^2(\mathbb{X})^{r,s}$ for the collection of functions of K -type (r,s) . The projection $\mathbf{P}^{r,s}$ of $L^2(\mathbb{X})$ onto $L^2(\mathbb{X})^{r,s}$ is given by

$$\mathbf{P}^{r,s}f(x) = d_{r,s} \int_K \chi_{r,s}(k^{-1})f(k \cdot x)dk, \quad f \in L^2(\mathbb{X}),$$

for $x \in \mathbb{X}$; see [10, Chapter V, §3] and [11, Chapter III, §5]. There are similar definitions and results for functions in $L^2(\mathbb{Y})$ and also for functions in $C^\infty(\mathbb{X})$ and $C^\infty(\mathbb{Y})$.

The algebra of left- G -invariant differential operators on $\mathbb{X}$ is generated by the Laplace–Beltrami operator $\Delta_{\mathbb{X}}$ (see [9, Part II, Example 4.1]), which in spherical coordinates is given by

$$\Delta_{\mathbb{X}}f = \frac{1}{J(t)} \frac{\partial}{\partial t} \left(J(t) \frac{\partial f}{\partial t} \right) - \frac{1}{\cosh^2 t} \Delta_m f + \frac{1}{\sinh^2 t} \Delta_n f, \quad f \in C^\infty(\mathbb{X});$$

see [16, p. 455]. It reduces to a differential operator $\Delta_{\mathbb{X}}^{r,s}$ in the t -variable when acting on functions of K -type (r,s) :

$$\begin{aligned} \Delta_{\mathbb{X}}^{r,s}f &= \Delta_{\mathbb{X}}f \\ &= \frac{1}{J(t)} \frac{\partial}{\partial t} \left(J(t) \frac{\partial f}{\partial t} \right) + \frac{r(r+m-2)}{\cosh^2 t} f - \frac{s(s+n-2)}{\sinh^2 t} f, \quad f \in C^\infty(\mathbb{X})^{r,s}. \end{aligned}$$

Consider the differential equation

$$(5) \quad \Delta_{\mathbb{X}}f = \Delta_{\mathbb{X}}^{r,s}f = (\lambda^2 - \rho^2)f, \quad f \in C^\infty(\mathbb{X})^{r,s},$$

where $\rho = \frac{1}{2}(m+n-2)$. Altering the proof of [11, Chapter I, Proposition 2.7] to fit our setup, we see that we can write any function $f \in C^\infty(\mathbb{X})^{r,s}$ in spherical coordinates as

$$(6) \quad f(x(t, y)) = \sum_i f_i(t) \phi_i^{r,s}(y),$$

where $\{\phi_i^{r,s}\} = \{\phi^r \otimes \phi^s\}_i$ is a (finite) basis for $\mathcal{H}^{r,s}$, and f_i is a function of the form $f_i(t) = t^{|s|} f_{i,o}(t)$, with $f_{i,o}$ even. Let $x = -\sinh^2 t$ and $g = (1-x)^{-|r|/2} (-x)^{-|s|/2} f_i$. Then g is a solution to the hypergeometric differential equation with parameters $1/2(\lambda + \rho + |r| + |s|)$, $1/2(-\lambda + \rho + |r| + |s|)$ and $q/2 + |s|$. Let $\Phi_\lambda^{r,s}$ denote the regular (for generic λ) solution to this hypergeometric differential equation satisfying the asymptotic condition $\Phi_\lambda^{r,s}(t) \sim e^{(\lambda-\rho)t}$ for $t \rightarrow \infty$ (for $\Re \lambda > 0$ and when defined). Then

$$\begin{aligned} \Phi_\lambda^{r,s}(t) &= 2^{\lambda-\rho-|r|-|s|} \cosh^{|r|} t \sinh^{|s|} t \\ &\times \frac{\Gamma(\frac{1}{2}(\lambda + \rho + |r| + |s|)) \Gamma(\frac{1}{2}(\lambda - \rho + n - |r| + |s|))}{\Gamma(\lambda) \Gamma(\frac{n}{2} + |s|)} \\ &\times {}_2F_1 \left(\frac{1}{2}(\lambda + \rho + |r| + |s|), \frac{1}{2}(-\lambda + \rho + |r| + |s|); \frac{n}{2} + |s|; -\sinh^2 t \right), \end{aligned}$$

for $\Re \lambda > 0$; see [1, pp. 72 and 76]. We also note that the function $x(t, y) \mapsto \Phi_{\lambda}^{r,s}(t)\phi(y)$ extends to a solution of (5) on $\mathbb{X}$ for any $\phi \in \mathcal{H}^{r,s}$.

Let $\epsilon \in \{0, 1\}$ and define $C_{\epsilon}^{\infty}(\mathbb{Y}) := \{\phi \in C^{\infty}(\mathbb{Y}) \mid \phi(-y) = (-1)^{\epsilon}\phi(y)\}$. The Poisson transform, $F_{\epsilon,\lambda} : C_{\epsilon}^{\infty}(\mathbb{Y}) \rightarrow C^{\infty}(\mathbb{X})$, is defined as

$$F_{\epsilon,\lambda}\phi(x) := \int_{\mathbb{Y}} |\langle x, y \rangle|^{(-\lambda-\rho)} \text{sign}^{\epsilon} \langle x, y \rangle \phi(y) dy, \quad \phi \in C_{\epsilon}^{\infty}(\mathbb{Y}),$$

when $-\Re \lambda \geq \rho$.

Lemma 3.1. *Let $\phi \in C_{\epsilon}^{\infty}(\mathbb{Y})$. The (meromorphic extension of the) function $F_{\epsilon,\lambda}\phi$ is an eigenfunction of the Laplace–Beltrami operator $\Delta_{\mathbb{X}}$ with eigenvalue $\lambda^2 - \rho^2$ (when defined), i.e.,*

$$\Delta_{\mathbb{X}} F_{\epsilon,\lambda}\phi = (\lambda^2 - \rho^2) F_{\epsilon,\lambda}\phi.$$

The asymptotic behaviour of $F_{\epsilon,\lambda}\phi$ for $t \rightarrow \infty$ is given by (when defined)

$$F_{\epsilon,\lambda}\phi(x(t, y)) \sim e^{(\lambda-\rho)t} c(\epsilon, \lambda) \phi(y)$$

for $\Re \lambda > 0$, where $c(\epsilon, \lambda)$ is the so-called c -function for $\mathbb{X}$ given by

$$c(\epsilon, \lambda) = \frac{2^{2\rho-1} \Gamma\left(\frac{m}{2}\right) \Gamma\left(\frac{n}{2}\right)}{\pi} \frac{\Gamma(\lambda)}{\Gamma(\lambda + \rho)} \begin{cases} \tan\left(\frac{\pi}{2}(\lambda + \rho + \epsilon)\right) & \text{if } m \text{ is even,} \\ 1 & \text{if } m \text{ is odd.} \end{cases}$$

Proof. The function $F_{\epsilon,\lambda}\phi$ extends meromorphically to $\mathbb{C}$ by distribution theory; see [16, Lemma 5(a)]. Differentiating under the integral sign for $\Re(\lambda + \rho)$ very negative and then using meromorphic continuation shows that it is an eigenfunction of the Laplace–Beltrami operator $\Delta_{\mathbb{X}}$ with eigenvalue $\lambda^2 - \rho^2$. The asymptotic behaviour is computed in [16, Appendix A]; see also [16, Lemma 4 and Lemma 5]. $\square$

We define the (normalised) Fourier transform $\mathcal{F}f$ of any function $f \in C_c^{\infty}(\mathbb{X})$ as

$$\mathcal{F}f(\epsilon, \lambda, y) := c(\epsilon, -\lambda)^{-1} \int_{\mathbb{X}} |\langle x, y \rangle|^{(\lambda-\rho)} \text{sign}^{\epsilon} \langle x, y \rangle f(x) dx,$$

for $\epsilon \in \{0, 1\}$, $\Re \lambda \geq \rho$ and $y \in \mathbb{Y}$. Let $f \in C_c^{\infty}(\mathbb{X})^{r,s}$ for some fixed K -type (r, s) . We can (re)write the Fourier transform of f as (with $\epsilon \equiv r + s \pmod{2}$)

$$\mathcal{F}f(\epsilon, \lambda, y) = \int_{\mathbb{R}_+} \Phi_{-\lambda}^{r,s}(t) f(x(t, y)) J(t) dt,$$

using spherical coordinates, Schur's Lemma and properties of the Poisson transform; see [1, pp. 74–76] for details. We see that $\mathcal{F}f(\epsilon, \lambda, y)$ extends to a meromorphic function in the λ -variable, with zeros and poles completely determined by the above expression for $\Phi_{\lambda}^{r,s}$. Due to the factor $\Gamma(\lambda)$ in the denominator there are no poles for purely imaginary λ .

Theorem 3.2. *Let $1 \leq p \leq \infty$, $1 \leq q < \infty$. Assume that f is a measurable function on $\mathbb{X}$ satisfying*

$$e^{\alpha|\cdot|^2} e^{(1-\frac{2}{p})\rho|\cdot|} f \in L^p(\mathbb{X}) \quad \text{and} \quad e^{\beta|\cdot|^2} \mathcal{F}f \in L^q(\{0, 1\} \times i\mathbb{R} \times \mathbb{Y})$$

for positive constants α, β such that $\alpha\beta \geq \frac{1}{4}$. Then $f = 0$ almost everywhere.

Proof. Let f be a measurable function satisfying the above growth conditions. The very rapid decay again implies that $f \in L^1(\mathbb{X}) \cap L^2(\mathbb{X})$ and that the Fourier transform $\mathcal{F}f$ is well-defined.

Define $\tilde{\rho} = \rho + |r| + |s|$, $a = |s| + \frac{n}{2} - 1$ and $b = |r| + \frac{m}{2} - 1$ (i.e., $\tilde{\rho} = a + b + 1$). Then

$$\Phi_{\lambda}^{r,s}(t) = 2^{\lambda-\tilde{\rho}} \cosh^{|r|} t \sinh^{|s|} t \frac{\Gamma(\frac{1}{2}(\lambda + \tilde{\rho}))\Gamma(\frac{1}{2}(\lambda - \tilde{\rho} + n + 2|s|))}{\Gamma(\frac{n}{2} + |s|)\Gamma(\lambda)} \varphi_{-i\lambda}^{a,b}(t).$$

Let $f_{r,s}(t, y) := \mathbf{P}^{r,s} f(x(t, y)) / \cosh^{|r|} t \sinh^{|s|} t$. By (6) and continuity of the projection $\mathbf{P}^{r,s}$ we see that $f_{r,s}$ is a measurable function on $\mathbb{R} \times \mathbb{Y}$, even in the t -variable. Also let

$$Q_{r,s}(\lambda) := 2^{\lambda-3\tilde{\rho}} \frac{\Gamma(\frac{1}{2}(\lambda + \tilde{\rho}))\Gamma(\frac{1}{2}(\lambda - \tilde{\rho} + n + 2|s|))}{\Gamma(\frac{n}{2} + |s|)\Gamma(\lambda)}.$$

Then

$$\widehat{f}_{r,s}^{a,b}(i\lambda, y) := \int_{\mathbb{R}_+} f_{r,s}(t, y) \varphi_{i\lambda}^{a,b}(t) \Delta^{a,b}(t) dt = Q_{r,s}(\lambda)^{-1} \mathcal{F} \mathbf{P}^{r,s} f(\epsilon, \lambda, y).$$

We note that $\widehat{f}_{r,s}^{a,b}(\lambda, y)$ is well-defined for all $\lambda \in \mathbb{C}$. Using spherical coordinates and the definition of $\mathbf{P}^{r,s}$, we get the following estimates of $f_{r,s}$ and $\widehat{f}_{r,s}^{a,b}$:

$$e^{\alpha|\cdot|^2} e^{(1-\frac{2}{p})\tilde{\rho}|\cdot|} f_{r,s}(\cdot, y) \in L^p(\mathbb{R}, |\Delta^{a,b}(t)|) \quad \text{and} \quad |Q_{r,s}(i\cdot)| e^{\beta|\cdot|^2} \widehat{f}_{r,s}^{a,b}(\cdot, y) \in L^q(\mathbb{R}),$$

for $y \in \mathbb{Y}$. We now note that $|Q_{r,s}(\lambda)|$ is bounded away from zero for $|\Im \lambda| > 0$ (see [6, 1.18]), which implies that

$$e^{\alpha|\cdot|^2} e^{(1-\frac{2}{p})\tilde{\rho}|\cdot|} f_{r,s}(\cdot, y) \in L^p(\mathbb{R}, |\Delta^{a,b}(t)|) \quad \text{and} \quad e^{\beta|\cdot|^2} \widehat{f}_{r,s}^{a,b}(\cdot, y) \in L^q(\mathbb{R}),$$

for $y \in \mathbb{Y}$. It follows from Theorem 2.5 that $f_{r,s}^{a,b}$ is zero almost everywhere, and thus also that $\mathbf{P}^{r,s} f$ is zero almost everywhere. We conclude the theorem since $f = \sum_{r,s} \mathbf{P}^{r,s} f$. $\square$

4. REMARKS AND FURTHER RESULTS

Hardy's uncertainty principle (the $p = q = \infty$, $\alpha\beta > \frac{1}{4}$ case) for the Jacobi transform and for the Fourier transform on real hyperbolic spaces was proved by the author in [2].

The space $\mathbb{X} = SO_o(m, n)/SO_o(m-1, n)$ is a semisimple Riemannian symmetric space of the non-compact type when $m = 1$ and of the non-Riemannian type when $m > 1$. Our statement and proof generalise to all rank 1 Riemannian symmetric spaces of the non-compact type, using the fact that the Fourier transform of K -finite functions can be expressed by Jacobi functions.

Versions of Theorem 3.2 for Riemannian symmetric spaces were proved by J. Sengupta (see [17, Theorem 2]) and by E. K. Narayanan and S. K. Ray; see [15, Theorem 3.2] (as a corollary of this theorem, valid for functions on semisimple Lie groups). The former result does not have the $e^{(1-\frac{2}{p})\rho|x|}$ factor in the estimate for the function f and is only valid for $\alpha\beta > \frac{1}{4}$. The latter result is stated using the Harish-Chandra function instead of the factor $e^{-\rho|x|}$, and thus by [15, Proposition 2.1 iii)] differs from our result by a factor $(1 + |t|)$.

Let $a = b = \frac{1}{2}$, i.e., $\rho = 2$. In this case the Jacobi transform can be viewed as the spherical Helgason–Fourier transform on $SL(2, \mathbb{C})$. Let $\alpha\beta = \frac{1}{4}$ and assume that $2 < p < \infty$. Following [15], it can be shown that the $e^{(1-\frac{2}{p})2|t|}$ factor is optimal;

there exists for any number s such that $0 < s < 1 - \frac{2}{p}$ a non-zero even function f such that

$$\int_{\mathbb{R}} \left(|f(t)| e^{\alpha|t|^2} e^{s2|t|} \right)^p |\Delta^{a,b}(t)| dt < \infty \quad \text{and} \quad \int_{\mathbb{R}} \left(|\widehat{f}^{a,b}(\lambda)| e^{\beta|\lambda|^2} \right)^q d\lambda < \infty.$$

Let $\mathbb{F}$ be either $\mathbb{C}$ or $\mathbb{H}$ and let $x \mapsto \bar{x}$ be the standard (anti)-involution of $\mathbb{F}$. Let m and n be two positive integers and let $[\cdot, \cdot]$ be the Hermitian form on $\mathbb{F}^{m+n}$ given by

$$[x, y] = x_1 \bar{y}_1 + \cdots + x_m \bar{y}_m - x_{m+1} \bar{y}_{m+1} - \cdots - x_{m+n} \bar{y}_{m+n},$$

for $x, y \in \mathbb{F}^{m+n}$. Let $G = U(m, n; \mathbb{F})$ denote the group of all $(m+n) \times (m+n)$ matrices over $\mathbb{F}$ preserving $[\cdot, \cdot]$. Thus $U(m, n; \mathbb{C}) = U(m, n)$ and $U(m, n; \mathbb{H}) = Sp(m, n)$ in standard notation. Let H be the subgroup of G stabilising the line $\mathbb{F}(1, 0, \dots, 0)$ in $\mathbb{F}^{m+n}$. We can identify H with $U(1, 0; \mathbb{F}) \times U(m-1, n; \mathbb{F})$ and the homogeneous space G/H (which is a reductive symmetric space) with the projective image of the space $\{z \in \mathbb{F}^{m+n} \mid [z, z] = 1\}$. The statement and proof of the L^p version of Hardy's uncertainty principle for the Fourier transform on G/H follows from above, either embedding G/H into $SO_o(dm, dn)/SO_o(dm-1, dn)$, with $d = \dim_{\mathbb{R}} \mathbb{F}$, or again expressing the Fourier transform of K -finite functions using modified Jacobi functions. See [1, p. 117] for more details.

REFERENCES

1. N. B. Andersen, Paley–Wiener Theorems for Hyperbolic Spaces, *J. Funct. Anal.* **179** (2001), 66–119. MR **2002f**:43008
2. N. B. Andersen, Hardy's uncertainty principle on Hyperbolic Spaces, *Bull. Austral. Math. Soc.* **66** (2002), 163–170.
3. E. P. van den Ban, H. Schlichtkrull, Expansions for Eisenstein integrals on semisimple symmetric spaces, *Ark. Mat.* **35** (1997), 59–86. MR **98e**:22003
4. M. Cowling, J. F. Price, Generalisations of Heisenberg's inequality, *Harmonic analysis (Cortona, 1982)*, Lecture Notes in Math., 992, Springer, Berlin, 1983, 443–449. MR **86g**:42002b
5. M. Cowling, A. Sitaram, M. Sundari, Hardy's uncertainty principle on semisimple groups, *Pacific J. Math.* **192** (2000), 293–296. MR **2001c**:22007
6. Erdélyi et al, *Higher Transcendental Functions, Volume I*, McGraw–Hill, New York, 1973. MR **49**:9265
7. M. Flensted-Jensen, Spherical functions on a simply connected semisimple Lie group. II. The Paley–Wiener theorem for the rank one case, *Math. Ann.* **228** (1977), 65–92. MR **56**:16267
8. G. H. Hardy, A theorem concerning Fourier transforms, *J. London Math. Soc.* **8** (1933), 227–231.
9. G. Heckman and H. Schlichtkrull, *Harmonic Analysis and Special Functions on Symmetric Spaces*, Academic Press, San Diego, 1994. MR **96j**:22019
10. S. Helgason, *Groups and Geometric Analysis*, Academic Press, Orlando, 1984. MR **86c**:22017
11. S. Helgason, *Geometric Analysis on Symmetric Spaces*, Mathematical Surveys and Monographs, Vol. 39, American Mathematical Society, Providence, Rhode Island, 1994. MR **96h**:43009
12. L. Hörmander, A uniqueness theorem of Beurling for Fourier transform pairs, *Ark. Mat.* **29** (1991), 237–240. MR **93b**:42016
13. T. H. Koornwinder, A new proof of a Paley–Wiener type theorem for the Jacobi transform, *Ark. Mat.* **13** (1975), 145–159. MR **51**:11028
14. T. H. Koornwinder, Jacobi functions and analysis on noncompact semisimple groups, *Special Functions: Group Theoretical Aspects and Applications*, Reidel, 1984, 1–84. MR **86m**:33018
15. E. K. Narayanan, S. K. Ray, L^p versions of Hardy's theorem on semisimple Lie groups, *Proc. Amer. Math. Soc.* **130** (2002), 1859–1866. MR **2003a**:22009
16. W. Rossmann, Analysis on Real Hyperbolic Spaces, *J. Funct. Anal.* **30** (1978), 448–477. MR **80f**:43021

17. J. Sengupta, The uncertainty principle on Riemannian symmetric spaces of the noncompact type, *Proc. Amer. Math. Soc.* **130** (2002), 1009–1017. MR **2003a**:43009
18. A. Sitaram, M. Sundari, An analogue of Hardy's theorem for very rapidly decreasing functions on semisimple Lie groups, *Pacific J. Math.* **177** (1997), 187–200. MR **99a**:22018

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